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CONVERGENCE OF VORTEX METHODS FOR 3-D EULER EQUATIONS*

Jia-fu Lin
(*Institute of System Science, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, Beijing, 100080, China; School of Mathematical Sciences, Peking University, Beijing 100871, China*)

Abstract

In this paper we apply an approach introduced in [6] [7], where continuous norms and hign order estimates and extension are used, to study the convergence of vortex methods for the 3-D Euler equations in bounded domains as the initial vorticity ω_0 and the curl of the body force f are non-compactly supported functions. Convergence results are proved.

Key words: Euler equations, Vortex methods, Convergence, Initial-boundary value problem.

1. Introduction

The convergence problem of vortex methods for the Euler equations has been studied by many authors. Hald and DelPrete proved the convergence for two-dimensional initial value problems [3]. Three-dimensional initial value problems were studied by Beale and Majda [2] and Beale [1]. Ying [4] and Ying and Zhang [5], [6] proved the convergence of vortex methods for two-dimensional initial-boundary value problems of the Euler equations. Ying [7] proved the convergence of vortex methods for three-dimensional initial-boundary value problems of the Euler equations under the assumption that the initial vorticity ω_0 and the curl of the body force f are compactly supported.

In this paper, we will prove the convergence of the vortex method for three-dimensional initial-boundary value problems without assuming that the ω_0 and $\nabla\times f$ are compactly supported. In contrast to [7], there are two new difficulties. One is how to extend the physical quantities such as velocity and the force function outside Ω. The other one is that the approximate velocity g^ϵ (ie. eqn. (20)) is no longer divergence-free outside Ω. We will use the approach in [4] [5] and [7] to perform the extension of the physical quantites outside Ω. Although $\nabla \cdot g^\epsilon \neq 0$ outside Ω, we will show that $\nabla \cdot g^\epsilon$ is small (see eqn. (48) (49)). This is sufficient for our convergence analysis.

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2. Formulation of Vortex Methods

We consider the initial-boundary value problems of inviscid incompressible flow as follows:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla) u + \frac{\nabla p}{\rho} = f, \tag{1}$$

$$\nabla \cdot u = 0, \tag{2}$$

$$u \cdot n|_{x \in \partial\Omega} = 0, \tag{3}$$

$$u|_{t=0} = u_0(x), \tag{4}$$

where $u = (u_1, u_2, u_3) \in \mathbf{R}^3$ is velocity, $p \in \mathbf{R}$ is pressure, ρ is a constant standing for density, f is body force, u_0 is the initial distribution of the velocity satisfying $\nabla \cdot u_0 = 0$ and $u_0 \cdot n|_{\partial\Omega} = 0$ and ∇ is the gradient operator $(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$. The domain Ω is assumed bounded with a sufficiently smooth boundary $\partial\Omega$ and n is the unit outward normal vector along the boundary. For simplicity we assume Ω is simply connected and convex. Let $\omega = \nabla \times u$ be the vorticity and F be the curl of the body force f, then applying the operator curl to the equation (1) and the initial condition (4), we obtain

$$\frac{\partial \omega}{\partial t} + (u \cdot \nabla)\omega - (\omega \cdot \nabla) u = F, \tag{5}$$

$$\omega|_{t=0} = \omega_0 \equiv \nabla \times u_0. \tag{6}$$

To get velocity u from the vorticity ω, we need the stream function, which is not unique for three-dimensional problems. We accept the formulation in [7]

$$-\triangle \psi + \nabla z = \omega, \ u = \nabla \times \psi, \tag{7}$$

$$\nabla \cdot \psi = 0, \tag{8}$$

$$\psi \times n \,|_{x \in \partial\Omega} = 0, \ z \,|_{x \in \partial\Omega} = 0, \tag{9}$$

where ψ is the stream function. Problems (7)-(9) admit a unique smooth solution for any smooth ω (see [7], §4).

We assume that the problem (1)-(4) admits a sufficiently smooth solution (u, p) on $\overline{\Omega} \times [0, T]$. Under this assumption, we consider the vortex method formulation.

Let positive constants h and ϵ be mesh sizes. $j = (j_1, j_2, j_3) \in \mathbf{Z}^3$, $B_j = \{x; j_i h < x_i < (j_i + 1)h, i = 1, 2, 3\}$, $X_j = ((j_1 + \frac{1}{2})h, (j_2 + \frac{1}{2})h, (j_3 + \frac{1}{2})h)$. We define a "vortex blob" function $\zeta(x)$ with a support in ball $\{x; |x| \leq 1\}$, which satisfies

$$\int \zeta(x) dx = 1. \tag{10}$$

Consider the following scheme: Set $\Omega_d = \{x; \mathrm{dist}(x, \overline{\Omega}) < d\}$, where $d > 0$ is a parameter. We solve the problem in $\overline{\Omega}_d \times [0, T]$. u_0, F are not defined outside Ω and $\Omega \times [0, T]$ and we extend u_0, F in the following way. By (7)-(9), we get $\psi(x, 0)$, then extend it smoothly to $\mathbf{R}^3$. $\psi(x, 0)$ needn't satisfy the divergence free condition outside Ω and we assume $\psi(x, 0)$ is compactly supported. Set $u_0 = \nabla \times \psi(x, 0)$, then u_0 is extended