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Penalized Schemes for Hamilton-Jacobi-Bellman Quasi-Variational Inequalities Arising in Regime Switching Utility Maximization with Optimal Stopping

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Abstract. The aim of this paper is to solve the Hamilton-Jacobi-Bellman (HJB) quasi-variational inequalities arising in regime switching utility maximization with optimal stopping. The HJB quasi-variational inequalities are penalized into the HJB equations and the convergence of the viscosity solution of the penalized HJB equations to that of the HJB variational inequalities is proved. The finite difference methods with iteration policy are used to solve the penalized HJB equations and the convergence is proved. The approach is implemented via numerical examples and the figures for the exercise boundaries and optimal strategies with sample paths are sketched.

AMS subject classifications: 65C20, 65C40, 65M06, 91G20, 91G60

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1. Introduction

The utility maximization is a kind of stochastic control problems. The dynamic programming approach is often applied to the optimal value function and the so-called HJB equation is derived (see the books Pham [34], Yong and Zhou [44] for the stochastic control and its applications). Since the HJB equation is a fully nonlinear PDE, the closed-form classical solution cannot be found except for some simple cases: a Black-Scholes complete market model with particular utility functions, see Bian *et al.* [8], Bian and Zheng [9]. For constrained market models it has to use numerical methods

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to solve the HJB equations. The standard approach to solve HJB equation by finite difference schemes is to discretize the derivatives in HJB equation and to solve the resulting finite dimensional control problem. The nonlinear discretized equations are often solved using policy iteration schemes (see e.g., [2, 3, 13, 15–19, 26, 27, 35, 36, 40, 41]). Among them, the work by Huang *et al.* [26, 27] and Babbin *et al.* [3] outlines the theory and implementation of the schemes for solving the coupled HJB equations arising in the American options under regime switching models. The convergence proofs are given therein.

A variant of utility maximization of terminal wealth is that investors may stop the investment before or at the maturity to achieve the overall maximum of the expected utility, which naturally leads to a mixed optimal control and stopping problem. The early work on this line includes Karatzas and Wang [29] and Dayanik and Karatzas [14] for properties of the value function at the initial time, Ceci and Bassan [12] for existence of viscosity solution of the variational equation, Henderson and Hobson [24] for equivalence of the value function in the presence of a Markov chain process and power utility. None of the above papers discusses the free boundary problem. Jian *et al.* [28] apply the dual transformation method to convert the nonlinear variational equation with power utility into an equivalent free boundary problem of a linear PDE and analyze qualitatively the properties of the free boundary and optimal strategies. The work is further extended in Guan *et al.* [21] to a problem with a call option type terminal payoff and power utility. Ma *et al.* [33] give rigorous analysis of the properties of the free boundary and construct the global approximation. It is well known that it is challenging to find the free boundary of a variational equation. The free boundary separates the exercise region from the continuation region and satisfies an integral equation which can be hardly solved. Finding the free boundary is much more difficult for the optimal investment stopping problem than for the American options pricing problem. The problems are often casted into a nonlinear quasi-variational inequality and the penalization methods are used to solve the nonlinear quasi-variational inequality. Witte and Reisinger [42] study the discrete quasi-variational inequalities arising from the discretization of an elliptic quasi-variational inequality using the penalty approach and Newton iterations. Azimzadeh *et al.* [1] study parabolic HJB quasi-variational inequalities, penalize it into a nonlinear HJB equation and use the policy iteration finite difference methods (FDMs) to solve the penalized HJB equations. Numerical implementations are not given in [1]. This paper extends the work of Azimzadeh *et al.* [1] to the regime-switching system of the HJB parabolic quasi-variational inequalities. Both the convergence analyses and the numerical implementations are given in this paper. Reisinger and Zhang [37] give the error estimates of penalty schemes for quasi-variational inequalities arising from impulse control problems. The setting of the problems is quite different from this paper, as the HJB operator in [37] is not time-dependent.

There has been active research in portfolio optimization with regime switching models. The regime switching model allows parameters of asset price dynamics to depend on a finite state Markov chain process. It provides good flexibility for characterizing macro market uncertainties while preserves analytic tractability for underlying asset price dynamics. Hamilton [22] introduces a regime switching model for non-stationary time series and business cycles. Hardy [23] applies a two-regime model to provide a good fit to monthly stock market returns. Zhang *et al.* [46] and Yin *et al.* [43] study the trading rules in a regime switching market. Zhou and Yin [47] investigate the mean-variance portfolio optimization in regime switching model. Canakoğlu and Özekici [11] discuss the HARA utility maximization in a regime switching model. Honda [25], Sass and Haussmann [39], and Rieder and Bäuerle [38] solve portfolio optimization problems with partial information and regime switching drift processes. Bäuerle and Rieder [7] and Fu *et al.* [20] show that the value function satisfies the HJB system of fully coupled nonlinear PDEs and prove the verification theorem. For a power or logarithmic utility function, the HJB equations can be reduced to a system of linear ODEs which are then solved with matrix exponentials. For general utility functions, it seems not possible to solve the system of HJB equations analytically. Ma *et al.* [31] develop the dual control Monte-Carlo methods to compute the tight bounds of value function in regime switching utility maximization, but it is not possible to guarantee the convergence in theory and the computation of the lower bound is rather time-consuming.

In this paper we study the optimal investment stopping problem for general utility functions in regime switching which can be written as a system of parabolic HJB quasi-variational inequalities. We first penalize the corresponding parabolic HJB quasi-variational inequalities into the penalized HJB equations and prove that the viscosity solutions of the penalized HJB equations converge to that of the HJB quasi-variational inequalities. We then solve the penalized HJB equations using the FDM with iteration policy and prove the convergence of the scheme. In the end, we implement the approach and draw the figures for the exercise boundaries and optimal strategies with sample paths via numerical examples.

The remaining parts of the paper are arranged as follows. In Section 2, we introduce the system of parabolic HJB quasi-variational inequalities arising in the regime switching utility maximization with optimal stopping. In Section 3, we study the penalized equations for the HJB quasi-variational inequalities and prove the convergence. In Section 4, we study the finite difference methods with iteration policy and prove the convergence. In Section 5, we verify the convergence of the approach and draw the exercise boundaries and optimal strategies with sample paths via numerical examples. Conclusions are given in the final section.

2. HJB quasi-variational inequalities

In this section, we introduce the system of HJB quasi-variational inequalities arising in the regime switching utility maximization with optimal stopping.

Consider a fixed time horizon $[0,T]$. Let $(\Omega,\mathcal{F},P)$ be a complete probability space, W a standard brownian motion, $\boldsymbol{\alpha}$ a continuous time finite state observable Markov chain processes, which are independent of each other, and let $\{\mathcal{F}_t\}$ be the natural filtration generated by W and $\boldsymbol{\alpha}$ completed with all P-null sets. We identify the state space of $\{\boldsymbol{\alpha_t}\}$ as a finite set of unit vector $\mathbb{E} := \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_d\}$ where $\mathbf{e}_i \in \mathbb{R}^d$ is a column vectors with one in the i-th position and zeros elsewhere, $j = 1, \ldots, d$. Denote by $\mathbf{Q} = (q_{ij})_{d\times d}$ the generator of the Markov chain $\{\boldsymbol{\alpha}_t\}$ with $q_{ij} \geq 0$ for $i \neq j$ and $\sum_{j=1}^{d} q_{ij} = 0$ for each $j \in \mathbb{D} := \{1, \ldots, d\}$. The Markov chain $\boldsymbol{\alpha}$ has the semi-martingale representation

$$\boldsymbol{\alpha}_t = \boldsymbol{\alpha}_0 + \int_0^t \mathbf{Q}'\boldsymbol{\alpha}_v dv + \mathbf{M}_t, \quad 0 \leq t \leq T,$$

where $\mathbf{Q}'$ is the transpose of $\mathbf{Q}$, and $\mathbf{M}$ is a purely discontinuous square integrable martingale with initial value zero.

Assume the financial market consists of one risk-free bond and one risky stock. The bond and stock price processes B and S are assumed to follow the stochastic differential equations (SDEs)

$$dB_t = r_t B_t dt, \quad dS_t = S_t(\mu_t dt + \sigma_t dW_t), \quad 0 \leq t \leq T,$$

where $r_t = \mathbf{r}\boldsymbol{\alpha}_t$, $\mu_t = \boldsymbol{\mu}\boldsymbol{\alpha}_t$, $\sigma_t = \boldsymbol{\sigma}\boldsymbol{\alpha}_t$ and $\mathbf{r} = (r_1, \ldots, r_d)$ is a vector of risk-free interest rates with r_i being the rate in regime i, and $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_d)$ and $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_d)$ are vectors of return and volatility rates of the risky asset. Assume all rates are positive constants. Denote by $\boldsymbol{\theta} := (\theta_1, \ldots, \theta_d)$ the vector of market prices of risk with $\theta_i = (\mu_i - r_i)/\sigma_i$ for $i \in \mathbb{D}$.

Let X be the wealth process of a portfolio comprising the bond B and the stock S. The wealth process X satisfies the SDE

$$dX_t = X_t\big(r_t dt + \pi_t \sigma_t \left(\theta_t dt + dW_t\right)\big), \quad 0 \leq t \leq T,$$

where π_t is a progressively measurable control process and represents the proportion of wealth X_t invested in risky asset S_t and $\theta_t = \boldsymbol{\theta}\boldsymbol{\alpha}_t$ is the market price of risk at time t. The optimal investment stopping problem is defined by

$$\sup_{\pi,\tau} E\left[\exp(-\beta\tau)U(X_\tau^{t,x,\pi} - K)\right], \tag{2.1}$$

where U is a utility function that is continuous, increasing and concave on $[0, \infty]$, $\tau \in [0, T]$ is an $\{\mathcal{F}_t\}$ adapted stopping time, $\beta > 0$ the utility discount factor, $K \geq 0$ the minimum wealth threshold value.

To solve the problem (2.1), we define the value functions

$$V(\varsigma, x, j) := \sup_{\pi,\tau} E_{t,x,j}\left[\exp(-\beta(\tau - t))U(X_\tau^{t,x,\pi} - K)\right], \quad j \in \mathbb{D},$$

where $\varsigma := T - t$ and $E_{t,x,j}$ is the conditional expectation operator given $X_t = x$, $\boldsymbol{\alpha}_t = \mathbf{e}_j$ for $j \in \mathbb{D}$. It follows from the dynamic programming principle that V satisfies the following system of HJB variational inequality (see [21]):

$$\min\left\{V_\varsigma(\varsigma, x, j) - \sup_\pi \mathcal{L}^\pi[V(\varsigma, x, j)],\, V(\varsigma, x, j) - U(x - K)\right\} = 0, \quad j \in \mathbb{D}, \tag{2.2}$$

where

$$\mathcal{L}^{\pi}[V(\varsigma,x,j)] = \frac{1}{2}\sigma_j^2\pi^2x^2 \cdot V_{xx}(\varsigma,x,j) + (\pi(\mu_j - r_j) + r_j)\, xV_x(\varsigma,x,j) \\ - \beta V(\varsigma,x,j) - q_j V(\varsigma,x,j) + \sum_{\ell=1,\ell\neq j}^{d} q_{j\ell} V(\varsigma,x,\ell) \tag{2.3}$$

on $[0,T] \times (K,+\infty)$ and the terminal and boundary conditions are given by

$$V(0,x,j) = U(x-K), \qquad x \in [K,+\infty), \tag{2.4}$$
$$V(\varsigma,K,j) = 0, \qquad \varsigma \in [0,T], \tag{2.5}$$
$$V(\varsigma,x_{\max},j) = \phi(\varsigma,x_{\max},j), \quad \varsigma \in [0,T], \tag{2.6}$$

where $q_j := \sum_{\ell=1,\ell\neq j}^{d} q_{j\ell}$, and the boundary condition (2.6) will be specified case by case in the following sections.

