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A TWO-GRID FINITE-ELEMENT METHOD FOR THE NONLINEAR SCHRÖDINGER EQUATION*

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Abstract

In this paper, some two-grid finite element schemes are constructed for solving the nonlinear Schrödinger equation. With these schemes, the solution of the original problem is reduced to the solution of the same problem on a much coarser grid together with the solutions of two linear problems on a fine grid. We have shown, both theoretically and numerically, that our schemes are efficient and achieve asymptotically optimal accuracy.

Mathematics subject classification: 65N30, 65N55
Key words: Nonlinear Schrödinger equation, Finite element method, Two-grid.

1. Introduction

Nonlinear Schrödinger equations arise from mathematical modelling of problems in various areas such as fluid dynamics, nonlinear optics, plasma physics, protein chemistry, etc., see, e.g., [2, 12, 13]. In this paper, we will study two-grid finite element discretization schemes for the boundary value problem of the nonlinear Schrödinger equation:

$$-\Delta\psi(\mathbf{x}) + V(\mathbf{x})\psi(\mathbf{x}) + |\psi(\mathbf{x})|^2\psi(\mathbf{x}) = f(\mathbf{x}), \quad \forall \mathbf{x} \in \Omega, \tag{1.1}$$

$$\psi(\mathbf{x}) = 0, \quad \forall \mathbf{x} \in \partial\Omega, \tag{1.2}$$

where $\Omega \subset R^2$ is a convex polygonal domain, $f(\mathbf{x}), V(\mathbf{x})$ and unknown function $\psi(\mathbf{x})$ are complex-valued. For any complex-valued function w, we denote its real part by w_1, the imaginary part by w_2. Then problem (1.1)-(1.2) is equivalent to the following coupled nonlinear equations:

$$-\Delta\psi_1(\mathbf{x}) + V_1(\mathbf{x})\psi_1(\mathbf{x}) + \psi_1^3(\mathbf{x}) + \psi_2^2(\mathbf{x})\psi_1(\mathbf{x}) - V_2(\mathbf{x})\psi_2(\mathbf{x}) = f_1(\mathbf{x}), \quad \forall \mathbf{x} \in \Omega, \tag{1.3}$$

$$-\Delta\psi_2(\mathbf{x}) + V_1(\mathbf{x})\psi_2(\mathbf{x}) + \psi_2^3(\mathbf{x}) + \psi_1^2(\mathbf{x})\psi_2(\mathbf{x}) + V_2(\mathbf{x})\psi_1(\mathbf{x}) = f_2(\mathbf{x}), \quad \forall \mathbf{x} \in \Omega, \tag{1.4}$$

$$\psi_j(\mathbf{x}) = 0, j = 1, 2, \quad \forall \mathbf{x} \in \partial\Omega. \tag{1.5}$$

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The two-grid discretization method, proposed originally by Xu [17] in 1992, is an efficient numerical method. Later, Xu [18, 19] introduced the two-grid finite element approach to solve nonlinear elliptic equations efficiently, where the basic idea is to use a coarse space to produce a rough approximation of the solution, and then use it as the initial guess for one Newton iteration on the fine grid. This procedure involves a nonlinear solver on coarse space and a linear solver on fine space. Now the idea of the two-grid discretization method has already been applied to solving many problems, such as nonlinear parabolic equations [5], nonlinear elasticity problems [1], two-phase mixed-domain PEMFC model [9], and the Schrödinger equations [3, 6–8, 10, 11, 15, 16, 20].

In the literature, there have been only a few papers on two-grid discretization methods for Schrödinger equation. Ignat et al. [10] constructed two-grid finite difference scheme for nonlinear Schrödinger equations, where the equations on the fine grid are linearizated, but not decoupled. Jin et al. [11] successfully extended the two-grid finite element method to solve coupled partial differential equations such as linear Schrödinger equation, where the equations on fine grid are decoupled, so that the computational complexity of solving Schrödinger equation is comparable to solving two decoupled Poisson equations on the same fine grid. Zhou et al. [7, 8] proposed a two-scale finite element discretization scheme for eigenvalue problem of Schrödinger equation. The approach is a powerful technique in obtaining accurate and efficient approximations for large scale quantum eigenvalue problems. Chang et al. [3] combined the two-grid discretization together with the predictor-corrector method and developed an algorithm for computing the extremum eigenpairs of the discrete Schrödinger eigenvalue problem. Chien et al. [6] proposed two-grid discretization schemes with two-loop continuation algorithms for nonlinear Schrödinger equations, where the centered difference approximations, the six-node triangular elements and the Adini elements are used to discretize the PDEs. Numerical experiments have shown that these schemes were efficient, but no rigorous error analysis were given. Wu [15, 16] constructed two-grid mixed finite element schemes for nonlinear Schrödinger equations, where a linear and indefinite (the typical nature of mixed finite element) discretization systems are solved on the fine grid. Numerical experiments using these schemes have been shown to be efficient, however, no rigorous error analysis has been conducted. Recently, Zhang et al. [20] extended the approach given in [11] to time-dependent linear Schrödinger equation.

In this paper,we follow the idea of [11, 20] to apply two-grid finite element method to solve the nonlinear problem (1.3)-(1.5). Specifically, we use Newton iteration method to solve the original problem directly on coarse grid, and algebraic multigrid method to solve the discrete systems of the linearized and decoupled equations on fine grid. The resulting solution, verified by theoretical analysis and numerical experiments, achieves optimal accuracy in H^1-norm.

The rest of the paper is organized as follows: Section 2 is a description and analysis of the finite element method for Schrödinger equation. In Section 3, we construct the two-grid finite element schemes and derive the error estimates. In Section 4, we demonstrate numerical examples to verify the efficiency and effectiveness of the schemes.

2. The Finite Element Approximation

For any real-valued and Lebesgue square integrable functions $\varphi_1(\mathbf{x})$ and $\varphi_2(\mathbf{x})$, let (φ_1, φ_2) denote the inner product

$$(\varphi_1, \varphi_2) = \int_\Omega \varphi_1(\mathbf{x})\varphi_2(\mathbf{x})d\mathbf{x}.$$