THE GENERALIZED PATCH TEST FOR ZIENKIEWICZ'S TRIANGLES*[1)]

SHI ZHONG-CI (石钟慈)
(China University of Science and Technology, Hefei, China)

Abstract

It is proved that Zienkiewicz's triangles for plate bending problems pass Stummel's generalized patch test——a necessary and sufficient condition for convergence of nonconforming finite elements——for mesh (a) generated by three sets of parallel lines, but do not pass it when "union jack" mesh (b) or when another mesh (c) is used. In the latter two cases the approximations are divergent.

1. Introduction

It is well known that Zienkiewicz's triangles[1] for plate bending problems are nonconforming, since the gradients of the shape functions are discontinuous at interelement boundaries. Concerning the convergence of this element, numerical experiments in [1, 2] have shown that mesh (a) of Figure 1, generated by three sets of parallel lines (called for brevity the condition of parallel lines), guarantees convergence, whereas mesh (b) of Figure 2 composed of "union jack" figures does not give convergence. In order to explain why Zienkiewicz's triangles were convergent in one configuration but not in others, Irons-Razzaque created the patch test[3] and showed that Zienkiewicz's triangles pass the test under the condition of parallel lines, but do not pass it for the union jack configuration.

Later on, Lascaux and Lesaint[4] gave a mathematical proof of the convergence of Zienkiewicz's triangles under the condition of parallel lines and derived corresponding error estimates for the plate problem. More recently, Stummel[5,6] pointed out that the patch test of Irons is neither necessary nor sufficient for convergence of nonconforming elements, and proposed a generalized patch test instead, which does indeed give both necessary and sufficient conditions for convergence. Stummel proved in [5] that various nonconforming elements pass this generalized patch test; however, Zienkiewicz's triangles were not analysed in that paper.

Since passing the patch test is no longer necessary for convergence, it is not proved yet whether mesh (b) and mesh (c) of Figure 3, that do not pass Irons patch test, diverge or not. Concerning mesh (c), the authors in [1] state: "the convergence is most unlikely, and this case has not been investigated numerically".

We shall prove in this paper that:

(i) Zienkiewicz's triangles pass the generalized patch test under the condition of

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parallel lines;

(ii) mesh (b) and mesh (c) do not pass the generalized patch test and, thus, do not converge.

According to Stummel's theory[7], the validity of the generalized patch test, together with the approximability condition and strong continuity condition at interelement boundaries (the latter two conditions are satisfied by Zienkiewicz's triangles for arbitrary decompositions), provide the preconditions for the validity of a generalized Rellich compactness theorem. As a consequence thereof, very general stability and convergence theorems are valid about approximations of general coercive elliptic variational equations and eigenvalue problems with variable, not necessarily smooth coefficients.

2. Zienkiewicz's Triangles under the Condition of Parallel Lines

We consider a triangulation $\mathscr{K}_h$ of a given polyhedral domain $G\subset\mathbb{R}^2$ with finite elements K. Let $h(K)$ =diameter of K, $h=\max\limits_{K\in\mathscr{K}_h}\{h(K)\}$, $\rho(K)$ = the greatest diameter of the circles inscribed in K. We assume that the triangulation $\mathscr{K}_h$ is regular[8], that is, there exists a constant σ independent of h such that

$$h(K)\leqslant\sigma\rho(K), \quad K\in\mathscr{K}_h, \tag{1}$$

when the greatest diameter h approaches zero.

Given a triangle K with vertices $p_i=(x_i, y_i)$, $1\leqslant i\leqslant 3$, we let λ_i denote the area coordinates relative to the vertices p_i, $\triangle$ the area of K, and

$$\begin{aligned} &\xi_1=x_{23}=x_2-x_3, \quad \xi_2=x_{31}=x_3-x_1, \quad \xi_3=x_{12}=x_1-x_2,\\ &\eta_1=y_{23}=y_2-y_3, \quad \eta_2=y_{31}=y_3-y_1, \quad \eta_3=y_{12}=y_1-y_2. \end{aligned} \tag{2}$$

Zienkiewicz's triangles are thus defined as follows (see [9, p. 187]):

(i) Nodal parameters are the function values and the values of the gradients at the vertices of K (In case of Dirichlet boundary conditions nodal parameters are zero at the vertices on the boundary.);

(ii) The space $\mathscr{P}(K)$ of the shape functions w is a space of polynomials of third degree having the following form:

$$\begin{aligned} w(p)=a_1\lambda_1+a_2\lambda_2+a_3\lambda_3&+a_4\left(\lambda_1^2\lambda_2+\frac{1}{2}\lambda_1\lambda_2\lambda_3\right)+a_5\left(\lambda_2^2\lambda_1+\frac{1}{2}\lambda_1\lambda_2\lambda_3\right)\\ &+a_6\left(\lambda_2^2\lambda_3+\frac{1}{2}\lambda_1\lambda_2\lambda_3\right)+a_7\left(\lambda_3^2\lambda_2+\frac{1}{2}\lambda_1\lambda_2\lambda_3\right)\\ &+a_8\left(\lambda_3^2\lambda_1+\frac{1}{2}\lambda_1\lambda_2\lambda_3\right)+a_9\left(\lambda_1^2\lambda_3+\frac{1}{2}\lambda_1\lambda_2\lambda_3\right). \end{aligned} \tag{3}$$

The unique polynomial in $\mathscr{P}(K)$, determined by its nodal parameters described above, is

$$w(p)=\sum_{i=1}^{3}[\varphi_i w(p_i)+\psi_i w_x(p_i)+\rho_i w_y(p_i)], \tag{4}$$

where

$$\varphi_i=\lambda_i^2(3-2\lambda_i)+2\lambda_1\lambda_2\lambda_3, \tag{5}$$

$$\psi_i=\xi_{i+1}\left(\lambda_i^2\lambda_{i+2}+\frac{1}{2}\lambda_1\lambda_2\lambda_3\right)-\xi_{i+2}\left(\lambda_i^2\lambda_{i+1}+\frac{1}{2}\lambda_1\lambda_2\lambda_3\right), \tag{6}$$