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A LEVEL SET METHOD FOR SOLVING FREE BOUNDARY PROBLEMS ASSOCIATED WITH OBSTACLES

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Abstract. A level set method is proposed for solving free boundary problems coming from contact with obstacles. Two different approaches are described and applied for solving an unilateral obstacle problem. The cost functionals coming from the new approach are nonsmooth. For solving the nonsmooth minimization problems, two methods are applied: firstly, a proximal bundle method, which is a method for solving general nonsmooth optimization problems. Secondly, a gradient method is proposed for solving the regularized problems. Numerical experiments are included to verify the convergence of the methods and the quality of the results.

Key Words. Level set methods, free boundary problems, obstacle problem.

1. Introduction

The level set method initiated by Osher and Sethian [14] has proven to be an efficient numerical device for capturing moving fronts, see [11, 12, 17]. There are many industrial problems where interfaces need to be identified, which can be formulated as moving front problems. We mention, for example, image segmentation problems [2], inverse problems [1, 3], and optimal shape design problem [13, 15]. In free boundary problems, it is often needed to find the boundary of some domains. It is natural to use level set method for this kind of applications. In [12, 18], the level set method was used for Stefan type of free boundary problems. In this work, we shall propose an alternative approach for the level set idea of [14, 12, 18] and use it for tracing the free boundaries from obstacle contact type of problems.

The contents of the paper are as follows. In Section 2, a model free boundary problem is described and the proposed level set approaches are introduced for solving the problem. In Section 3, the solution algorithms are described that are applied for realizing the proposed level set approaches. In Section 4, numerical experiments are presented to verify the convergence of the methods and the quality of the results. Finally, in Section 5, the conclusions are stated.

2. A modified level set method

Consider a model free boundary problem which comes from the minimization problem:

$$\min_{v\in K} F(v), \tag{1}$$

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with

$$F(v) = \int_\Omega \left(\frac{1}{2}|\nabla v|^2 - fv\right) dx, \qquad K = \{v|\ v \in H_0^1(\Omega), v \geq \psi\}. \tag{2}$$

In the above, $\Omega \subset R^p$, $p = 1, 2$, ψ is the obstacle function satisfying $\psi \leq 0$ on $\partial\Omega$, and f typically represents external force for physical problems. The solution u for (2) is unique and it can be formally written as the function satisfying

$$-\Delta u \geq f, \quad u \geq \psi, \quad (-\Delta u - f) \cdot (u - \psi) = 0.$$

To find the solution u, we need to find the contact region $\Omega^+ = \{x|\ \ u(x) = \psi(x), x \in \Omega\}$. Once we know Ω^+, the value of u in $\Omega\backslash\Omega^+$ can be obtained from solving

$$-\Delta u = f \text{ in } \Omega\backslash\Omega^+, \quad u = 0 \text{ on } \partial\Omega, \quad u = \psi \text{ on } \partial\Omega^+.$$

In order to find u, we essentially just need to find $\Gamma = \partial\Omega^+$. Inside Γ, $u = \psi$ and outside Γ, u is the solution of the Poisson equation.

Based on the above observation, we see that it is essentially enough to find the curve in order to solve the free boundary problem (1). Starting with an initial curve, we shall slowly evolve the curve to the true free boundary. We can use the level set method to represent the curve, i.e., we try to find a function $\varphi(t, x)$ such that

$$\Gamma(t) = \{x|\ \varphi(t, x) = 0\}.$$

In the above, $\Gamma(0)$ is the initial curve and $\Gamma(t)$ converges to the true free boundary, when $t \to \infty$. One of the essential ingredient of the level set method is to find the velocity field $V_n(t, x)$ in the normal direction of $\Gamma(t)$, which is then used to move the level set function $\varphi(t, x)$ by solving

$$\varphi_t - V_n|\nabla\varphi| = 0, \quad \varphi(0, x) = \varphi_0(x) = \pm\text{distance}(x, \Gamma(0)).$$

In this work, we propose an alternative approach, which seems to be simpler than the approach outlined above. Define the Heaviside function $H(\varphi)$ as

$$H(\varphi) = \begin{cases} 1, & \varphi > 0, \\ 0, & \varphi \leq 0. \end{cases}$$

For any $v \in K$, there is exists $\varphi \in H^1(\Omega)$ such that

$$v = \psi + \varphi H(\varphi). \tag{3}$$

It is easy to see that

$$v = \begin{cases} \psi, & if\ \ \varphi \leq 0 \ \ \text{(i.e., in the contact region)} \\ \psi + \varphi, & if\ \ \varphi > 0 \ \ \text{(i.e., outside the contact region).} \end{cases} \tag{4}$$

Thus, the sign of the function φ tells the information of the contact region. The curve, which separates the regions where φ is positive or negative, gives the free boundary. In the traditional level set method, the function φ is only used to represent the curve Γ. In our approach, φ is not only used to represent the curve, but also to carry information about the solution u outside the contact region, i.e., φ is used to indicate that $u = \psi$ inside the contact region and its value outside the contact region shall be $\varphi = u - \psi$. We use iterative type of methods to find the correct values of φ both inside and outside the contact region. Note that the value of φ inside the contact region is not unique.