To define the viscosity solution of the HJB variational equations (2.2), it is convenient to write it as the following form:

$$\mathcal{F}_j\left(V_{xx}(j), V_x(j), V_\varsigma(j), V(j), -\sum_{\ell=1,\ell\neq j}^{d} q_{j\ell}V(\ell), x, \varsigma\right) = 0, \tag{2.7}$$

where we denote $V(j) = V(\varsigma,x,j)$, $j \in \mathbb{D}$ is the current regime state.

Next we give the definitions of the upper and lower semi-continuous envelopes of function $\mathcal{F}_j$ and the viscosity sub-solution, the viscosity super-solution and the viscosity solution of (2.7).

Definition 2.1. *The upper and lower semi-continuous envelopes of function $\mathcal{F}_j$ are defined respectively by*

$$\overline{\mathcal{F}}_j \equiv \limsup_{\substack{\widetilde{\varsigma}\to\varsigma \\ \widetilde{x}\to x \\ \widetilde{\varsigma}\in B(\varsigma,\rho) \\ \widetilde{x}\in B(x,h)}} \mathcal{F}_j\left(V_{\widetilde{x}\widetilde{x}}(j), V_{\widetilde{x}}(j), V_{\widetilde{\varsigma}}(j), V(j), -\sum_{\ell=1,\ell\neq j}^{d} q_{j\ell}V(\ell), \widetilde{x}, \widetilde{\varsigma}\right)$$

and

$$\underline{\mathcal{F}}_j \equiv \liminf_{\substack{\widetilde{\varsigma}\to\varsigma \\ \widetilde{x}\to x \\ \widetilde{\varsigma}\in B(\varsigma,\rho) \\ \widetilde{x}\in B(x,h)}} \mathcal{F}_j\left(V_{\widetilde{x}\widetilde{x}}(j), V_{\widetilde{x}}(j), V_{\widetilde{\varsigma}}(j), V(j), -\sum_{\ell=1,\ell\neq j}^{d} q_{j\ell}V(\ell), \widetilde{x}, \widetilde{\varsigma}\right),$$

where $B(\cdot,\circ)$ denotes the neighborhood with center $\cdot$ and size $\circ$.

Definition 2.2. *Let $V : \overline{\Omega} \to \mathbb{R}$ be locally bounded function.*

(i) If for all $\varphi \in C^{1,2}(\overline{\Omega})$ and $(\overline{\varsigma}, \overline{x}) \in \overline{\Omega}$ such that $\overline{V} - \varphi$ has a local maximum at $(\overline{\varsigma}, \overline{x})$, we have

$$\underline{\mathcal{F}}_j\left(\varphi_{xx}(\overline{\varsigma}, \overline{x}, j), \varphi_x(\overline{\varsigma}, \overline{x}, j), \varphi_\varsigma(\overline{\varsigma}, \overline{x}, j), \overline{V}, -\sum_{\ell=1,\ell\neq j}^{d} q_{j\ell}\overline{V}(\overline{\varsigma}, \overline{x}, \ell), \overline{x}, \overline{\varsigma}\right) \leq 0,$$

then V is called the viscosity sub-solution of (2.7).

(ii) If for all $\varphi \in C^{1,2}(\overline{\Omega})$ and $(\underline{\varsigma}, \underline{x}) \in \underline{\Omega}$ such that $\underline{V} - \varphi$ has a local minimum at $(\underline{\varsigma}, \underline{x})$, we have

$$\overline{\mathcal{F}}_j\left(\varphi_{xx}(\underline{\varsigma}, \underline{x}, j), \varphi_x(\underline{\varsigma}, \underline{x}, j), \varphi_\varsigma(\underline{\varsigma}, \underline{x}, j), \underline{V}, -\sum_{\ell=1,\ell\neq j}^{d} q_{j\ell}\underline{V}(\underline{\varsigma}, \underline{x}, \ell), \underline{x}, \underline{\varsigma}\right) \geq 0,$$

then V is called the viscosity super-solution of (2.7).

(iii) If it is both a viscosity sub-solution and viscosity super-solution of (2.7)*, then we call that V is the viscosity solution of* (2.7).

3. Penalized equations for the HJB variational inequalities

In this section, we transform the HJB variational inequalities (2.2) into HJB equations using the penalty approach (see [1,30,45]) and prove that the viscosity solution of the penalized equations converges to that of the HJB variational inequalities.

We define the penalized HJB equations for the HJB variational inequalities (2.2) as

$$V^\varepsilon_\varsigma(\varsigma, x, j) - \sup_\pi \mathcal{L}^\pi[V^\varepsilon(\varsigma, x, j)] - \varepsilon\big(U(x-K) - V^\varepsilon(\varsigma, x, j)\big)^+ = 0, \quad j \in \mathbb{D} \tag{3.1}$$

on $[0,T] \times (0,+\infty)$ with terminal and boundary conditions

$$\begin{aligned} &V^\varepsilon(0, x, j) = U(x-K), && x \in [K, +\infty), && (3.2)\\ &V^\varepsilon(\varsigma, K, j) = 0, && \varsigma \in [0,T], && (3.3)\\ &V^\varepsilon(\varsigma, x_{\max}, j) = \phi(\varsigma, x_{\max}, j), && \varsigma \in [0,T], && (3.4) \end{aligned}$$

where $(a)^+ := \max(a, 0)$ and $\varepsilon > 0$. Denote (3.1) as

$$F_j\left(V^\varepsilon_{xx}(j), V^\varepsilon_x(j), V^\varepsilon_\varsigma(j), V^\varepsilon(j), -\sum_{\ell=1,\ell\neq j}^{d} q_{j\ell}V^\varepsilon(\ell), x, \varsigma\right) = 0, \tag{3.5}$$

where $V^\varepsilon(j)$ denotes a function $V^\varepsilon(j) = V^\varepsilon(\varsigma, x, j)$, $j \in \mathbb{D}$ is the current regime state. The definition of the viscosity solution of (3.1) is similar to that for (2.2) (see Definition 2.2) by replacing $\mathcal{F}_j$ by F_j.

Lemma 3.1 (Comparison Principle). *Let $\overline{V}^{\varepsilon}$ (resp. $\underline{V}^{\varepsilon}$) be a upper-semi-continuous viscosity sub-solution (resp. lower-semi-continuous viscosity super-solution) with polynomial growth condition to* (3.1) *for every ε. If boundary function ϕ in* (3.4) *is bounded and $\overline{V}^{\varepsilon}(T,.) \leq \underline{V}^{\varepsilon}(T,.)$ on $[0,+\infty)$. Then $\overline{V}^{\varepsilon} \leq \underline{V}^{\varepsilon}$ on $[0,T]\times[0,+\infty)$.*

Proof. Since both $b(x,\pi^j) = [\pi^{(j)}(\mu_j - r_j) + r_j]x$ and $a(x,\pi^j) = \sigma_j \pi^{(j)} x$ satisfy the Lipschitz condition in x, the proof follows from [34, Theorem 4.4.5]. □

In the follow-up, we prove that the viscosity solution of (3.1) converges to that of (2.2) as $\varepsilon \to \infty$.

Definition 3.1. *The upper and lower weak limits of function $V^{\varepsilon}(\varsigma,x,j)$ for $j\in\mathbb{D}$ are defined respectively by*

$$V^*(\varsigma,x,j) \equiv \limsup_{\substack{\widetilde{\varsigma}\to\varsigma\\ \widetilde{x}\to x\\ \widetilde{\varsigma}\in B(\varsigma,1/\varepsilon)\\ \widetilde{x}\in B(x,1/\varepsilon)\\ \varepsilon\to\infty}} V^{\varepsilon}(\widetilde{\varsigma},\widetilde{x},j) \quad \textit{and} \quad V_*(\varsigma,x,j) \equiv \liminf_{\substack{\widetilde{\varsigma}\to\varsigma\\ \widetilde{x}\to x\\ \widetilde{\varsigma}\in B(\varsigma,1/\varepsilon)\\ \widetilde{x}\in B(x,1/\varepsilon)\\ \varepsilon\to\infty}} V^{\varepsilon}(\widetilde{\varsigma},\widetilde{x},j),$$

where $B(\cdot,\circ)$ denotes the neighborhood with center $\cdot$ and size $\circ$.

Lemma 3.2. *Let $V^{\varepsilon}(\varsigma,x,j)$ be the solution of the penalized HJB equations* (3.1) *and $V^*(\varsigma,x,j)$ and $V_*(\varsigma,x,j)$ be respectively the upper and lower weak limits of $V^{\varepsilon}(\varsigma,x,j)$. Then it has the following conclusions:*

(i) $V^*(\varsigma,x,j)$ and $V_*(\varsigma,x,j)$ are respectively the upper semi-continuous and the lower semi-continuous functions.

(ii) Let $V^{\varepsilon}(\varsigma,x,j)$ be the upper semi-continuous function (respectively the lower semi-continuous functions) and $(\overline{\varsigma},\overline{x})\in[0,T]\times[K,x_{\max}]$ be a strict local maximum point of $V^*-\varphi$ (respectively minimum point of $V_*-\varphi$) for a function $\varphi\in C^{1,2}([0,T]\times[K,x_{\max}])$. Then there exist subsequences $(\varsigma_\varepsilon,x_\varepsilon)\to(\overline{\varsigma},\overline{x})$ and $V^{\varepsilon}(\varsigma_\varepsilon,x_\varepsilon,j)\to V^*(\overline{\varsigma},\overline{x},j)$ (respectively $V_*(\overline{\varsigma},\overline{x},j)$) as $\varepsilon\to\infty$, such that $(\varsigma_\varepsilon,x_\varepsilon)$ is a local maximum (respectively minimum) point of $V^{\varepsilon}-\varphi$ for every ε.

Proof. The proof follows Bardi and Capuzzo [4]. □

Lemma 3.3. *Let $V^{\varepsilon}(\varsigma,x,j)$ be the unique viscosity solution of* (3.1) *for every ε. Suppose that $V^*(\varsigma,x,j)$ and $V_*(\varsigma,x,j)$ are respectively the upper and lower weak limits of $V^{\varepsilon}(\varsigma,x,j)$. Then $V^*(\varsigma,x,j)$ and $V_*(\varsigma,x,j)$ are respectively a sub-solution and super-solution of* (2.2).

