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New Results for the BBM Equation

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Abstract. The BBM equation posed on $\mathbb{R}$ and $\mathbb{R}^+$ is revisited. Improving on earlier results, global well-posedness and bounds for the growth in time of relevant norms of solutions corresponding to very general auxiliary data are derived.

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1 Introduction

The regularized long wave equation, or BBM-equation,

$$u_t+u_x-u_{xxt}+uu_x=0 \tag{1.1}$$

was first introduced by Peregrine [7] to model small amplitude, long waves propagating in one direction. Here $u=u(x,t)$ is a real-valued function defined on $\mathbb{R}\times\mathbb{R}^+$. The equation with initial condition

$$u(x,0)=\varphi(x), \quad \text{for} \quad x\in\mathbb{R} \tag{1.2}$$

in the L_2-based Sobolev space $H^k(\mathbb{R}),k=1,2,\cdots$, was first rigorously investigated by Benjamin *et al.* [1], they showed that (1.1)-(1.2) is globally well-posed, the solution $u\in C^\infty([0,\infty);H^k(\mathbb{R}))$. Bona-Tzvetkov [6] extended the global well-posedness result for the initial data $\varphi\in H^k(\mathbb{R})$, $k=1,2,\cdots$, to $H^s(\mathbb{R})$ for all $s\geq 0$. It is worth pointing out that when $0\leq s<1$, the method they used is high-low frequency decomposition.

While using the high-low frequency approach to show the global well-posedness, the upper bound for the growth in time of the relevant Sobolev norms $\|u(\cdot,t)\|_{H^s(\mathbb{R})}$ of the solution u cannot be obtained. In this paper, a new approach is introduced, so this issue is resolved.

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Modeling waves generated in a laboratory at Fluid Mechanics Research Institute at the University of Essex, the regularized long-wave, or BBM equation (1.1) reappeared, see Bona-Bryant [2], Bona-Pritchard-Scott [5]. That is to say, the domain of the BBM-equaiton (1.1) is $(x,t)\in\mathbb{R}^+\times\mathbb{R}^+$. Hence the problem has both initial and boundary condition:

$$u(x,0)=\varphi(x) \quad \text{and} \quad u(0,t)=g(t) \quad \text{for} \quad x,t\geq 0. \tag{1.3}$$

Eq. (1.1) together with (1.3) is some time called the BBM quarter plane problem, or wave maker problem.

Assuming that $g\in C^1(\mathbb{R}^+)$ and $\varphi\in H^1(\mathbb{R}^+)\cap C_b^2(\mathbb{R}^+)$ with compatibility condition $\varphi(0)=g(0)$, Bona-Bryant [2] showed that the Eq. (1.1) with the initial-boundary condition (1.3) is globally well-posed, the solution u lies in space $C^1([0,\infty);H^1(\mathbb{R}^+)\cap C_b^2(\mathbb{R}^+))$ and it is a classical solution.

Later, under assumptions that $\varphi=0$ and $g\in C(\mathbb{R}^+)$ with compatibility $g(0)=0$, Bona *et al.* [4] showed that (1.1) & (1.3) is globally well posed, the solution u is a member of $C([0,\infty);H^\infty(\mathbb{R}^+))$.

Most recently, assuming that $\varphi\in L_2(\mathbb{R}^+)$ and $g\in L_\infty^{loc}(\mathbb{R}^+)$ are locally continuous at $x,t=0$ with compatibility condition $\varphi(0)=g(0)$, Bona *et al.* [3] showed that the initial-boundary-value problem (1.1) & (1.3) is well-posed globally in time, the solution $u\in L_\infty^{loc}([0,\infty);L_2(\mathbb{R}^+))$. The method used was high-low frequency as Bona-Tzvetkov introduced in [6]. Hence, there is no estimate on the growth bound in time of the norm $\|u(\cdot,t)\|_{L_2(\mathbb{R}^+)}$ in terms of auxiliary data.

Improving and completing the earlier results, in this paper, new results are summarized in following.

Theorem 1.1. *The BBM equation* (1.1) *post for* $(x,t)\in\mathbb{R}\times\mathbb{R}^+$ *with the initial condition* (1.2) *is globally well-posed if the initial data* $\varphi\in H^s(\mathbb{R})$ *for any* $s\geq 0$. *Moreover,* $u\in C([0,\infty);H^s(\mathbb{R}))$ *has the following bounds.*

$$\|u(\cdot,t)\|_{H^s(\mathbb{R})}\leq c(\|\varphi\|_{H^s(\mathbb{R})})(1+t)^{\frac{2}{3}(s-1)+\frac{1}{3}(s-\lfloor s\rfloor)} \text{ if } s\geq 1,$$

$$\|u(\cdot,t)\|_{H^s(\mathbb{R})}<c(\|\varphi\|_{L_2(\mathbb{R})},\|\varphi\|_{H^s(\mathbb{R})})e^{\|\varphi\|_{L_2(\mathbb{R})}t} \text{ if } \frac{1}{4}<s<1,$$

and

$$\|u(\cdot,t)\|_{H^s(\mathbb{R})}\leq e^{p_2(t)} \quad \text{if} \quad 0\leq s\leq\frac{1}{4}$$

where $c(\|\varphi\|_{H^s(\mathbb{R})})$, $c(\|\varphi\|_{L_2(\mathbb{R})},\|\varphi\|_{H^s(\mathbb{R})})$ *are constants dependent on* $\|\varphi\|_{H^s(\mathbb{R})}$, *and* $\|\varphi\|_{L_2(\mathbb{R})}$ *and* $\|\varphi\|_{H^s(\mathbb{R})}$, *respectively,* $p_2(t)$ *is a polynomial function of degree* 2 *with coefficients only dependent on* $\|\varphi\|_{H^s(\mathbb{R})}$.

Theorem 1.2. *Considered here is BBM equation* (1.1) *post for* $(x,t)\in\mathbb{R}^+\times\mathbb{R}^+$ *with the initial-boundary condition* (1.3). *If for any given* $s\geq 0$, *the initial data* $\varphi\in H^s(\mathbb{R}^+)$ *and it is required to be continuous locally at* $x=0$ *when* $0\leq s\leq\frac{1}{2}$, *and if the boundary data* $g\in L_\infty^{loc}([0,\infty))$ *is continuous*