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A NUMERICAL METHOD FOR A SYSTEM OF GENERALIZED NONLINEAR SCHRÖDINGER EQUATIONS*

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§ 1. Introduction

In this paper, we consider the following initial-boundary value problem of the system of generalized nonlinear Schrödinger equations

$$\begin{cases} i\boldsymbol{u}-\dfrac{\partial}{\partial x}A(x)\dfrac{\partial \boldsymbol{u}}{\partial x}+\beta(x)q(|\boldsymbol{u}|^2)\boldsymbol{u}+F(x,t)\boldsymbol{u}=\boldsymbol{G}(x,t), \quad t>0,\ 0<x<1, & (1.1)\\ \boldsymbol{u}|_{x=0}=\boldsymbol{u}|_{x=1}=0, \quad t\geqslant 0, & (1.2)\\ \boldsymbol{u}|_{t=0}=\boldsymbol{u}_0(x), \quad 0<x<1, & (1.3)\end{cases}$$

where $\boldsymbol{u}(x,t)$ is an unknown complex functional vector, $A(x)=(a_m(x))$ is a real diagonal matrix, $\beta(x)$ and $q(s)$ are real functions, $\boldsymbol{u}_0(x)$ and $\boldsymbol{G}(x,t)$ are complex functional vectors. In [1] and [2] a class of stable and convergent finite difference schemes of (1.1) have been proved. In [3] the existence and uniqueness of the generalized solution for system (1.1) have been obtained. Now we consider a wide class of functions, which should satisfy one of the following conditions for the nonlinear terms $\beta(x)q(s)$:

(i) $K_\beta\geqslant\beta(x)\geqslant 0$; $Q(s)\geqslant 0$, $s\in[0,\infty)$, $Q(s)=\int_0^s q(z)dz$,

(ii) $|\beta(x)|\leqslant K_\beta$, $|q'(s)|\leqslant K_q$, $s\in[0,\infty)$,

(iii) $|\beta(x)|\leqslant K_\beta$, $q(|\boldsymbol{u}|^2)=|\boldsymbol{u}|^{2p}$, $0\leqslant p<2$.

We will construct a new finite difference scheme, which possesses two conservation quantities, and will prove that it is unconditionally stable and convergent. Furthermore, by using the Galerkin method, we will prove the existence and uniqueness of the generalized solution for problem (1.1)—(1.3), and the convergence of the iteration method in finding a solution of the finite difference scheme under the condition $k=O(h)$. The notations and conventions here are adopted as in [1].

§ 2. Convergence and Stability

The schemes in [1] and [2] have a common defect: they can not preserve the conservation of energy. Now we propose the following scheme to tackle this

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problem:

$$\begin{cases} i(\phi^{n+1}_{m,j})_{\bar t}-\frac{1}{2}\{a_{m,j+1/2}[(\phi^{n+1}_{m,j})_x+(\phi^{n}_{m,j})_x]\}_{\bar x} \\ \qquad +\frac{\beta_j}{2}\frac{Q(|\phi^{n+1}_j|^2)-Q(|\phi^n_j|^2)}{|\phi^{n+1}_j|^2-|\phi^n_j|^2}(\phi^{n+1}_{m,j}+\phi^n_{m,j}) \\ \qquad +\frac{1}{2}\sum_{l=1}^{M} f^{n+1/2}_{m,l,j}(\phi^{n+1}_{m,j}+\phi^n_j)=G^{n+1/2}_{m,j}, \quad 1\le m\le M, 1\le j\le J-1, & (2.1) \\ \phi^n_{m,0}=\phi^n_{m,J}=0, \quad 1\le m\le M, & (2.2) \\ \phi^0_{m,j}=u_{0,m}(x_j), \quad 1\le m\le M, 1\le j\le J-1, & (2.3) \end{cases}$$

where $Q(s)=\int_0^s q(z)dz$. First we take a priori estimates for the finite difference solution.

Lemma 1. *Suppose that $f_{m,l}(x,t)=f_{l,m}(x,t)$, $\|G_m(x,t)\|_{L_2}\le K_G$, $u_{0,m}\in L_2[0,1]$, $1\le m, l\le M$, where K_G is a positive constant. Then there is an estimate for the solution of problem (2.1)—(2.3):*

$$\|\phi^n_m\|\le C_2,\ 0\le nk\le T,\ 1\le m\le M,$$

where C_2 is a positive constant.

Proof. Computing the inner product of both sides of (2.1) with $(\overline{\phi^{n+1}_{m,j}}+\overline{\phi^n_{m,j}})$, summing up for m from 1 to M, and taking the imaginary part in the resulting relation, we have

$$h\sum_{m=1}^{M}\sum_{j=1}^{J-1}(|\phi^{n+1}_{m,j}|^2)_{\bar t}=h\sum_{m=1}^{M}\sum_{j=1}^{J-1}I_m[G^{n+1/2}_{m,j}(\overline{\phi^{n+1}_{m,j}}+\overline{\phi^n_{m,j}})]. \tag{2.4}$$

By using the discrete Gronwall inequality, the conclusion of the lemma can be obtained.

Lemma 2 (Sobolev estimate [4]). *Suppose $u\in L_q(R^n)$, $D^m u\in L_r(R^n)$, $1\le q, r\le\infty$. Then for $0\le j\le m$, $\frac{j}{m}\le\alpha\le 1$, we have*

$$\|D^j u\|_{L_p}\le C\|D^m u\|^{\alpha}_{L_r}\|u\|^{1-\alpha}_{L_q},$$

where $\frac{1}{p}=\frac{j}{n}+\alpha\left(\frac{1}{r}-\frac{m}{n}\right)+(1-\alpha)\frac{1}{q}$, and C is a positive constant.

Lemma 3. *Suppose that the conditions of Lemma 1 are satisfied, $0<a\le a_m(x)\le A$, $|f_{m,l}(x,t)|\le K_F$, $\left|\frac{\partial f_{m,l}(x,t)}{\partial t}\right|\le K_F$, $\left\|\frac{\partial G_m(x,t)}{\partial t}\right\|_{L_2}\le K_G$, $u_{0,m}(x)\in H^1_0[0,1]$, $1\le m, l\le M$, where a, A, K_F and K_G are positive constants and $q(s)\in C^1$, and assume that one of the following conditions are satisfied:*

(i) $K_\beta\ge\beta(x)\ge 0$, $Q(s)\ge 0$, $s\in[0,\infty)$;

(ii) $|\beta(x)|\le K_\beta$, $|q'(s)|\le K_q$, $s\in[0,\infty)$;

(iii) $|\beta(x)|\le K_\beta$, $q(s)=s^p$, $0\le p<2$,

where p is a real number, K_β and K_q are positive constants. Then for the solution of problem (2.1)—(2.3), there is an estimate

$$\|(\phi^n_m)_x\|\le C_3,\ 0\le nk\le T,\ 1\le m\le M,$$

where C_3 is a positive constant.

Proof. Computing the inner product of both sides of (2.1) with $(\overline{\phi^{n+1}_{m,j}})_{\bar t}$, summing up for m from 1 to M, and taking the real part, we have