Proof. We first prove that $V^*(\varsigma,x,j)$ is a sub-solution of (2.2). To this end, we need to show that, for all $\varphi\in C^{1,2}([0,T]\times[K,x_{\max}])$ and a strict local maximum point $(\overline{\varsigma},\overline{x})$ of $V^*(\varsigma,x,j)-\varphi(\varsigma,x,j)$, the following inequality holds:

$$\underline{\mathcal{F}}_j\left(\varphi_{xx}(\overline{\varsigma},\overline{x},j),\varphi_x(\overline{\varsigma},\overline{x},j),\varphi_\varsigma(\overline{\varsigma},\overline{x},j),V^*,-\sum_{\ell=1,\ell\neq j}^{d} q_{j\ell}V^*(\overline{\varsigma},\overline{x},\ell),\overline{x},\overline{\varsigma}\right)\leq 0, \qquad (3.6)$$

i.e.,

$$\min\left\{\varphi_\varsigma(\overline{\varsigma},\overline{x},j)-\sup_{\pi}\left\{\frac{1}{2}\sigma_j^2\pi^2\overline{x}^2\cdot\varphi_{xx}(\overline{\varsigma},\overline{x},j)+\big(\pi(\mu_j-r_j)+r_j\big)\overline{x}\varphi_x(\overline{\varsigma},\overline{x},j)\right.\right.$$
$$\left.-\beta V^*(\overline{\varsigma},\overline{x},j)-q_jV^*(\overline{\varsigma},\overline{x},j)+\sum_{\ell=1,\ell\neq j}^{d}q_{j\ell}V^*\left(\overline{\varsigma},\overline{x},\ell\right)\right\},$$
$$\left.V^*(\overline{\varsigma},\overline{x},j)-U(\overline{x}-K)\right\}\leq 0. \tag{3.7}$$

Assume it is not the case of (3.6) (or (3.7)), i.e.,

$$\underline{\mathcal{F}}_j\left(\varphi_{xx}(\overline{\varsigma},\overline{x},j),\varphi_x(\overline{\varsigma},\overline{x},j),\varphi_\varsigma(\overline{\varsigma},\overline{x},j),V^*,-\sum_{\ell=1,\ell\neq j}^{d}q_{j\ell}V^*(\overline{\varsigma},\overline{x},\ell),\overline{x},\overline{\varsigma}\right)>0. \tag{3.8}$$

Then from Lemma 3.2, there exists a subsequence $(\varsigma_\varepsilon,x_\varepsilon)\to(\overline{\varsigma},\overline{x})$ and $V^\varepsilon(\varsigma_\varepsilon,x_\varepsilon,j)\to V^*(\overline{\varsigma},\overline{x},j)$ as $\varepsilon\to\infty$, such that $(\varsigma_\varepsilon,x_\varepsilon)$ is a local maximum point of $V^\varepsilon(\varsigma,x,j)-\varphi(\varsigma,x,j)$ for every ε. Since $\varphi\in C^{1,2}([0,T]\times[K,x_{\max}])$, by continuity and (3.8), there exists a neighborhood of $(\overline{\varsigma},\overline{x})$, $B(\overline{\varsigma},\overline{x})\subset[0,T]\times[K,x_{\max}]$ such that for all $(\varsigma_\varepsilon,x_\varepsilon)\in B(\overline{\varsigma},\overline{x})$,

$$\underline{\mathcal{F}}_j\left(\varphi_{xx}(\varsigma_\varepsilon,x_\varepsilon,j),\varphi_x(\varsigma_\varepsilon,x_\varepsilon,j),\varphi_\varsigma(\varsigma_\varepsilon,x_\varepsilon,j),V^*,-\sum_{\ell=1,\ell\neq j}^{d}q_{j\ell}V^*(\varsigma_\varepsilon,x_\varepsilon,\ell),x_\varepsilon,\varsigma_\varepsilon\right)>0. \tag{3.9}$$

From the definitions of $\underline{\mathcal{F}}_j$ and $\underline{F}_j$, (3.9) leads to

$$\underline{F}_j\left(\varphi_{xx}(\varsigma_\varepsilon,x_\varepsilon,j),\varphi_x(\varsigma_\varepsilon,x_\varepsilon,j),\varphi_\varsigma(\varsigma_\varepsilon,x_\varepsilon,j),V^*,-\sum_{\ell=1,\ell\neq j}^{d}q_{j\ell}V^*(\varsigma_\varepsilon,x_\varepsilon,\ell),x_\varepsilon,\varsigma_\varepsilon\right)>0. \tag{3.10}$$

Since we have assumed that $V^\varepsilon(\varsigma,x,j)$ is the unique viscosity solution of (3.1) and $(\varsigma_\varepsilon,x_\varepsilon)$ is a local maximum point of $V^\varepsilon(\varsigma,x,j)-\varphi(\varsigma,x,j)$ for every ε, from the definition of the viscosity solution for penalized equation (3.1), we have

$$\underline{F}_j\left(\varphi_{xx}(\varsigma_\varepsilon,x_\varepsilon,j),\varphi_x(\varsigma_\varepsilon,x_\varepsilon,j),\varphi_\varsigma(\varsigma_\varepsilon,x_\varepsilon,j),V^\varepsilon,-\sum_{\ell=1,\ell\neq j}^{d}q_{j\ell}V^\varepsilon(\varsigma_\varepsilon,x_\varepsilon,\ell),x_\varepsilon,\varsigma_\varepsilon\right)\leq 0. \tag{3.11}$$

Inequalities (3.10) and (3.11) are contradictory. Therefore we conclude that the assumption (3.8) is not right, instead, (3.6) holds true which means that $V^*(\varsigma,x,j)$ is a sub-solution of (2.2).

We next prove that $V_*(\varsigma,x,j)$ is a super-solution of (2.2). Let $\varphi\in C^{1,2}([0,T]\times[K,x_{\max}])$ and suppose that $(\overline{\varsigma},\overline{x})$ is a strict local minimum point of $V_*(\varsigma,x,j)-\varphi(\varsigma,x,j)$. We need to show that

$$\overline{\mathcal{F}}_j\left(\varphi_{xx}(\overline{\varsigma},\overline{x},j),\varphi_x(\overline{\varsigma},\overline{x},j),\varphi_\varsigma(\overline{\varsigma},\overline{x},j),V_*,-\sum_{\ell=1,\ell\neq j}^{d}q_{j\ell}V_*(\overline{\varsigma},\overline{x},\ell),\overline{x},\overline{\varsigma}\right)\geq 0, \tag{3.12}$$

i.e.,

$$\min\left\{\varphi_\varsigma(\overline{\varsigma},\overline{x},j)-\sup_\pi\left\{\frac{1}{2}\sigma_j^2\pi^2\overline{x}^2\varphi_{xx}(\overline{\varsigma},\overline{x},j)+\big(\pi(\mu_j-r_j)+r_j\big)\overline{x}\varphi_x(\overline{\varsigma},\overline{x},j)\right.\right.$$
$$\left.-\beta V_*(\overline{\varsigma},\overline{x},j)-q_jV_*(\overline{\varsigma},\overline{x},j)+\sum_{\ell=1,\ell\neq j}^{d}q_{j\ell}V_*\left(\overline{\varsigma},\overline{x},\ell\right)\right\},$$
$$\left.V_*(\overline{\varsigma},\overline{x},j)-U(\overline{x}-K)\right\}\geq 0. \tag{3.13}$$

Assume the following two cases:

Case 1. Assume that

$$\varphi_\varsigma(\overline{\varsigma},\overline{x},j)-\sup_\pi\left\{\frac{1}{2}\sigma_j^2\pi^2\overline{x}^2\varphi_{xx}(\overline{\varsigma},\overline{x},j)+\big(\pi(\mu_j-r_j)+r_j\big)\overline{x}\varphi_x(\overline{\varsigma},\overline{x},j)\right.$$
$$\left.-\beta V_*(\overline{\varsigma},\overline{x},j)-q_jV_*(\overline{\varsigma},\overline{x},j)+\sum_{\ell=1,\ell\neq j}^{d}q_{j\ell}V_*\left(\overline{\varsigma},\overline{x},\ell\right)\right\}<0. \tag{3.14}$$

By Lemma 3.2, we see that there exists a subsequence $(\varsigma_\varepsilon,x_\varepsilon)\to(\overline{\varsigma},\overline{x})$ and $V^\varepsilon(\varsigma_\varepsilon,x_\varepsilon,j)\to V_*(\overline{\varsigma},\overline{x},j)$ as $\varepsilon\to\infty$, such that $(\varsigma_\varepsilon,x_\varepsilon)$ is a local minimum point of $V^\varepsilon(\varsigma,x,j)-\varphi(\varsigma,x,j)$ for every ε. Since $\varphi\in C^{1,2}([0,T]\times[K,x_{\max}])$, by continuity and (3.14), there exists a neighborhood of $(\overline{\varsigma},\overline{x})$, $B(\overline{\varsigma},\overline{x})\subset[0,T]\times[K,x_{\max}]$, such that for all $(\varsigma_\varepsilon,x_\varepsilon)\in B(\overline{\varsigma},\overline{x})$,

$$\varphi_\varsigma(\varsigma_\varepsilon,x_\varepsilon,j)-\sup_\pi\left\{\frac{1}{2}\sigma_j^2\pi^2x_\varepsilon^2\varphi_{xx}(\varsigma_\varepsilon,x_\varepsilon,j)+\big(\pi(\mu_j-r_j)+r_j\big)x_\varepsilon\varphi_x(\varsigma_\varepsilon,x_\varepsilon,j)\right.$$
$$\left.-\beta V_*(\varsigma_\varepsilon,x_\varepsilon,j)-q_jV_*(\varsigma_\varepsilon,x_\varepsilon,j)+\sum_{\ell=1,\ell\neq j}^{d}q_{j\ell}V_*\left(\varsigma_\varepsilon,x_\varepsilon,\ell\right)\right\}<0.$$

For $\varepsilon>0$ this inequality leads to

$$\varphi_\varsigma(\varsigma_\varepsilon,x_\varepsilon,j)-\sup_\pi\left\{\frac{1}{2}\sigma_j^2\pi^2x_\varepsilon^2\varphi_{xx}(\varsigma_\varepsilon,x_\varepsilon,j)+\big(\pi(\mu_j-r_j)+r_j\big)x_\varepsilon\varphi_x(\varsigma_\varepsilon,x_\varepsilon,j)\right.$$
$$\left.-\beta V_*(\varsigma_\varepsilon,x_\varepsilon,j)-q_jV_*(\varsigma_\varepsilon,x_\varepsilon,j)+\sum_{\ell=1,\ell\neq j}^{d}q_{j\ell}V_*\left(\varsigma_\varepsilon,x_\varepsilon,\ell\right)\right\}$$
$$-\varepsilon\big(U(x-K)-V_*(\varsigma_\varepsilon,x_\varepsilon,j)\big)^+<0,\quad j\in\mathbb{D},$$

i.e.,

$$\overline{F}_j\left(\varphi_{xx}(\varsigma_\varepsilon,x_\varepsilon,j),\varphi_x(\varsigma_\varepsilon,x_\varepsilon,j),\varphi_\varsigma(\varsigma_\varepsilon,x_\varepsilon,j),V_*,-\sum_{\ell=1,\ell\neq j}^{d}q_{j\ell}V_*(\varsigma_\varepsilon,x_\varepsilon,\ell),x_\varepsilon,\varsigma_\varepsilon\right)<0. \tag{3.15}$$

Since we assumed that V^ε is a viscosity solution of (3.1), and $(\varsigma_\varepsilon, x_\varepsilon)$ is a local minimum point of $V^\varepsilon(\varsigma,x,j)-\varphi(\varsigma,x,j)$, for every ε we have

$$\overline{F}_j\left(\varphi_{xx}(\varsigma_\varepsilon,x_\varepsilon,j),\varphi_x(\varsigma_\varepsilon,x_\varepsilon,j),\varphi_\varsigma(\varsigma_\varepsilon,x_\varepsilon,j),V^\varepsilon,-\sum_{\ell=1,\ell\neq j}^{d}q_{j\ell}V^\varepsilon(\varsigma_\varepsilon,x_\varepsilon,\ell),x_\varepsilon,\varsigma_\varepsilon\right)\geq 0.$$

Therefore, for sufficiently large $\varepsilon>0$, we have the following inequality:

$$\overline{F}_j\left(\varphi_{xx}(\varsigma_\varepsilon,x_\varepsilon,j),\varphi_x(\varsigma_\varepsilon,x_\varepsilon,j),\varphi_\varsigma(\varsigma_\varepsilon,x_\varepsilon,j),V_*,-\sum_{\ell=1,\ell\neq j}^{d}q_{j\ell}V_*(\varsigma_\varepsilon,x_\varepsilon,\ell),x_\varepsilon,\varsigma_\varepsilon\right)\geq 0. \quad (3.16)$$

Inequalities (3.15) and (3.16) are contradictory. Therefore we conclude that assumption (3.14) is not true.

