J. Math. Study
doi: 10.4208/jms.v49n1.16.03

Vol. **49**, No. 1, pp. 23-32
March 2016

The Distortion Theorems for Harmonic Mappings with Negative Coefficient Analytic Parts

Mengkun Zhu and Xinzhong Huang*

School of Mathematical Science, Huaqiao University, Quanzhou 362021, Fujian, P.R. China.

Received 22 May 2015; Accepted 9 July 2015

Abstract. Some sharp estimates for coefficients, distortion and the growth order are obtained for harmonic mappings $f \in TL_H^\alpha$, which are locally univalent harmonic mappings in the unit disk $\mathbb{D} := \{z : |z| < 1\}$. Moreover, denoting the subclass TS_H^α of the normalized univalent harmonic mappings, we also estimate the growth of $|f|, f \in TS_H^\alpha$, and their covering theorems.

AMS subject classifications: 30D15, 30D99

Key words: Harmonic mapping, coefficient estimate, distortion theorem, covering problem

1 Introduction

Let S denote the class of functions of the form $F(z) = z + \sum_{n=2}^{\infty} a_n z^n$, that are analytic and univalent in the unit disk $\mathbb{D} := \{z : |z| < 1\}$. Denoting T to be the subclass of S consisting of functions whose nonzero coefficients, from the second on, are negative. That is, an univalent analytic function $F \in T$ if and only if it can be written in the form

$$F(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad z \in \mathbb{D}. \tag{1.1}$$

A complex-valued harmonic function f in the unit disk $\mathbb{D}$ has a canonical decomposition

$$f(z) = h(z) + \overline{g(z)} \tag{1.2}$$

where h and g are analytic in $\mathbb{D}$ with $g(0) = 0$. Usually, we call h the analytic part of f and g the co-analytic part of f. A complete and elegant account of the theory of planar harmonic mappings is given in Duren's monograph [1].

*Corresponding author. *Email addresses:* ZMK900116@163.com (M. Zhu), huangXZ@hqu.edu.cn (X. Huang)

In [2], Ikkei Hotta and Andrzej Michalski denoted the class L_H of all normalized locally univalent and sense-preserving harmonic functions in the unit disk with $h(0)=g(0)=h'(0)-1=0$. Which means every function $f\in L_H$ is uniquely determined by coefficients of the following power series expansions

$$h(z)=z+\sum_{n=2}^{\infty}a_nz^n,\ \ g(z)=\sum_{n=1}^{\infty}b_nz^n,\ \ z\in\mathbb{D}, \tag{1.3}$$

where $a_n,\ b_n\in\mathbb{C},n=2,3,4,\dots$ Clunie and Sheil-small introduced in [3] the class S_H of all normalized univalent harmonic mappings in $\mathbb{D}$, obviously, $S_H\subset L_H$.

Lewy [4] proved that a necessary and sufficient condition for f to be locally univalent and sense-preserving in $\mathbb{D}$ is $J_f(z)>0$, where

$$J_f(z)=|h'(z)|^2-|g'(z)|^2,\ \ z\in\mathbb{D}. \tag{1.4}$$

To such a function f, not identically constant, let

$$\omega(z)=\frac{g'(z)}{h'(z)},\ \ z\in\mathbb{D}, \tag{1.5}$$

then $\omega(z)$ is analytic in $\mathbb{D}$ with $|\omega(z)|<1$, it is called the second complex dilatation of f.

In [5], Silverman investigated the subclass of T which denoted by $T^*(\beta)$, starlike of order $\beta(0\le\beta<1)$. That is, a function $F(z)\in T^*(\beta)$ if $\mathrm{Re}\{zF'(z)/F(z)\}>\beta$, $z\in\mathbb{D}$. It was proved in [5] that

Corollary 1.1.

$$T=T^*(0).$$

In [7-8], Dominika Klimek and Andrzej Michalski studied the cases when the analytic parts h is the identity mapping or a convex mapping, respectively. The paper [2] was devoted to the case when the analytic h is a starlike analytic mapping. In [9], Qin Deng got sharp results concerning coefficient estimate, distortion theorems and covering theorems for functions in T. The main idea of this paper is to characterize the subclasses of L_H and S_H when $h\in T$.

In order to establish our main results, we need the following theorems and lemmas.

Theorem 1.1. ([8]) *A function $f(z)=z-\sum_{n=2}^{\infty}|a_n|z^n$ is in T if and only if*

$$\sum_{n=2}^{\infty}n|a_n|\le 1,\ \ z\in\mathbb{D}. \tag{1.6}$$

Lemma 1.1. ([10]) *If $f(z)=a_0+a_1z+\dots+a_nz^n+\dots$ is analytic and $|f(z)|\le 1$ on $\mathbb{D}$, then*

$$|a_n|\le 1-|a_0|^2,\ \ n=1,2,\dots. \tag{1.7}$$