INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 2, Number 2, Pages 197–220

CONVERGENCE AND STABILITY OF BALANCED IMPLICIT METHODS FOR SYSTEMS OF SDES

HENRI SCHURZ

(Communicated by Edward J. Allen)

This paper is dedicated to academic fairness and honesty.

Abstract. Several convergence and stability issues of the balanced implicit methods (BIMs) for systems of real-valued ordinary stochastic differential equations are thoroughly discussed. These methods are linear-implicit ones, hence easily implementable and computationally more efficient than commonly known nonlinear-implicit methods. In particular, we relax the so far known convergence condition on its weight matrices c^j. The presented convergence proofs extend to the case of nonrandom variable step sizes and show a dependence on certain Lyapunov-functionals $V : \mathbb{R}^d \to \mathbb{R}^1_+$. The proof of L^2-convergence with global rate 0.5 is based on the stochastic Kantorovich-Lax-Richtmeyer principle proved by the author (2002). Eventually, p-th mean stability and almost sure stability results for martingale-type test equations document some advantage of BIMs. The problem of weak convergence with respect to the test class $C^2_{b(\kappa)}(\mathbb{R}^d, \mathbb{R}^1)$ and with global rate 1.0 is tackled too.

Key Words. Balanced implicit methods, linear-implicit methods, conditional mean consistency, conditional mean square consistency, weak V-stability, stochastic Kantorovich-Lax-Richtmeyer principle, L^2-convergence, weak convergence, almost sure stability, p-th mean stability.

1. Introduction

There are plenty of numerical methods for systems of ordinary stochastic differential equations (SDEs)

$$dX_t = a(t, X_t)\, dt + \sum_{j=1}^{m} b^j(t, X_t) dW_t^j \tag{1}$$

driven by standard one-dimensional Wiener processes $W^j = (W_t^j)_{0 \le t \le T}$ and interpreted in Itô sense (for the sake of simplicity of this representation), where $a, b^j \in C^0([0, T] \times \mathbb{R}^d, \mathbb{R}^d)$. For an overview, e.g. see Kloeden, Platen and Schurz [8], Milstein [10], Talay [18] or Schurz [13]. However, only a few of them can tackle the problem of almost sure stochastic stability (as seen section 3) or of invariances with respect to certain subsets of $\mathbb{R}^d$ as commonly met in mathematical finance or biology. One of the successful approximation techniques in this respect is given

Received by the editors April 11, 2004 and, in revised form, July 30, 2004.
2000 *Mathematics Subject Classification.* 65C30, 65L20, 65D30, 34F05, 37H10, 60H10.
This research was supported by research grant "Logistic Equations" from ORDA at SIU.

by the class of *balanced implicit methods* (BIMs) as introduced by Milstein, Platen and Schurz [11]. They follow the iteration scheme

$$Y_{k+1} = Y_k + \sum_{j=0}^{m} b^j(t_k, Y_k)\Delta W_k^j + \sum_{j=0}^{m} c^j(t_k, Y_k)|\Delta W_k^j|(Y_k - Y_{k+1}) \tag{2}$$

where $\Delta W_k^j = W_{t_{k+1}}^j - W_{t_k}^j$, $c^j \in C^0([0,T]\times\mathbb{R}^d, \mathbb{R}^{d\times d})$ with the convention $W_t^0 = t$ and $b^0(t,x) = a(t,x)$ along discretizations

$$0 \le t_0 < t_1 < ... < t_k < ... < t_{n_T} \le T \tag{3}$$

with both variable or constant step sizes $\Delta_k = t_{k+1} - t_k$, finite, nonrandom (fixed) terminal time $T > 0$ and maximum step size

$$\Delta = \Delta_{max} = \max_{k=0,1,...,n_T-1} |t_{k+1} - t_k|. \tag{4}$$

For the sake of abbreviation, we use the identities $b^0(t,x) = a(t,x)$ and $W_t^0 = t$ throughout this paper. In fact, these numerical methods (2) can be implemented in explicit form thanks to their linear-implicit structure. Therefore, they are easily and efficiently implementable. They can guarantee enlarged stability regions compared to the forward Euler methods with the matrix-valued weights $c^j \equiv \mathcal{O}, j = 1,2,...,m$ ($\mathcal{O}$ denotes the $d \times d$-zero matrix) contained in the family of BIMs (2). BIMs (2) possess the *one-step representations*

$$Y_{s,y}(t) = y + M_{s,y}^{-1}(t) \sum_{j=0}^{m} b^j(s,y)(W_t^j - W_s^j) \quad \text{with} \tag{5}$$

$$M_{s,y}(t) = I_d + \sum_{j=0}^{m} c^j(s,y)|W_t^j - W_s^j| \tag{6}$$

while assuming the existence of $M_{s,y}^{-1}(t)$ for all $0 \le t - s \le \delta_0 \le T$ and all $y \in \mathbb{R}^d$ and all $s,t \in [0,T]$, where I_d denotes the $d \times d$ unit matrix of $\mathbb{R}^{d\times d}$. Using the one-step representation (5), the *continuous polygonal representation* of the scheme (2) can recursively be written as

$$Y_{0,y_0}(t) = Y_k + M_{t_k,Y_k}^{-1}(t) \sum_{j=0}^{m} b^j(t_k, Y_k)(W_t^j - W_{t_k}^j) \quad \text{if} \quad t_k \le t \le t_{k+1} \tag{7}$$

for all times $t \in [0,T]$, started at $Y_0 = Y_{0,y_0}(t_0) = y_0 \in \mathbb{R}^d$, where we have the identity $Y_{0,y_0}(t_{k+1}) = Y_{t_k,Y_k}(t_{k+1}) = Y_{k+1}$ for all $k = 0,1,...,n_T - 1$.

The main interest of this paper is to prove rigorously convergence and stability of BIMs (2) applied to systems of SDEs (1). In detail we are going to discuss the issues of almost sure stability, exponential p-th mean and weak V-stability, conditional mean consistency with rate $r_0 \ge 1.5$, conditional mean square consistency with rate $r_2 \ge 1.0$, global L^2-convergence with rate $r_g \ge 0.5$ and weak convergence of these methods for the test class $C^2_{b(\kappa)}$ with coefficients $b^j \in C^0_{b(\kappa)} \cap C^0_{Lip}$ along nonrandom partitions of time-intervals $[0,T]$ with both variable and constant step sizes with maximum step size $\Delta_{max} \le \delta_0 \le \min(1,T)$. Due to the necessarily immense volume, we refrain from a systematic comparison study comparing with the pool of other, commonly known numerical methods in this paper. Such a more laborious work is left to the future and needs extensive simulation studies.

The paper is organized as follows. After this introduction, Section 2 investigates the class of BIMs (2) with respect to conditional mean and mean square consistency. Thereafter, we study weak V-stability, exponential p-th mean and almost sure