Case 2. Assume that

$$\begin{aligned}\varphi_\varsigma(\overline{\varsigma},\overline{x},j)-\sup_\pi\bigg\{&\frac{1}{2}\sigma_j^2\pi^2\overline{x}^2\cdot\varphi_{xx}(\overline{\varsigma},\overline{x},j)+\big(\pi(\mu_j-r_j)+r_j\big)\overline{x}\varphi_x(\overline{\varsigma},\overline{x},j) \qquad (3.17)\\ &-\beta V_*(\overline{\varsigma},\overline{x},j)-q_jV_*(\overline{\varsigma},\overline{x},j)+\sum_{\ell=1,\ell\neq j}^{d}q_{j\ell}V_*\left(\overline{\varsigma},\overline{x},\ell\right)\bigg\}\equiv k\geq 0,\end{aligned}$$

$$V_*(\overline{\varsigma},\overline{x},j)-U(\overline{x}-K)\equiv l<0. \quad (3.18)$$

By Lemma 3.2, we see that there exists a subsequence $(\varsigma_\varepsilon,x_\varepsilon)\to(\overline{\varsigma},\overline{x})$ as $\varepsilon\to\infty$, such that $(\varsigma_\varepsilon,x_\varepsilon)$ is a local minimum point of $V^\varepsilon(\varsigma,x,j)-\varphi(\varsigma,x,j)$ for every ε. Since $\varphi\in C^{1,2}([0,T]\times[K,x_{\max}])$, by continuity, (3.17) and (3.18), there exists a neighborhood of $(\overline{\varsigma},\overline{x})$, $B(\overline{\varsigma},\overline{x})\subset[0,T]\times[K,x_{\max}]$, such that for all $(\varsigma_\varepsilon,x_\varepsilon)\in B(\overline{\varsigma},\overline{x})$,

$$\begin{aligned}k-\frac{|l|}{2}\leq\varphi_\varsigma(\varsigma,x,j)-\sup_\pi\bigg\{&\frac{1}{2}\sigma_j^2\pi^2x^2\varphi_{xx}(\varsigma,x,j)+\big(\pi(\mu_j-r_j)+r_j\big)x\varphi_x(\varsigma,x,j)\\ &-\beta V_*(\varsigma,x,j)-q_jV_*(\varsigma,x,j)+\sum_{\ell=1,\ell\neq j}^{d}q_{j\ell}V_*\left(\varsigma,x,\ell\right)\bigg\}\\ \leq k+\frac{|l|}{2},& \qquad (3.19)\end{aligned}$$

$$l-\frac{|l|}{2}\leq V_*(\varsigma,x,j)-U(x-K)\leq l+\frac{|l|}{2}. \quad (3.20)$$

Therefore, for $\varepsilon>(2k+|l|)/|l|$, inequalities (3.19) and (3.20) give that

$$\begin{aligned}&\overline{F}_j\left(\varphi_{xx}(\varsigma_\varepsilon,x_\varepsilon,j),\varphi_x(\varsigma_\varepsilon,x_\varepsilon,j),\varphi_\varsigma(\varsigma_\varepsilon,x_\varepsilon,j),V_*,-\sum_{\ell=1,\ell\neq j}^{d}q_{j\ell}V_*(\varsigma_\varepsilon,x_\varepsilon,\ell),x_\varepsilon,\varsigma_\varepsilon\right)\\ &<k+\frac{|l|}{2}+\frac{2k+|l|}{|l|}\left(l+\frac{|l|}{2}\right)=k-\frac{l}{2}+\frac{2k-l}{-l}\frac{l}{2}=0. \qquad (3.21)\end{aligned}$$

Since we have assumed that $V^\varepsilon(\varsigma,x,j)$ be the unique viscosity solution of (3.1) and $(\varsigma_\varepsilon,x_\varepsilon)$ is a local minimum point of $V^\varepsilon(\varsigma,x,j)-\varphi(\varsigma,x,j)$ for every ε. So

$$\overline{F}_j\left(\varphi_{xx}(\varsigma_\varepsilon,x_\varepsilon,j),\varphi_x(\varsigma_\varepsilon,x_\varepsilon,j),\varphi_\varsigma(\varsigma_\varepsilon,x_\varepsilon,j),V^\varepsilon,-\sum_{\ell=1,\ell\neq j}^{d}q_{j\ell}V^\varepsilon(\varsigma_\varepsilon,x_\varepsilon,\ell),x_\varepsilon,\varsigma_\varepsilon\right)\geq 0.$$

Therefore for sufficiently large ε, we have the following inequality:

$$\overline{F}_j\left(\varphi_{xx}(\varsigma_\varepsilon,x_\varepsilon,j),\varphi_x(\varsigma_\varepsilon,x_\varepsilon,j),\varphi_\varsigma(\varsigma_\varepsilon,x_\varepsilon,j),V_*,-\sum_{\ell=1,\ell\neq j}^{d}q_{j\ell}V_*(\varsigma_\varepsilon,x_\varepsilon,\ell),x_\varepsilon,\varsigma_\varepsilon\right)\geq 0. \quad (3.22)$$

Inequalities (3.21) and (3.22) are contradictory. Therefore we conclude that assumptions (3.17) and (3.18) are not true.

From the above two-case discussions, we conclude that (3.12) (or (3.13)) holds true. Therefore $V_*(\varsigma,x,j)$ is a super-solution of (2.2).

Theorem 3.1 (Convergence of Viscosity Solutions). *Let $V(\varsigma,x,j)$ be the unique viscosity solution of* (2.2)*, and $V^\varepsilon(\varsigma,x,j)$ be the unique viscosity solution of* (3.1) *for every ε. Then, $V^\varepsilon(\varsigma,x,j)\to V(\varsigma,x,j)$ as $\varepsilon\to\infty$.*

Proof. Let $V^*(\varsigma,x,j)$ and $V_*(\varsigma,x,j)$ be respectively the upper and lower weak limits of $V^\varepsilon(\varsigma,x,j)$. Then, from Definition 3.1, we have

$$V^*(\varsigma,x,j)\geq V_*(\varsigma,x,j). \quad (3.23)$$

From Lemma 3.3, we known that $V^*(\varsigma,x,j)$ and $V_*(\varsigma,x,j)$ are respectively a sub-solution of (2.2) and super-solution of (2.2). By Lemma 3.1, we known that $V^*(\varsigma,x,j)$ and $V_*(\varsigma,x,j)$ satisfies $V^*(\varsigma,x,j)\leq V_*(\varsigma,x,j)$. Therefore it has that $V^*(\varsigma,x,j)=V_*(\varsigma,x,j)=V(\varsigma,x,j)$. From the definition of viscosity solutions and the upper and lower weak limits, we have $V^\varepsilon(\varsigma,x,j)\to V(\varsigma,x,j)$ as $\varepsilon\to\infty$. □

4. Finite difference methods

We now use the finite-difference scheme to solve the penalized HJB equations (3.1). A grid is constructed consisting of a set of $M+1$ nodes $\{x_0,\ldots,x_M\}$ with $x_0=K$, $x_M=x_{\max}$, $\Delta x=(x_{\max}-K)/M$, following a sequence of N time steps $\{\varsigma_0,\ldots,\varsigma_N\}$ with $\Delta\varsigma=T/N$, $\varsigma_n=n\Delta\varsigma$. Let $V_i^{\varepsilon,n}(j)$ be the approximation to $V^\varepsilon(\varsigma_n,x_i,j)$. Eq. (3.1) can be discretized by a standard finite difference method to give

$$\begin{aligned}&\frac{V_i^{\varepsilon,n+1}(j)-V_i^{\varepsilon,n}(j)}{\Delta\varsigma}\\&=\sup_{\pi^{(j)}}\left\{\frac{1}{2}\sigma_j^2\left(\pi^{(j)}\right)^2x_i^2\frac{V_{i+1}^{\varepsilon,n+1}(j)-2V_i^{\varepsilon,n+1}(j)+V_{i-1}^{\varepsilon,n+1}(j)}{\Delta x^2}\right.\end{aligned}$$

$$
\begin{aligned}
&+\xi\left[\pi^{(j)}(\mu_j-r_j)+r_j\right]x_i\cdot\frac{V_{i+1}^{\varepsilon,n+1}(j)-V_i^{\varepsilon,n+1}(j)}{\Delta x}\\
&+(1-\xi)\left[\pi^{(j)}(\mu_j-r_j)+r_j\right]x_i\cdot\frac{V_i^{\varepsilon,n+1}(j)-V_{i-1}^{\varepsilon,n+1}(j)}{\Delta x}\\
&-\beta V_i^{\varepsilon,n+1}(j)-q_jV_i^{\varepsilon,n+1}(j)+\sum_{\ell=1,\ell\neq j}^{d}q_{j\ell}V_i^{\varepsilon,n+1}(\ell)\\
&+\varepsilon\left(U(x_i-K)-V_i^{\varepsilon,n+1}(j)\right)^+\Bigg\}
\end{aligned}
\tag{4.1}
$$

with $V_0^{\varepsilon,n}(j)=0$ and $V_M^{\varepsilon,n}(j)=\phi(\varsigma_n,x_M,j)$. For the convenience of analysis, (4.1) is re-written as

$$
\begin{aligned}
&\frac{V_i^{\varepsilon,n+1}(j)-V_i^{\varepsilon,n}(j)}{\Delta\varsigma}\\
=&\Big[\Big(-\alpha_i^{n+1}\big(\pi_{n+1}^{(j)}\big)-\beta_i^{n+1}\big(\pi_{n+1}^{(j)}\big)-\beta-q_j\Big)V_i^{\varepsilon,n+1}(j)\\
&\quad+\alpha_i^{n+1}\big(\pi_{n+1}^{(j)}\big)V_{i-1}^{\varepsilon,n+1}(j)+\beta_i^{n+1}\big(\pi_{n+1}^{(j)}\big)V_{i+1}^{\varepsilon,n+1}(j)\Big]\\
&\quad+\sum_{\ell=1,\ell\neq j}^{d}q_{j\ell}V_i^{\varepsilon,n+1}(\ell)+\varepsilon\left(U(x_i-K)-V_i^{\varepsilon,n+1}(j)\right)^+,
\end{aligned}
\tag{4.2}
$$

