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Separable Determination of the Fixed Point Property of Convex Sets in Banach Spaces

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Abstract. In this paper, we first show that for every mapping f from a metric space Ω to itself which is continuous off a countable subset of Ω, there exists a nonempty closed separable subspace $S\subset\Omega$ so that $f|_S$ is again a self mapping on S. Therefore, both the fixed point property and the weak fixed point property of a nonempty closed convex set in a Banach space are separably determined. We then prove that every separable subspace of $c_0(\Gamma)$ (for any set Γ) is again lying in c_0. Making use of these results, we finally presents a simple proof of the famous result: Every non-expansive self-mapping defined on a nonempty weakly compact convex set of $c_0(\Gamma)$ has a fixed point.

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1 Introduction

A mapping f from a nonempty set D to itself is said to be a self-mapping of D. If $D=(D,d)$ is a metric space, then we call the mapping f "Lipschitz" whenever there exists a constant $L>0$, so that

$$d(f(x),f(y))\le Ld(x,y),\quad \forall x,y\in D.$$

In this case, we also say precisely that f is L-Lipschitz on D. A mapping $f:D\to D$ is nonexpansive if it is 1-Lipschitz.

We say that a nonempty closed convex set C of a Banach space X has the fixed point property, if every nonexpansive mapping f defined on each nonempty closed bounded convex subset D of C has a fixed point, i.e. there is $x_0\in D$ so that $f(x_0)=x_0$. Mathematicians also often consider the weak fixed point property of a closed convex set C of

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a Banach space X: a Banach space X has the weak fixed point property, if every non-expansive self-mapping defined on a nonempty weakly compact convex set of X has a fixed point. As is well known, the metric fixed point theory is a well-developed branch of fixed point theory, which not only has its own methods and problems, but also relates with other fields such as geometry of Banach spaces, integral and differential equations, multivalued analysis and so on. See, for example, Kirk [1].

Denote $c_0=\{(x_n)_{n\in\mathbb{N}}: \lim_{n\to\infty} x_n=0, \quad \|x\|_\infty=\sup_n |x_n|\}$. Let Γ be a nonempty set and $c_0(\Gamma)$ denote the Banach space(supremum norm) of all real-valued function x on Γ, such that for any $\varepsilon>0$, $\{r\in\Gamma, |x(r)|>\varepsilon\}$ is finite, or, equivalently, $c_0(\Gamma)$ consists of all those functions defined on the set Γ with countable support and with its range as a null sequence. The spaces $c_0(\Gamma)$ have received renewed interest, because of a powerful mapping theorem of Lindenstrauss [2]: (1) They are weakly compactly generated Banach spaces; and (2) if E is a weakly compactly generated Banach space, then there exists a set Γ and a continuous one-to-one linear map T of E into $c_0(\Gamma)$.

Maurey ([3], 1980) first showed that c_0 has weak fixed point property by means of ultrapower techniques. Later, Odell and Sternfeld ([4], 1981), Haydon, Odell and Sternfeld ([5], 1981), among many other things, gave Maurey's theorem different proofs. Borwein and Sims ([6], 1984), and Domínguez Benavides ([7], 1996) proved the same conclusion holds for $c_0(\Gamma)$. Fuster and Sims ([8], 1998) showed that c_0 fails to have the fixed point property and conjectured that every closed bounded convex subset of c_0 with the fixed point property is weakly compact. An example of Pineda ([9], 2003) showed that the fixed point property in c_0 can not go beyond the class of all nonempty weakly compact convex subsets, even if to a compact convex subset in some topology slightly weaker than the weak topology. An affirmative answer to this conjecture was given by Dowling, Lennard and Turett ([10], 2004).

In this paper, we first show that for every mapping f from a metric space Ω to itself which is continuous off a countable subset of Ω, there exists a nonempty closed separable subspace $S\subset\Omega$ so that $f|_S$ is again a self mapping on S. Therefore, the fixed point property of a closed convex set in a Banach space is separably determined. We then prove that every separable subspace of $c_0(\Gamma)$ (for any set Γ) is again lying in c_0. Making use of these results, we finally present a simple proof of the famous result: Every non-expansive self-mapping defined on a nonempty weakly compact convex set of $c_0(\Gamma)$ has a fixed point.

2 Separable determination of the fixed point property

In this section, we shall show that the fixed point properties for continuous (in particular, non-expansive) mappings is separably determined.

For subset A in a topological space, we denote by $\overline{A}$, the closure of A.

Lemma 2.1. *Suppose that D is a nonempty closed subset of a metric space Ω, and $f: D\to D$ is*