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ON THE COLLOCATION METHODS FOR HIGH-ORDER VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS*

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Abstract

We study the numerical solution of high-order Volterra integro-differential equations by means of collocation techniques in certain polynomial spline spaces. The attainable order of global convergence and local superconvergence of these methods is analyzed.

§1. Introduction

In this paper we shall be concerned with the approximate solution of the initial-value problem for a nonlinear (high-order) Volterra integro-differential equation (VIDE)

$$y^{(r)}(t) = f(t, y(t), \cdots, y^{(r-1)}(t)) + \int_0^t k(t,s,y(s),\cdots,y^{(r-1)}(s))ds, \ t \in I := [0,T], \quad (1.1)$$

with initial conditions $y^{(j)}(0) = y_{0j}, 0 \le j \le r-1$. Here, $r \ge 1$ is a natural number; the given functions $f : I \times I\!R^r \to I\!R$ and $k : S \times I\!R^r \to I\!R$ (with $S := \{(t,s) : 0 \le s \le t \le T\}$) are assumed to be continuous and such that (1.1) has a unique solution $y \in C^r(I)$ satisfying the given intial conditions.

The analysis of the convergence properties of any numerical method for (1.1) will necessarily involve the linearization of the given equation and lead to a problem of the form

$$y^{(r)}(t) = \sum_{j=0}^{r-1} a_j(t)y^{(j)}(t) + b(t) + \int_0^t (\sum_{j=0}^{r-1} K_j(t,s)y^{(j)}(s))ds, \quad t \in I. \quad (1.2)$$

Equations of the form (1.1) (or (1.2)) have a frequent use in the mathematical modeling of various physical and biological phenomena. Many authors studied the first-order problem using collocation methods. A complete convergence theory of collocation approximations in $S_m^{(0)}(Z_N)$ (see (1.5) for the symbol) for (1.1) when $r = 1$, including local superconvergence results and the discretization of the collocation equations, may be found in Brunner (1984) and Brunner & Houwen (1986). For $r = 2$, Aguilar & Brunner (1986) considered the equations of the form

$$y''(t) = f(t, y(t)) + \int_0^t k(t,s,y(s))ds, \quad t \in I. \quad (1.3)$$

The equations of the form (1.3) arise, for example, in one-dimensional visco-elastic problems, in the construction of a field-theoretical model for electron-beam devices, and in problems of one-dimensional heat flow in materials with memory (see Burton (1983), and Hrusa & Nohel (1984)). For the linear counterpart of (1.3), the attainable order of (local) convergence

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of the numerical method used is analysed in Aguilar & Brunner (1986). Furthermore, the convergence analysis of high-order equations of the form

$$y^{(r)}(t) = p(t)y(t) + q(t) + \int_0^t k(t,s)y(s)ds, \quad t \in I, \tag{1.4}$$

could be found in Aguilar (1986).

In practical applications, one occasionally encounters high-order integro-differential equations which are of the form (1.1) or (1.2) (see also Burton (1983)). Pouzet (1962) introduced a special class of explicit Runge-Kutta methos for the numerical solutions of certain classes of (1.1) when $r = 2$. Wahr (1977) investigated the convergence and application of collocation methods to high-order linear VIDEs. The recent paper by Bellen (1985) and recent book by Bruuner & Houwen (1986) contatin, among other things, a concise survey of recent advances in the numerical solution of VIDEs by collocation and related methods.

In this paper, VIDEs of the form (1.1) will be solved numerically in certain polynomial spline spaces. In order to describe these approximation spaces, let

$$\Pi_N : 0 = t_0 < t_1 < \cdots < t_N = T, \qquad \text{where } t_n = t_n^{(N)}$$

be a mesh for the given interval I, and set

$$\sigma_n := [t_n, t_{n+1}], \quad h_n := t_{n+1} - t_n, \quad n = 0, \cdots, N-1,$$

$$Z_n := \{t_n : n = 1, \cdots, N-1\} \text{ (interior mesh points, or knots)},$$

$$\bar{Z}_N := Z_N \cup T.$$

Moreover, let P_k denote the space of (real) polynomials of degree not exceeding k. We then define, for given integers k and d with $0 \le d \le k-1$,

$$\begin{aligned} S_k^{(d)}(Z_N) := \Big\{ u : u = u_n \in P_k \text{ on } \sigma_n, 0 \le n \le N-1; \\ u_{n-1}^{(j)}(t_n) = u_n^{(j)}(t_n) \text{ for } t_n \in Z_N \text{ and } 0 \le j \le d \Big\} \end{aligned} \tag{1.5}$$

to be the space of polynomial splines (or piecewise polynomials) of degree k whose elements possess the knots Z_N, and d times continuously differentiable on I. It is easily seen that the dimension of this linear vector space is equal to $N(k-d) + (d+1)$. In the following we shall deal with the space $S_{m+r-1}^{(r-1)}(Z_N)$, with $r \ge 1$, whose dimension is given by $Nm + \Gamma$.

In order to determine an approximation $u \in S_{m+r-1}^{(r-1)}(Z_N)$ to the solution of the VIDE (1.1), let $\{c_j\}$ be a given set of parameters satisfying

$$0 \le c_1 < c_2 < \cdots < c_m \le 1,$$

and define the points

$$t_{nj} := t_n + c_j h_n, \quad j = 1, \cdots, m; \ n = 0, \cdots, N-1, \tag{1.6a}$$

with

$$X_n := \{t_{nj} : j = 1, \cdots, m\}, \tag{1.6b}$$