where

$$
\begin{aligned}
\pi_{n+1}^{(j)}\in\arg\sup_{\pi^{(j)}}\Big[&\alpha_i^{n+1}(\pi^{(j)})V_{i-1}^{\varepsilon,n+1}(j)+\beta_i^{n+1}(\pi^{(j)})V_{i+1}^{\varepsilon,n+1}(j)\\
&+\big(-\alpha_i^{n+1}(\pi^{(j)})-\beta_i^{n+1}(\pi^{(j)})-\beta-q_j\big)V_i^{\varepsilon,n+1}(j)\Big]
\end{aligned}
$$

and

$$
\begin{aligned}
\alpha_i^{n+1}\big(\pi_{n+1}^{(j)}\big)&=\frac{\sigma_j^2\big(\pi_{n+1}^{(j)}\big)^2x_i^2}{2\Delta x^2}-\frac{(1-\xi)\big[\pi_{n+1}^{(j)}(\mu_j-r_j)+r_j\big]x_i}{\Delta x},\\
\beta_i^{n+1}\big(\pi_{n+1}^{(j)}\big)&=\frac{\sigma_j^2\big(\pi_{n+1}^{(j)}\big)^2x_i^2}{2\Delta x^2}+\frac{\xi\big[\pi_{n+1}^{(j)}(\mu_j-r_j)+r_j\big]x_i}{\Delta x},\quad \xi\in\{0,1\}.
\end{aligned}
$$

At each node, in order to ensure $\alpha_i^{n+1}(\pi_{n+1}^{(j)})$ and $\beta_i^{n+1}(\pi_{n+1}^{(j)})$ are positive, we need a reasonable choice ξ. Specifically, if $\pi_{n+1}^{(j)}(\mu_j-r_j)x_i/\Delta x\geq 0$, we choose $\xi=1$, and if $\pi_{n+1}^{(j)}(\mu_j-r_j)x_i/\Delta x<0$, $\xi=0$.

For ease of analysis, we can also write Eqs. (4.1) into the matrix form. Let

$$
\mathbf{V}^{\varepsilon,n+1}=\left[V_0^{\varepsilon,n+1}(1),\ldots,V_M^{\varepsilon,n+1}(1),\ldots,V_0^{\varepsilon,n+1}(d),\ldots,V_M^{\varepsilon,n+1}(d)\right]'.
$$

Define matrix operator $A(\pi_{n+1})$ by

$$\begin{aligned}
&\big[A(\pi_{n+1})\mathbf{V}^{\varepsilon,n+1}\big]_{i+1+(j-1)(M+1)}\\
=&\bigg[\Big(-\alpha_i^{n+1}(\pi_{n+1}^{(j)})-\beta_i^{n+1}(\pi_{n+1}^{(j)})-\beta-q_j\Big)V_i^{\varepsilon,n+1}(j)\\
&\quad+\alpha_i^{n+1}(\pi_{n+1}^{(j)})V_{i-1}^{\varepsilon,n+1}(j)+\beta_i^{n+1}(\pi_{n+1}^{(j)})V_{i+1}^{\varepsilon,n+1}(j)\bigg]\\
&\quad+\sum_{\ell=1,\ell\neq j}^{d}q_{j\ell}V_i^{\varepsilon,n+1}(\ell),\quad i=1,\dots,M-1,\quad j=1,\dots,d.
\end{aligned}$$

Then (4.1) can be written as

$$\big[\mathbf{I}-\Delta\varsigma A(\pi_{n+1})\big]\mathbf{V}^{\varepsilon,n+1}=\mathbf{V}^{\varepsilon,n}+\boldsymbol{\phi}^{n+1}-\boldsymbol{\phi}^{n}+\varepsilon\Delta\varsigma\mathcal{M}\mathbf{V}^{\varepsilon,n+1}, \tag{4.3}$$

where

$$\begin{aligned}
\mathcal{M}\mathbf{V}^{\varepsilon,n+1}=&\Big[0,\Big(U(x_1-K)-V_1^{\varepsilon,n+1}(1)\Big)^+,\dots,\\
&\Big(U(x_{M-1}-K)-V_{M-1}^{\varepsilon,n+1}(1)\Big)^+,0,\dots,\\
&0,\Big(U(x_1-K)-V_1^{\varepsilon,n+1}(d)\Big)^+,\dots,\\
&\Big(U(x_{M-1}-K)-V_{M-1}^{\varepsilon,n+1}(d)\Big)^+,0,\Big]',\\
\boldsymbol{\phi}^{n+1}(j)=&\big[0,\dots,0,\phi_M^{n+1}(1),\dots,0,\dots,0,\phi_M^{n+1}(d)\big]',\\
\phi_M^{n+1}(j):=&\,\phi(\varsigma_{n+1},x_M,j).
\end{aligned}$$

It is known from [5, 6, 17] that the stability, consistency and monotonicity of the discretization can ensure the convergence to the viscosity solution. So we will analyze the stability, consistency and monotonicity of (4.2) or equivalent form (4.3). The next proposition presents the monotonicity of the scheme (4.2) or (4.3), which plays an important role in the stability and convergence analysis of the discrete equation.

To proceed the analysis, it is convenient to denote (4.2) as

$$\begin{aligned}
G_j\bigg(&V_i^{\varepsilon,n+1}(j),V_{i-1}^{\varepsilon,n+1}(j),V_{i+1}^{\varepsilon,n+1}(j),V_i^{\varepsilon,n}(j),\\
&-\sum_{\ell=1,\ell\neq j}^{d}q_{j\ell}V_i^{\varepsilon,n+1}(\ell),\Big(U(x_i-K)-V_i^{\varepsilon,n+1}(j)\Big)^+\bigg)=0,
\end{aligned}\tag{4.4}$$

where G_j is defined by the left-hand side minus the right-hand side of (4.2).

Lemma 4.1 (Monotonicity of the FDMs). *If boundary function ϕ in* (3.4) *is bounded, then the implicit FDMs* (4.1) *are monotone in the sense that for any $\rho_1,\rho_2,\rho_3,\rho_4\geq 0$, it* *holds true that*

$$
\begin{aligned}
&G_j\Big(V_i^{\varepsilon,n+1}(j), V_{i-1}^{\varepsilon,n+1}(j)+\rho_1, V_{i+1}^{\varepsilon,n+1}(j)+\rho_2, V_i^{\varepsilon,n}(j)+\rho_3,\\
&\qquad -\sum_{\ell=1,\ell\neq j}^{d} q_{j\ell}(V_i^{\varepsilon,n+1}(\ell)+\rho_4), \Big(U(x_i-K)-V_i^{\varepsilon,n+1}(j)\Big)^+\Big)\\
&\le G_j\Big(V_i^{\varepsilon,n+1}(j), V_{i-1}^{\varepsilon,n+1}(j), V_{i+1}^{\varepsilon,n+1}(j), V_i^{\varepsilon,n}(j),\\
&\qquad -\sum_{\ell=1,\ell\neq j}^{d} q_{j\ell}V_i^{\varepsilon,n+1}(\ell), \Big(U(x_i-K)-V_i^{\varepsilon,n+1}(j)\Big)^+\Big).
\end{aligned}
$$

Proof. The operator G_j defined in (4.4) has one more penalty term than the operator defined in [32], and the penalty term has no effect on the proof of monotonicity. Therefore, the proof of monotonicity is referred to [32]. □

The next lemma presents the stability of scheme (4.2) or (4.3).

Lemma 4.2 (Stability of the FDMs). *If boundary function ϕ in* (3.4) *is bounded, then the fully implicit FDMs* (4.2) *are stable,*

$$
\|\mathbf{V}^{\varepsilon,n+1}\|_\infty \le \max\left\{\frac{1}{1+\Delta\varsigma\beta}\|\mathbf{V}^{\varepsilon,0}\|_\infty, C_1, C_2\right\},
$$

where $C_1 := \max_i|U(x_i-K)|$, $C_2 := \max_{j,n}|\phi_M^{n+1}(j)|$.

Proof. For time step $n<N$, let $\bar{i}$ and $\bar{j}$ be indices such that $V_{\bar{i}}^{\varepsilon,n}(\bar{j}) = \min_{i,j}V_i^{\varepsilon,n}(j)$. Furthermore since $\alpha_{\bar{i}}^{n+1}$ and $\beta_{\bar{i}}^{n+1}$ are positive, we have that

$$
\begin{aligned}
&\Big(-\alpha_{\bar{i}}^{n+1}(\pi_{n+1}^{(\bar{j})}) - \beta_{\bar{i}}^{n+1}(\pi_{n+1}^{(\bar{j})})\Big) V_{\bar{i}}^{\varepsilon,n+1}(\bar{j}) + \alpha_{\bar{i}}^{n+1}(\pi_{n+1}^{(\bar{j})})V_{\bar{i}-1}^{\varepsilon,n+1}(\bar{j})\\
&\quad + \beta_{\bar{i}}^{n+1}(\pi_{n+1}^{(\bar{j})})V_{\bar{i}+1}^{n+1}(\bar{j}) \ge 0.
\end{aligned}
$$

Therefore, it follows from (4.2) that

$$
\begin{aligned}
0 =\ & \frac{V_{\bar{i}}^{\varepsilon,n+1}(\bar{j}) - V_{\bar{i}}^{\varepsilon,n}(\bar{j})}{\Delta\varsigma} - \Big[\Big(-\alpha_{\bar{i}}^{n+1}(\pi_{n+1}^{(\bar{j})}) - \beta_{\bar{i}}^{n+1}(\pi_{n+1}^{(\bar{j})}) - \beta - q_{\bar{j}}\Big) V_{\bar{i}}^{\varepsilon,n+1}(\bar{j})\\
&\qquad + \alpha_{\bar{i}}^{n+1}(\pi_{n+1}^{(\bar{j})})V_{\bar{i}-1}^{\varepsilon,n+1}(\bar{j}) + \beta_{\bar{i}}^{n+1}(\pi_{n+1}^{(\bar{j})})V_{\bar{i}+1}^{\varepsilon,n+1}(\bar{j})\Big]\\
& - \sum_{\ell=1,\ell\neq\bar{j}}^{d} q_{\bar{j}\ell}V_{\bar{i}}^{\varepsilon,n+1}(\ell) - \varepsilon\Big(U(x_{\bar{i}}-K) - V_{\bar{i}}^{\varepsilon,n+1}(\bar{j})\Big)^+\\
\le\ & \frac{V_{\bar{i}}^{\varepsilon,n+1}(\bar{j}) - V_{\bar{i}}^{\varepsilon,n}(\bar{j})}{\Delta\varsigma} + \beta V_{\bar{i}}^{\varepsilon,n+1}(\bar{j}) + q_{\bar{j}}V_{\bar{i}}^{\varepsilon,n+1}(\bar{j}) - V_{\bar{i}}^{\varepsilon,n+1}(\bar{j})\sum_{\ell=1,\ell\neq\bar{j}}^{d} q_{\bar{j}\ell}
\end{aligned}
$$

$$= \frac{V_{\bar{i}}^{\varepsilon,n+1}(\bar{j}) - V_{\bar{i}}^{\varepsilon,n}(\bar{j})}{\Delta\varsigma} + \beta V_{\bar{i}}^{\varepsilon,n+1}(\bar{j}).$$

