Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **5**, No. 6, pp. 791-808

DOI: 10.4208/aamm.12-m12142
December 2013

A Solver for Helmholtz System Generated by the Discretization of Wave Shape Functions

Long Yuan[1] and Qiya Hu[2,*]

[1] *Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China*
[2] *LSEC, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China*

Received 25 October 2012; Accepted (in revised version) 19 March 2013

Available online 6 September 2013

Abstract. An interesting discretization method for Helmholtz equations was introduced in B. Després [1]. This method is based on the ultra weak variational formulation (UWVF) and the wave shape functions, which are exact solutions of the governing Helmholtz equation. In this paper we are concerned with fast solver for the system generated by the method in [1]. We propose a new preconditioner for such system, which can be viewed as a combination between a coarse solver and the block diagonal preconditioner introduced in [13]. In our numerical experiments, this preconditioner is applied to solve both two-dimensional and three-dimensional Helmholtz equations, and the numerical results illustrate that the new preconditioner is much more efficient than the original block diagonal preconditioner.

AMS subject classifications: 65F08, 65N55

Key words: Helmholtz equation, ultra weak variational formulation, wave shape functions, preconditioner, iteration counts.

1 Introduction

A wide range of physical problems (for example, the acoustic scattering) in steady-state oscillation can be characterized using the Helmholtz equation (i.e., harmonic wave problem). Yet the main difficulties in numerically solving harmonic wave problems lie in the non-coercive nature of the problem and in the fact that the solution is oscillatory with a wavelength $\lambda=2\pi/\omega$. There exist many numerical methods for solving Helmholtz equation in literature: finite volume [2] and finite difference methods [3], the finite element

*Corresponding author.
Email: yuanlong@lsec.cc.ac.cn (L. Yuan), hqy@lsec.cc.ac.cn (Q. Y. Hu)

method (FEM) [4, 5], the Galerkin least-squares FEM [6], the quasi-stabilized finite element method [7], the Partition of Unity Method (PUM) [8,9], and the predefined reduced bases [10], the boundary element method (BEM) [11, 12] and the wave shape functions method [1, 13–15]. Among these methods, the last one becomes popular in the recent years.

The UWVF method was first proposed for solving Helmholtz equation and Maxwell equations by Cessenat and Després, see [1, 13, 16]. In this method, the Ultra Weak Variational Formulation is associated with a triangulation on the underlying domain where the trace of the analytic solution and its normal derivative on the skeleton of the mesh need to be computed. In the discrete UWVF, one uses exact solutions of the Helmholtz equation (without boundary condition) on every elements as basis functions, which are usually called *wave shape functions*. Once the discrete variational problem based on UWVF is solved, the full approximate solution can be obtained by solving local problems on the elements. It was proved in [1, 13] that the approximate solutions generated by the discrete UWVF possess the optimal error estimate. The results reported in the subsequent study [14] show that the method permits the use of a relatively coarse mesh and can reduce the numbers of DOFs per wavelength in comparison with the standard FEM.

Since the stiffness matrix associated with the UWVF method is of high ill condition when the frequency ω is large (so the mesh is fine), how to iteratively solve the system generated by the UWVF method is a difficult problem. Because of this, a block diagonal preconditioner $\mathcal{D}$ for the system generated by UWVF was proposed in [13]. In the numerical experiments of [13], the preconditioned Richardson's iteration with such block diagonal preconditioner is applied to solve the linear system generated by UWVF for some two-dimensional Helmholtz equations, and the numerical results indicate that the preconditioner $\mathcal{D}$ is slightly effective only. In the numerical experiments of [16], the authors adopt the preconditioned Bi-Conjugate Gradients Stabilized Method (BICGSTAB) with the above block diagonal preconditioner to solve the corresponding linear system since the BICGSTAB is faster than the Richardson's iteration. It was pointed out in [16] (pp. 743) that *an interesting question is how to obtain a better preconditioner than* $\mathcal{D}$. However, to our knowledge, up to now there is no better preconditioner proposed in literature.

In this paper we try to construct a more effective preconditioner than $\mathcal{D}$. Motivated by the domain decomposition method, we construct a coarse solver associated with a non-overlapping domain decomposition to the underlying domain. By adding the coarse solver into the original block diagonal preconditioner $\mathcal{D}$, we obtain a new preconditioner $\mathcal{B}$. We would like to emphasize that the new preconditioner is *not a standard* substructuring preconditioner based on non-overlapping domain decomposition, and is simple and easy to implement. To illustrate the effectiveness of the new preconditioner, we apply the preconditioned BICGSTAB method with the new preconditioner (and the original block diagonal preconditioner) to solve the systems generated by the UWVF method for both two-dimensional and three-dimensional Helmholtz equations. Numerical results show that the new preconditioner $\mathcal{B}$ is much more effective than the original block diagonal preconditioner $\mathcal{D}$. It is well known that a satisfactory theoretical result on a solver for