ON STABILITY AND CONVERGENCE FOR DISCRETE-DISCONTINUOUS FINITE ELEMENT METHOD*

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Abstract

In this paper, we deal with the discrete-discontinuous finite element method for solving the time-dependent neutron transport equation in two-dimensional planar geometry. Its stability and convergence are proved. The numerical results are given. Compared with SN method it is of higher accuracy and superconvergence.

The discrete-ordinate method[4] (DSN method) is an effective method for solving neutron transport equations. Its computing process is simpler and the amount of program and calculation is less than that of other methods. And its error, as can be demonstrated, is at most of order 2. But solving some physical problems, a more accurate approximate solution, and so a solution of higher accuracy with less store, are desired. Therefore, it is natural to adopt the finite element method in solving neutron transport problems. Although the variational method (Ritz method) can be used, the complexity of the formula and large amount of program and calculation have impeded its application. On the other hand, while the Galerkin method can be used in the finite element method for its simpler computing process and program, it requires to solve of a large system of linear algebraic equations and will give rise to more difficulties. As the discontinuous finite element method has the advantages of both DSN and FEM, it provides a better way to solving multidimensional transport problems.

The discontinuous finite element method, where the angular flux is assumed to be given by a low-order polynomial in each mesh, has been used to solve the discrete-ordinate equations. It has been considered in [1, 2, 3] for solving the simplified steady neutron transport equations.

In this paper, we describe the basic steps of this method for solving time-dependent neutron transport equations in two-dimensional planar geometry. Some estimations of solution and error are given and the stability and superconvergency of the method are proved. Here, Crank-Nicholson central difference approximation is used for the time variable, while the discrete ordinate approximation for the angular variables.

Furthermore, we have used the above method to calculate many numerical examples for one-dimensional slab problem. The results demonstrate higher accuracy, faster rate of convergence and higher efficiency.

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1. Numerical Method

We consider the initial-boundary value problem for the time-dependent neutron transport equation in two-dimensional planar geometry:

$$\begin{cases} A(\varphi) \equiv \dfrac{1}{v_g}\dfrac{\partial \varphi_g}{\partial t} + \boldsymbol{\Omega}\cdot \operatorname{grad}\varphi_g + \alpha_g\varphi_g = S_g(\varphi_g) + F_g, \text{ in } D = B_{xy}\times Q_\Omega \times E_t, \\ \varphi_g(t, x, y, \mu, \nu)\,|_{t=0} = \varphi_g^0(x, y, \mu, \nu), \\ \varphi_g(t, x, y, \mu, \nu) = 0, \text{ if } (x, y)\in \Gamma,\ \boldsymbol{\Omega}\cdot \boldsymbol{n}_\Gamma < 0, \\ \varphi_g(t, x, y, \mu, \nu)\,|_{x=0} = \varphi_g(t, x, y, -\mu, \nu)\,|_{x=0}, \\ \varphi_g(t, x, y, \mu, \nu)\,|_{y=0} = \varphi_g(t, x, y, \mu, -\nu)\,|_{y=0}, \end{cases} \tag{1.1}$$

where

$$\boldsymbol{\Omega}\cdot \operatorname{grad}\varphi_g = \mu\frac{\partial \varphi_g}{\partial x} + \nu\frac{\partial \varphi_g}{\partial y},$$

the function $\varphi_g\,(t,\ x,\ y,\ \mu,\ \nu)$ represents the flux of g-group neutron at the point $(t,\ x,\ y)$ in the angular direction $\boldsymbol{\Omega} = (\mu,\ \nu)$, α_g is the nuclear macroscopic total cross section, $S_g(\varphi)$ represents sources of neutrons due to scattering and fission, and F_g inhomogeneous source terms. Let us assume that the domains B_{xy}, Q_Ω, E_t take the forms: $B_{xy} = \{0 \leqslant x \leqslant X,\ 0\leqslant y \leqslant Y\}$, $Q_\Omega = \{0 \leqslant \mu^2+\nu^2 \leqslant 1\}$, $E_t = \{0 \leqslant t \leqslant T\}$. Γ is the boundary of B_{xy} which is $x = X$ or $y = Y$. Denote by $\boldsymbol{n}_\Gamma$ the unit vector in the direction of outward normal to Γ.

We choose a suitable set of discrete direction and weights $\{\boldsymbol{\Omega}_{ms},\ w_{ms}\}$, where $\boldsymbol{\Omega}_{ms} = (\mu_{ms},\ \nu_{ms})$, $s = 1,\ 2,\ \cdots,\ N$, $m = 1,\ 2,\ \cdots,\ M_s$. For simplicity, we omit the group index g and restrict our discussion to one-group and isotropic scattering. Hence, the discrete-ordinate equations can be written as

$$\begin{cases} A_{ms}(\varphi_{ms}) = \dfrac{1}{v}\dfrac{\partial \varphi_{ms}}{\partial t} + \mu_{ms}\dfrac{\partial \varphi_{ms}}{\partial x} + \nu_{ms}\dfrac{\partial \varphi_{ms}}{\partial y} + \alpha\varphi_{ms} = S_M(\varphi) + F_{ms}, \text{ in } B_{xy}\times E_t, \\ \varphi_{ms}|_{t=0} = \varphi^0(x,\ y,\ \mu_{ms},\ \nu_{ms}), \\ \varphi_{ms}|_\Gamma = 0, \text{ if } (x,\ y)\in\Gamma,\ \Omega_{ms}\cdot \boldsymbol{n}_\Gamma < 0, \\ \varphi_{ms}(t,\ 0,\ y,\ \mu_{ms},\ \nu_{ms}) = \varphi_{ms}(t,\ 0,\ y,\ -\mu_{ms},\ \nu_{ms}), \\ \varphi_{ms}(t,\ x,\ 0,\ \mu_{ms},\ \nu_{ms}) = \varphi_{ms}(t,\ x,\ 0,\ \mu_{ms},\ -\nu_{ms}), \end{cases} \tag{1.2}$$

where $s = 1,\ 2, \cdots,\ N$, $m = 1,\ 2, \cdots,\ M_s$, $\varphi_{ms} = \varphi(t,\ x,\ y,\ \mu_{ms},\ \nu_{ms})$, $S_M(\varphi) = \beta\sum\limits_{m,s}\varphi_{ms}w_{ms}$, $\sum\limits_{m,s} w_{ms} = 1$.

The boundary conditions can be written as

$$\begin{cases} \varphi_{ms}|_{x=X} = 0,\ s = 1,\ 2,\ \cdots,\ N,\ m = 1,\ 2,\ \cdots,\ \dfrac{M_s}{2}, \\ \varphi_{ms}|_{y=Y} = 0,\ s = 1,\ 2,\ \cdots, \dfrac{N}{2},\ m = 1,\ 2,\ \cdots,\ M_s, \\ \varphi_{ms}|_{x=0} = \varphi_{M_s+1-m,s}|_{x=0},\ s = 1,\ 2,\ \cdots,\ N,\ m = \dfrac{M_s}{2}+1,\ \cdots,\ M_s, \\ \varphi_{ms}|_{y=0} = \varphi_{m,N+1-s}|_{y=0},\ s = \dfrac{N}{2}+1,\ \cdots,\ N,\ m = 1,\ 2,\ \cdots,\ M_s. \end{cases} \tag{1.3}$$

Let $0 = x_0 < x_1 < \cdots < x_I = X$, $0 = y_0 < y_1 < \cdots < y_J = Y$, $0 = t_0 < t_1 < \cdots < t_{N_0} = T$ be the