It then gives that

$$V_{\bar{i}}^{\varepsilon,n}(\bar{j}) \le (1+\Delta\varsigma\beta)V_{\bar{i}}^{\varepsilon,n+1}(\bar{j}). \tag{4.5}$$

Now, let $\bar{\bar{i}}$ and $\bar{\bar{j}}$ be indices such that $V_{\bar{\bar{i}}}^{\varepsilon,n}(\bar{\bar{j}}) = \max_{i,j} V_i^{\varepsilon,n}(j)$, this case corresponds to the continuation region. Therefore, $V_{\bar{\bar{i}}}^{\varepsilon,n}(\bar{\bar{j}}) \ge U(x_{\bar{\bar{i}}} - K)$, namely,

$$\Big(U(x_{\bar{\bar{i}}} - K) - V_{\bar{\bar{i}}}^{\varepsilon,n}(\bar{\bar{j}})\Big)^+ = 0.$$

Furthermore, since $\alpha_{\bar{\bar{i}}}^{n+1}$ and $\beta_{\bar{\bar{i}}}^{n+1}$ are positive, we have

$$\begin{aligned}&\Big(-\alpha_{\bar{\bar{i}}}^{n+1}(\pi_{n+1}^{(\bar{\bar{j}})}) - \beta_{\bar{\bar{i}}}^{n+1}(\pi_{n+1}^{(\bar{\bar{j}})})\Big) V_{\bar{\bar{i}}}^{\varepsilon,n+1}(\bar{\bar{j}}) + \alpha_{\bar{\bar{i}}}^{n+1}(\pi_{n+1}^{(\bar{\bar{j}})}) V_{\bar{\bar{i}}-1}^{\varepsilon,n+1}(\bar{\bar{j}}) \\ &\quad + \beta_{\bar{\bar{i}}}^{n+1}(\pi_{n+1}^{(\bar{\bar{j}})}) V_{\bar{\bar{i}}+1}^{\varepsilon,n+1}(\bar{\bar{j}}) \le 0.\end{aligned}$$

Therefore, it follows from (4.2) that

$$\begin{aligned}0 &= \frac{V_{\bar{\bar{i}}}^{\varepsilon,n+1}(\bar{\bar{j}}) - V_{\bar{\bar{i}}}^{\varepsilon,n}(\bar{\bar{j}})}{\Delta\varsigma} - \Big[\Big(-\alpha_{\bar{\bar{i}}}^{n+1}(\pi_{n+1}^{(\bar{\bar{j}})}) - \beta_{\bar{\bar{i}}}^{n+1}(\pi_{n+1}^{(\bar{\bar{j}})}) - \beta - q_{\bar{\bar{j}}}\Big) V_{\bar{\bar{i}}}^{\varepsilon,n+1}(\bar{\bar{j}}) \\ &\qquad\qquad + \alpha_{\bar{\bar{i}}}^{n+1}(\pi_{n+1}^{(\bar{\bar{j}})}) V_{\bar{\bar{i}}-1}^{\varepsilon,n+1}(\bar{\bar{j}}) + \beta_{\bar{\bar{i}}}^{n+1}(\pi_{n+1}^{(\bar{\bar{j}})}) V_{\bar{\bar{i}}+1}^{\varepsilon,n+1}(\bar{\bar{j}})\Big] \\ &\quad - \sum_{\ell=1,\ell\ne\bar{\bar{j}}}^{d} q_{\bar{\bar{j}}\ell} V_{\bar{\bar{i}}}^{\varepsilon,n+1}(\ell) - \varepsilon\Big(U(x_{\bar{\bar{i}}} - K) - V_{\bar{\bar{i}}}^{\varepsilon,n+1}(\bar{\bar{j}})\Big)^+ \\ &\ge \frac{V_{\bar{\bar{i}}}^{\varepsilon,n+1}(\bar{\bar{j}}) - V_{\bar{\bar{i}}}^{\varepsilon,n}(\bar{\bar{j}})}{\Delta\varsigma} + \beta V_{\bar{\bar{i}}}^{\varepsilon,n+1}(\bar{\bar{j}}) + q_{\bar{\bar{j}}} V_{\bar{\bar{i}}}^{\varepsilon,n+1}(\bar{\bar{j}}) - V_{\bar{\bar{i}}}^{\varepsilon,n+1}(\bar{\bar{j}}) \sum_{\ell=1,\ell\ne\bar{\bar{j}}}^{d} q_{\bar{\bar{j}}\ell} \\ &= \frac{V_{\bar{\bar{i}}}^{\varepsilon,n+1}(\bar{\bar{j}}) - V_{\bar{\bar{i}}}^{\varepsilon,n}(\bar{\bar{j}})}{\Delta\varsigma} + \beta V_{\bar{\bar{i}}}^{\varepsilon,n+1}(\bar{\bar{j}}).\end{aligned}$$

It then gives that

$$(1+\Delta\varsigma\beta)V_{\bar{\bar{i}}}^{\varepsilon,n+1}(\bar{\bar{j}}) \le V_{\bar{\bar{i}}}^{\varepsilon,n}(\bar{\bar{j}}). \tag{4.6}$$

Combining (4.5) with (4.6) reaches

$$\begin{aligned}&\min_{i,j} V_i^{\varepsilon,n}(j) = V_{\bar{i}}^{\varepsilon,n}(\bar{j}) \\ &\le (1+\Delta\varsigma\beta)V_{\bar{i}}^{\varepsilon,n+1}(\bar{j}) \le (1+\Delta\varsigma\beta)V_{\bar{\bar{i}}}^{\varepsilon,n+1}(\bar{\bar{j}}) \\ &\le V_{\bar{\bar{i}}}^{\varepsilon,n}(\bar{\bar{j}}) = \max_{i,j} V_i^{\varepsilon,n}(j) \quad \text{for} \quad j \in \mathbb{D}.\end{aligned}$$

This leads to

$$\|\mathbf{V}^{\varepsilon,n+1}\|_\infty \le \frac{1}{1+\Delta\varsigma\beta}\|\mathbf{V}^{\varepsilon,n}\|_\infty.$$

Since $V_i^{\varepsilon,0}(j) = V^\varepsilon(0,x_i,j) = U(x_i - K)$, we have $\|\mathbf{V}^0\|_\infty = \max_i|U(x_i-K)|$. Let

$$C_1 := \max_i |U(x_i - K)|, \quad C_2 := \max_{j,n} |\phi_M^{n+1}(j)|.$$

Then we obtain

$$\|\mathbf{V}^{\varepsilon,n+1}\|_\infty \le \max\left\{\frac{1}{1+\Delta\varsigma\beta}\|\mathbf{V}^{\varepsilon,n}\|_\infty, C_1, C_2\right\}. \tag{4.7}$$

Iterating (4.7) gives that

$$\|\mathbf{V}^{\varepsilon,n+1}\|_\infty \le \max\left\{\frac{1}{1+\Delta\varsigma\beta}\|\mathbf{V}^{\varepsilon,0}\|_\infty, C_1, C_2\right\}.$$

The proof is thus complete. □

Lemma 4.3 (Consistency of the FDMs). *The implicit FDMs* (4.1) *are consistent, i.e., for* $\varphi \in C^{1,2}([0,T]\times[K,x_{\max}])$*, it holds true that*

$$\begin{aligned}
&\liminf_{\substack{\widetilde{\varsigma}\to\varsigma\\ \widetilde{x}\to x\\ \rho\to 0\\ h\to 0}} G_j\Bigg(\varphi(\widetilde{\varsigma},\widetilde{x},j),\varphi(\widetilde{\varsigma},\widetilde{x}-h,j),\varphi(\widetilde{\varsigma},\widetilde{x}+h,j),\varphi(\widetilde{\varsigma}-\rho,\widetilde{x},j),\\
&\qquad\qquad -\sum_{\ell=1,\ell\neq j}^{d} q_{j\ell}\varphi(\widetilde{\varsigma},\widetilde{x},\ell),\Big(U(\widetilde{x}-K)-\varphi(\widetilde{\varsigma},\widetilde{x},j)\Big)^+\Bigg)\\
&\ge \underline{F}_j\Bigg(\varphi_{xx}(\varsigma,x,j),\varphi_x(\varsigma,x,j),\varphi_\varsigma(\varsigma,x,j),\varphi(\varsigma,x,j),-\sum_{\ell=1,\ell\neq j}^{d} q_{j\ell}\varphi(\varsigma,x,\ell),x,\varsigma\Bigg),
\end{aligned}$$

and

$$\begin{aligned}
&\limsup_{\substack{\widetilde{\varsigma}\to\varsigma\\ \widetilde{x}\to x\\ \rho\to 0\\ h\to 0}} G_j\Bigg(\varphi(\widetilde{\varsigma},\widetilde{x},j),\varphi(\widetilde{\varsigma},\widetilde{x}-h,j),\varphi(\widetilde{\varsigma},\widetilde{x}+h,j),\varphi(\widetilde{\varsigma}-\rho,\widetilde{x},j),\\
&\qquad\qquad -\sum_{\ell=1,\ell\neq j}^{d} q_{j\ell}\varphi(\widetilde{\varsigma},\widetilde{x},\ell),\Big(U(\widetilde{x}-K)-\varphi(\widetilde{\varsigma},\widetilde{x},j)\Big)^+\Bigg)\\
&\le \overline{F}_j\Bigg(\varphi_{xx}(\varsigma,x,j),\varphi_x(\varsigma,x,j),\varphi_\varsigma(\tau,x,j),\varphi(\varsigma,x,j),-\sum_{\ell=1,\ell\neq j}^{d} q_{j\ell}\varphi(\varsigma,x,\ell),x,\varsigma\Bigg).
\end{aligned}$$

Proof. The operator G_j defined in (4.4) has one more penalty term than the operator defined in [32], and the penalty term does not cause any difficulty when we use the Taylor series expansion. Therefore, the proof of consistency is similar to [32]. □

From Lemmas 4.1-4.3, we know that (4.1) or (4.2) is a consistent, stable, monotone discretization. In Barles and Souganidis [6], Barles [5], Huang *et al.* [26,27], Reisinger and Forsyth [36], Pooley *et al.* [35], Forsyth and Labahn [17], they all mention that a consistent, stable, monotone discretization converges to the viscosity solution.

Theorem 4.1 (Convergence of the FDMs)**.** *Assumed that the original HJB equation* (3.1) *satisfies the conditions for Lemma* 3.1 *and discretization* (4.1) *satisfies all the conditions for Lemmas* 4.1-4.3. *Let* $V^{\varepsilon,h,\rho}$ *denote the continuous form of* (4.1) *with* $h = \Delta x$ *and* $\rho = \Delta\varsigma$. *Then* $V^{\varepsilon,h,\rho}$ *converges to the unique viscosity solution* V^{ε} *of the nonlinear PDE* (3.1) *for every* ε*, when* $\rho \to 0$ *and* $h \to 0$*.*

Proof. The proof follows from Lemmas 4.1-4.3, 3.1 and [32]. □

To implement the iterative FDM scheme (4.2), we need the following algorithm of iteration policy.

