NUMERICAL SOLUTION OF RADON'S PROBLEM IN A TWO DIMENSIONAL SPACE*

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§ 1

As in the Fourier transform of a function, we associate with a function $f(x)$ its Radon transform $g(\alpha, p)$, defined by the following integral of $f(x)$ over the hyperplane with unit normal α and distance p from the origin:

$$Rf=\int_{x\cdot\alpha=p} f(x)\,d\omega_x=g(\alpha, p) \tag{1}$$

for $p\in R^1$, x, $\alpha\in R^n$, $|\alpha|=1$. The Radon problem consists in solving equation (1) for $f(x)$ from $g(\alpha, p)$. This problem is of great importance in many applications, for instance, in the reconstruction of objects from X-ray pictures ([1], [2]).

In this paper we shall merely treat Radon's problem for $n=2$. Describing the unit normal α by its polar angle θ, we can rewrite (1) as

$$Rf=\int_{-\infty}^{\infty} f(\theta;\, p,\, r)\,dr=g(\theta, p), \tag{2}$$

$$f(\theta;\, p,\, r)=f(p\cos\theta+r\sin\theta,\ p\sin\theta-r\cos\theta),$$

or

$$4\pi\int_p^{\infty}\frac{\eta a(\eta)}{\sqrt{\eta^2-p^2}}\,d\eta=G(p),\quad G(p)=\int_0^{2\pi} g(\theta, p)\,d\theta,$$

where $a(\eta)$ is the average of f on the circle of radius η about the origin:

$$a(\eta)=\frac{1}{\omega\eta^{n-1}}\int_{|x|=\eta} f(x)\,ds_x.$$

The problem of determining the solution $a(\eta)$ (in particular, $f(\eta)$, if the function f has the property of circular symmetry[1], i.e. $f(x_1, x_2)=f(\eta)$, $x_1^2+x_2^2=\eta^2$) from the initial data $G(p)$ has been explored in [3] in greater detail.

Radon's problem (2) is not well-posed on the pair of spaces $(\bar{C}, L_2)$[4], where

$$L_2=L_2(H),$$

$$\bar{C}=\bar{C}(K_{\bar{T}})=\{f(x): f(x) \text{ is continuous and has compact support } K_\xi,\ 0<\xi\leqslant\bar{T}\},$$

$$H=\{(\alpha, p): p\in R^1,\ \alpha\in R^2,\ |\alpha|=1\}$$

is the unit cylinder in R^3 and K_ξ is the circle of radius ξ about the origin. This is because the range of Radon's integral operator R clearly does not coincide with L_2 and the inverse R^{-1} of the operator R is not continuous.

It should be pointed out that the reciprocity formula for $f(x)$ holds ([1], [2]):

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$$(2\pi i)^n f(z) = (\Delta_z)^{\frac{n-2}{2}} \int_{\Omega_x} d\omega_x \int_{p=-\infty}^{p=\infty} \frac{dg(x, p)}{p - z\cdot x}.$$

§ 2

Below, we shall use the finite-difference method to solve the Radon equation (2). In order to clarify the essentials of the method, in this section we study merely a semi-discrete scheme, where only the variable p is discretized:

$$p_i = i\tau,\ i = -n-1,\ -n,\ \cdots,\ n,\ n+1,\ \tau = \frac{\overline{T}}{n+1}$$

while the variable r is left continuous. The fully discrete case is discussed in section 3. Let f_τ be a vector-valued function:

$$f_\tau = f_\tau(\theta, r) = (f_{-n-1}(\theta, r), \cdots, f_{n+1}(\theta, r))$$

the components of which, i.e., $f_i(\theta, r)$, are defined on the segments:

$$\{(p_i, r):\ -\overline{T} \leqslant r \leqslant \overline{T}\} \quad (i = -n-1, \cdots, n+1).$$

Now let us consider the functional M_τ^α of the argument function f_τ, which may be taken to be an arbitrary continuous function with a continuous derivative:

$$M_\tau^\alpha[\theta; f_\tau, g_\tau] = \sum_{i=-n}^{n} \tau \left[\int_{-\overline{T}}^{\overline{T}} f_i(\theta, r)dr - g_i(\theta)\right]^2 + \alpha \sum_{i=-n}^{n} \tau \int_{-\overline{T}}^{\overline{T}} \left\{f_i^2(\theta, r) + \left[\frac{df_i}{dr}\right]^2\right\} dr$$
$$+ \alpha \sum_{i=-n-1}^{n} \tau \int_{-\overline{T}}^{\overline{T}} \left[\frac{f_{i+1} - f_i}{\tau}\right]^2 dr,$$

where g_τ is a given vector:

$$g_\tau = g_\tau(\theta) = (g_{-n}(\theta), \cdots, g_n(\theta)).$$

Theorem 1. *For every g_τ and every positive parameter α, there exists a unique continuous function $f_\tau^\alpha(\theta, r)$ with a continuous derivative for which the functional $M_\tau^\alpha[\theta; f_\tau, g_\tau]$ attains its greatest lower bound:*

$$M_\tau^\alpha[\theta; f_\tau^\alpha, g_\tau] = \inf M_\tau^\alpha[\theta; f_\tau, g_\tau].$$

Proof. 1) The desired function $f_\tau^\alpha(\theta, r)$ should be determined by the integro-differential equation of Euler:

$$\alpha L[f_\tau] = \int_{-\overline{T}}^{\overline{T}} f_\tau(\theta, s)ds - g_\tau(\theta), \tag{3}$$

$$L[f_\tau] = \frac{d^2 f_\tau}{dr^2} + Bf_\tau,$$

$$B = \begin{pmatrix} -\left(1+\frac{1}{\tau^2}\right) & \frac{1}{\tau^2} & & & \\ \frac{1}{\tau^2} & -\left(1+\frac{1}{\tau^2}\right) & \frac{1}{\tau^2} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{1}{\tau^2} & -\left(1+\frac{1}{\tau^2}\right) & \frac{1}{\tau^2} \\ & & & \frac{1}{\tau^2} & -\left(1+\frac{1}{\tau^2}\right) \end{pmatrix}$$