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An Example of Nonexistence of $\kappa-$solutions to the Ricci Flow

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Abstract. In this paper, we construct an example of three-dimensional complete smooth $\kappa-$noncollapsed manifold, which admits no short time smooth complete and $\kappa-$noncollapsed solutions to the Ricci flow.

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Key words: Ricci flow, nonexistence.

1 Introduction

In 1982, R. Hamilton introduced the Ricci flow [3] to the world of mathematics. In this seminal work [3], among other things, R. Hamilton proved that when the manifold M is compact, the Ricci flow has a solution for a short time. In 1989, W.X. Shi [10] extended the short time existence result to complete noncompact manifolds with bounded curvature.

In this paper, we will revisit the problem of existence of the Ricci flow. We attempt to show that, on a general manifold, the bounded curvature condition in Shi's theorem can not be dropped. More precisely, one can show the following:

Theorem 1.1. *There is a smooth complete $\kappa-$noncollapsed three-dimensional Riemannian manifold (M,g_{ij}) such that it admits no complete and $\kappa-$noncollapsed smooth solution to the Ricci flow for a short time with (M,g_{ij}) as initial data.*

Here, we say a Riemannian manifold of dimension n is $\kappa-$noncollapsed if there is a positive constant $\kappa>0$ such that for any $x_0\in M$ with $|Rm|\leq r^{-2}$ on $B(x_0,r)$, we have $vol(B(x_0,r))\geq\kappa r^n$. A solution to the Ricci flow is said to be $\kappa-$noncollapsed (see [7] section 8) if for any spacetime point (x_0,t_0) such that $|Rm|\leq r^{-2}$ on $B_t(x_0,r)$ for any $t\in[t_0-r_0^2,t_0]$, we have $vol(B_{t_0}(x_0,r))\geq\kappa r^n$.

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We remark that in the above theorem, besides the completeness of the metric, a $\kappa-$noncollapsed condition was imposed on the solutions. We expect that this technical condition could be removed.

It is well-known that the Ricci flow on spheres of dimension n with initial metric of constant curvature K will shrink the spheres to a point at time $T=\frac{1}{2(n-1)K}$. A tiny sphere will shrink to a point in short time. Intuitively, we can imagine that the Ricci flow on a manifold with many tiny necks N_j whose cross spheres have radius $r_j\to 0$ can not move for any short time. The purpose of this paper is trying to provide such an example.

Note that it is a nontrivial issue to control the behavior of the individual spheres or necks on the manifold during the evolution. In this paper, instead of dealing with each individual spheres, our idea is to choose and investigate the minimal surfaces in the homotopy classes of those spheres. The point is that we do not need to care about the shape of the minimal surfaces, we only care about their existence. Because whenever these minimal surfaces exist, their areas can be used to estimate the life span of the solution. Now the difficulty of the problem is to prove the existence of minimal surfaces which must be confined in some specific regions.

The minimal surface argument in Ricci flow was first used by Hamilton [4, 5], who proved the incompressibility of certain boundary tori of hyperbolic pieces in the long time nonsingular solutions. In [1] and [9], the argument with min-max constructions of minimal surfaces was used to prove the finite time extinction for the Ricci flow on homotopy three-spheres.

In the present paper, we will use [6] to construct minimal surfaces in certain domains. The difficulty is to prove that the boundaries of these domains are convex or mean convex for the evolving metrics. Ultimately, this will amount to certain technical local a priori curvature estimate, which is the main difficulty of the paper to overcome.

The paper is organized as follows. In Section 2, we first give the construction of the manifolds, then we outline the proof of nonexistence of the Ricci flow, modulo key Lemma 2.2. In Section 3, we prove Lemma 2.2.

2 Examples

2.1 Construction

We will construct a warped product metric on cylinder $M=\mathbb{R}\times\mathbb{S}^{n-1}$ in the form $ds^2=dr^2+f^2(r)ds^2_{\mathbb{S}^{n-1}}$, where $f(r)$ is a positive even smooth function on $\mathbb{R}$.

Let $X_1,X_2,\cdots,X_{n-1}$ be an orthogonal basis of M tangent to the sphere $\mathbb{S}^{n-1}$ and $T=\frac{\partial}{\partial r}$. It is well-known that