Algorithm 4.1 Iteration Policy for Solving (4.2) or (4.3)

for $n = 0, 1, 2, \dots, N$ **do**
  set $(\mathbf{V}^{\varepsilon,n+1})^0 = \mathbf{V}^{\varepsilon,n}$,
  set $(\hat{\mathbf{V}})^0 = (\mathbf{V}^{\varepsilon,n+1})^0$,
  for $k = 0, 1, 2, \dots$ **do**
    solve
    $\left[\mathbf{I} - \Delta\varsigma A\left((\pi_{n+1})^k\right)\right](\hat{\mathbf{V}})^{k+1} = \mathbf{V}^{\varepsilon,n} + \Delta\varsigma\varepsilon\mathcal{M}(\mathbf{V}^{\varepsilon,n}) + \boldsymbol{\phi}^{n+1} - \boldsymbol{\phi}^{n}$,
    where $(\pi_{n+1})^k \in \arg\sup_{\pi}\left[A(\pi)(\hat{\mathbf{V}})^k\right]$,
    if $\left[\max_{i,j}\left(\frac{|(\hat{V}_i(j))^{k+1} - (\hat{V}_i(j))^k|}{\max\left(1, |(\hat{V}_i(j))^{k+1}|\right)}\right) < \text{tolerance}\right]$ **then**
      Let $\mathbf{V}^{\varepsilon,n+1} = (\hat{\mathbf{V}})^{k+1}$, then quit
    else
      $(\hat{\mathbf{V}})^k = (\hat{\mathbf{V}})^{k+1}$,
      $k = k + 1$,
    end if
  end for
  $n = n + 1$,
end for

Theorem 4.2 (Convergence of the Algorithm of Iteration Policy)**.** *If boundary function* ϕ *in* (3.4) *is bounded, then the sequences* $(\hat{\mathbf{V}})^k$ *in Algorithm* 4.1 *converge monotonically to the unique solution of* (4.2) *or* (4.3) *for any initial iteration value* $(\hat{\mathbf{V}})^0$ *as* $k \to \infty$*.*

Proof. The proof immediately follows from [32]. □

5. Numerical examples

In the following examples, we solve the HJB variational inequalities (2.2) using the FDMs with policy iterations. Via the numerical tests, we verify the convergence of the approach and draw the exercise boundaries and optimal strategies with sample paths.

The boundary conditions (2.6) are constructed by

$$\begin{aligned} V(t, x_{\max}, j) &= \exp\big(-\beta(T-t)\big) E\left[\exp\left(\int_0^T < \mathbf{r}, \boldsymbol{\alpha}_t > dt\right)\right] U(x_{\max}) \\ &= \big\langle \exp\left[(\mathbf{Q} - \text{diag}(\mathbf{r}) - \beta)(T-t)\right] \cdot \mathbf{1},\, \mathbf{e_j} \big\rangle U(x_{\max}), \quad t \in [0, T], \end{aligned} \tag{5.1}$$

where the second equality is calculated by John *et al.* [10]. The construction of the boundary condition is motivated by that we allocate all of the wealth measured by the utility to the risk-free bond over $[t, T]$.

Example 5.1. We consider two-state Markov chain process (MCP) with generating matrix

$$\mathbf{Q}_1 = \begin{pmatrix} -0.005 & 0.005 \\ 0.005 & -0.005 \end{pmatrix} \quad \text{or} \quad \mathbf{Q}_2 = \begin{pmatrix} -0.5 & 0.5 \\ 0.5 & -0.5 \end{pmatrix},$$

which represent a small enough probability parameter of regimes switching and a large probability parameter, respectively. The riskless interest rates, return, volatility rates of risky asset and the discount rate are given by for different regimes

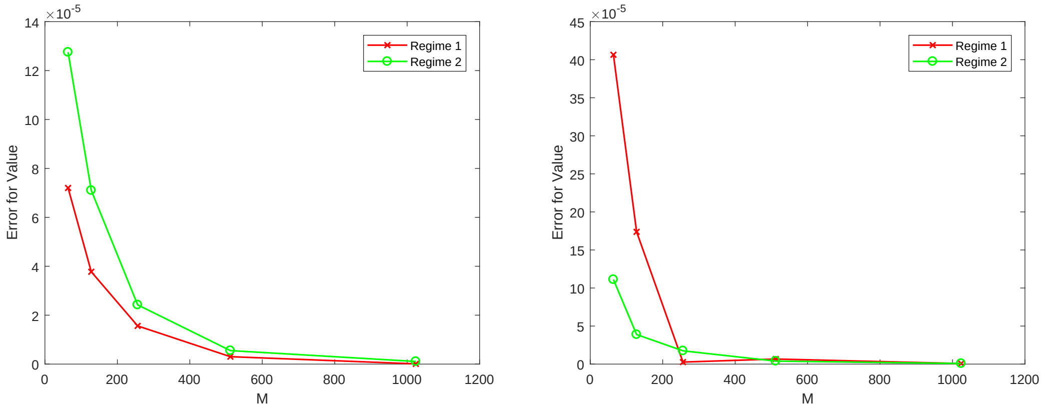
$$\mathbf{r} = (0.05, 0.06), \quad \boldsymbol{\mu} = (0.1, 0.12), \quad \boldsymbol{\sigma} = (0.30, 0.35), \quad \beta = 0.1.$$

The power utility function is $U(x) = 2x^{1/2}$. The initial wealth at time $t = 0$ is $x = 1.5$, the investment period $T = 1$ and the minimum wealth threshold $K = 1$, the boundary $x_{\max} = 9$.

In Table 1, the benchmark values are taken as the results calculated by the FDMs with the large number of spatial meshes and temporal meshes $M = 2048$ and $N = 10M$, and the penalty parameters $\varepsilon = 100$. In the computation of FDMs, the number of spatial meshes and temporal meshes are taken as $M = 64, 128, 256, 512, 1024$, $N = 10M$, and the penalty parameters $\varepsilon = 100$. The numerics in Table 1 are drawn in Fig. 1

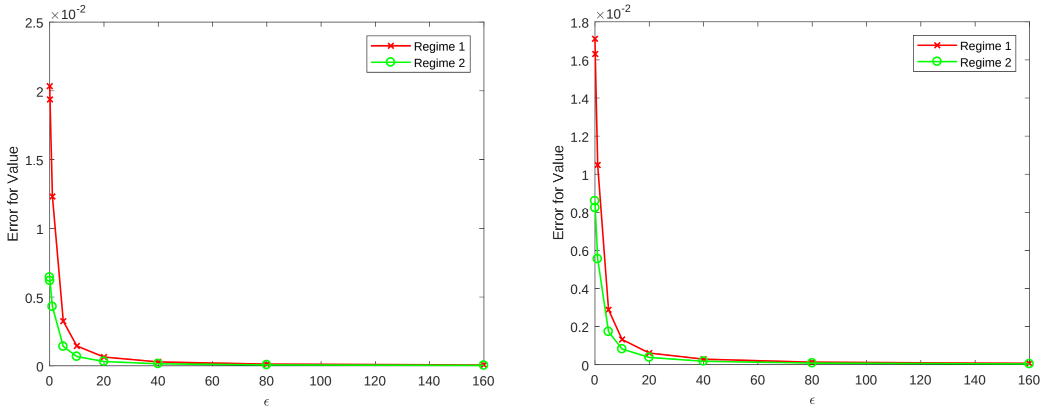
Table 1: Errors of the FDMs for Example 5.1 (power utility).

		Regime 1		Regime 2	
Generating matrix	M	Error	Time	Error	Time
$\mathbf{Q}_1$	64	4.9343e-04	0.3s	9.4233e-04	0.3s
	128	1.4726e-04	17.8s	5.5312e-04	17.8s
	256	6.2498e-05	244.1s	1.3580e-04	244.1s
	512	1.5350e-05	2800.7s	3.1161e-05	2800.7s
	1024	3.1921e-06	2259.5s	2.1321e-06	2259.5s
$\mathbf{Q}_2$	64	4.0649e-04	0.3s	1.1114e-04	0.3s
	128	1.7308e-04	18.7s	3.8839e-05	18.7s
	256	2.6412e-06	265.9s	1.7421e-05	265.9s
	512	6.6609e-06	3143.8s	3.9315e-06	3143.8s
	1024	7.3099e-07	2316.0s	7.8392e-07	2316.0s



(a) For $\mathbf{Q}_1$ (b) For $\mathbf{Q}_2$

Figure 1: Convergence of the FDMs for Example 5.1 (power utility).



(a) For $\mathbf{Q}_1$ (b) For $\mathbf{Q}_2$

Figure 2: The convergence of the penalized HJB equations to the HJB variational inequalities with respect to ε in Example 5.1.

and show that the FDMs are convergent. The errors between the values of the penalized HJB equations and the HJB variational inequalities versus the penalty parameter ε for $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are drawn in Fig. 2 with $M = 400$, $N = 4000$. It can be seen from Fig. 2 that the value of the penalized HJB equations converges to that of the HJB variational inequalities as the penalty parameters ε tends to ∞ both in the case of small and large probability parameters of regimes switching.

Example 5.2. We use example in [31], which considers the non-HARA utility function

$$U(x) = \frac{1}{3}H(x)^{-3} + H(x)^{-1} + xH(x)$$

for $x > 0$, where $H(x) = \sqrt{2}(-1 + \sqrt{1 + 4x})^{-1/2}$ and three-state MCP with generating matrix

$$\mathbf{Q}_1 = \begin{pmatrix} -0.0075 & 0.005 & 0.0025 \\ 0.005 & -0.01 & 0.005 \\ 0.0025 & 0.005 & -0.0075 \end{pmatrix} \quad \text{or} \quad \mathbf{Q}_2 = \begin{pmatrix} -0.75 & 0.5 & 0.25 \\ 0.5 & -1 & 0.5 \\ 0.25 & 0.5 & -0.75 \end{pmatrix}.$$

For regimes 1, 2 and 3, (r, μ, σ) are taken as respectively $(0.05, 0.1, 0.3)$, $(0.06, 0.12, 0.35)$, $(0.07, 0.14, 0.4)$. The initial wealth at time $t = 0$ is $x = 1.5$, the investment period $T = 1$, the minimum wealth threshold $K = 1$, and $\beta = 0.1$, the boundary $x_{\max} = 9$.

In Table 2, the benchmark values are taken as the results calculated by the FDMs with the number of spatial and temporal meshes $M = 1024$ and $N = 10M$, and the penalty parameters $\varepsilon = 100$. The numerics in Table 2 are shown in Fig. 3 and verify that the approach is convergent. Furthermore, it can be seen from Fig. 4 that the value of the penalized HJB equations converges to that of the HJB variational inequalities as the penalty parameters ε tends to ∞.

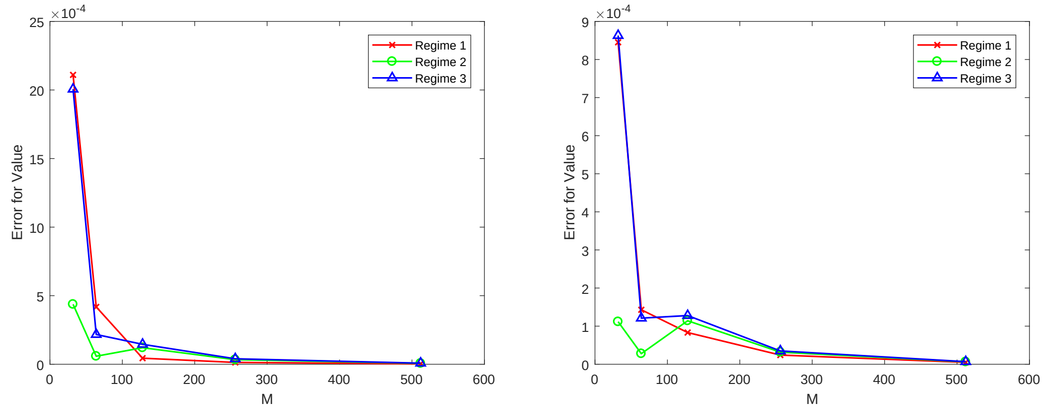
Randomly choosing the generating matrix of three-state MCP

$$\mathbf{Q} = \begin{pmatrix} -0.1500225 & 0.1000150 & 0.0500075 \\ 0.3499220 & -0.6998440 & 0.3499220 \\ 0.7678250 & 1.5356500 & -2.3034750 \end{pmatrix}$$

and taking the number of spatial meshes and temporal meshes $M = 1024$ and $N = 10M$, the penalty parameters $\varepsilon = 100$ and the initial value $x = 1.4$, we draw Fig. 5 to characterize the free boundaries, the continuation and exercise regions with different regime states and draw Fig. 6 for the optimal strategies with the sample paths of wealth. Here, with the strategies computed by Algorithm 4.1, we simulate the paths of wealth process by Euler-Maruyama scheme with the number of temporal meshes $N = 10240$. We only draw four typical sample paths including three paths that stop

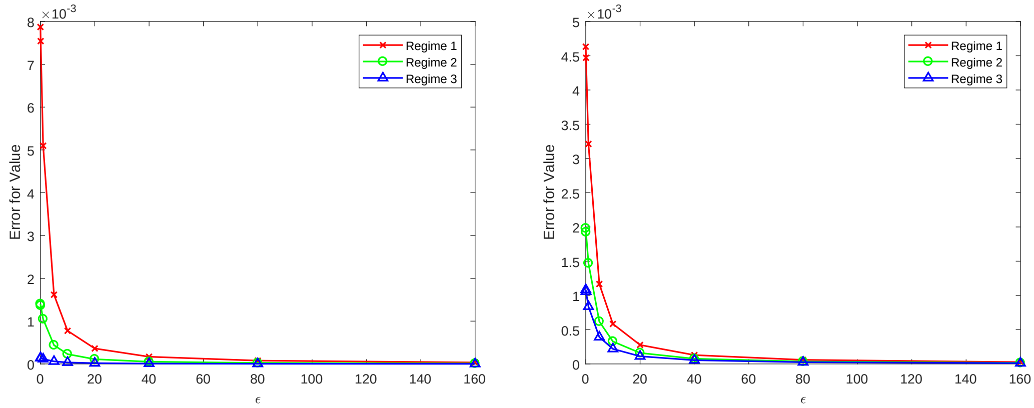
Table 2: Errors of the FDMs for Example 5.2 (non-HARA utility).

		Regime 1		Regime 2		Regime 3	
$\mathbf{Q}$	M	Error	Time	Error	Time	Error	Time
$\mathbf{Q}_1$	32	3.5613e-04	0.3s	7.8634e-04	0.3s	2.3161e-04	0.3s
	64	6.9383e-05	2.9s	9.2539e-05	2.9s	8.4594e-05	2.9s
	128	4.9818e-05	60.3s	1.8101e-05	60.3s	4.6756e-05	60.3s
	256	1.8072e-05	500.4s	8.8846e-06	500.4s	1.7992e-05	500.4s
	512	1.4625e-06	8426.8s	1.4685e-06	8426.8s	1.5045e-06	8426.8s
$\mathbf{Q}_2$	32	8.4514e-04	0.2s	1.1162e-04	0.2s	8.6291e-04	0.2s
	64	1.4299e-04	2.8s	2.7753e-05	2.8s	1.2100e-05	2.8s
	128	8.3203e-05	60.4s	1.1415e-04	60.4s	1.2780e-05	60.4s
	256	2.4051e-05	506.3s	3.1765e-05	506.3s	3.4924e-05	506.3s
	512	4.9017e-06	7866.4s	6.3160e-06	7866.4s	6.7710e-06	7866.4s



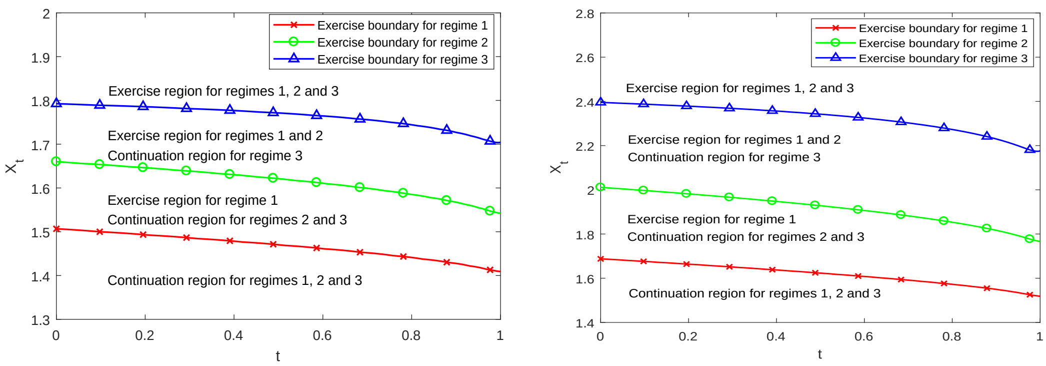
(a) For $\mathbf{Q}_1$ (b) For $\mathbf{Q}_2$

Figure 3: The convergence of the FDMs for Example 5.2.



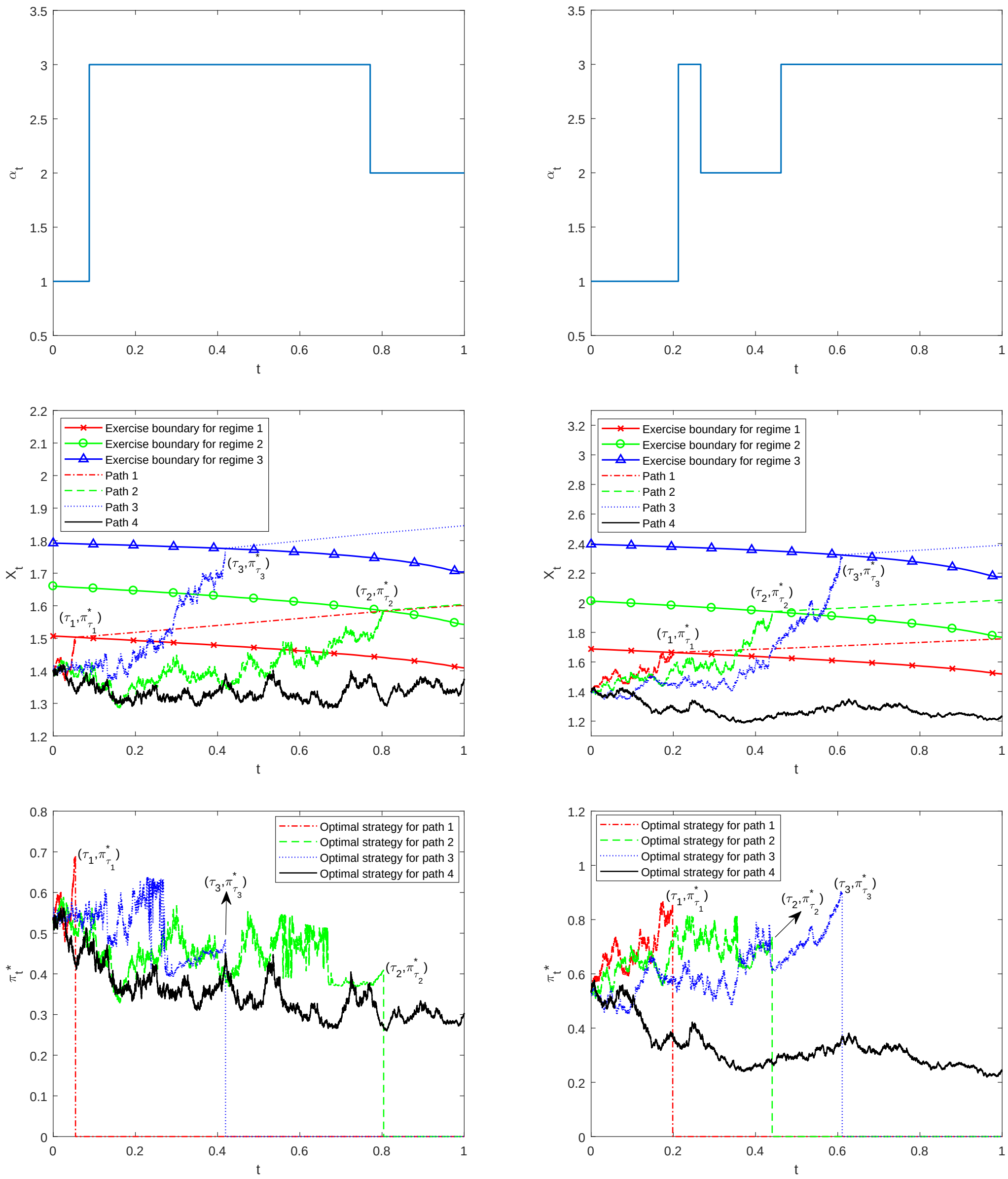
(a) For $\mathbf{Q}_1$ (b) For $\mathbf{Q}_2$

Figure 4: The convergence of the penalized HJB equations to the HJB variational inequalities with respect to ε in Example 5.2.



(a) For power utility (b) For non-HARA utility

Figure 5: The optimal exercise boundaries.



(a) For power utility (b) For non-HARA utility

Figure 6: Regime simulation, wealth sample paths and optimal strategies.

in three different regimes and one path that does not stop before the terminal time T. As observed in Fig. 6(a), if path 1, path 2 and path 3 hit the corresponding boundaries for regime state 1, regime state 2 and regime state 3, respectively, then the investment stops, namely, the corresponding optimal trading strategies become zero after hitting time. Meanwhile path 4 does not hit any boundaries during time horizon and thus the investment only terminates at time T. The explanation for Fig. 6(b) is similar.

6. Conclusions

In this paper we have studied the penalty methods and FDMs with iteration policy for the system of HJB quasi-variational inequalities. This problem arises in the continuous-time optimal investment with optimal stopping under regime switching models. The HJB quasi-variational inequalities are penalized into the HJB equations and then the FDMs with iteration policy are used to solve the penalized HJB equations. Both the convergence of the viscosity solution from the penalized HJB equations to the HJB quasi-variational inequalities and the convergence of the FDMs with iteration policy are proved. Numerical examples confirm the efficiency and reliability of the approach and the exercise boundaries and optimal strategies with sample paths are drawn.

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