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A Fast Compact Block-Centered Finite Difference Method on Graded Meshes for Time-Fractional Reaction-Diffusion Equations and Its Robust Analysis

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Abstract. In this article, an α-th $(0 < \alpha < 1)$ order time-fractional reaction-diffusion equation with variably diffusion coefficient and initial weak singularity is considered. Combined with the fast $L1$ time-stepping method on graded temporal meshes, we develop and analyze a fourth-order compact block-centered finite difference (BCFD) method. By utilizing the discrete complementary convolution kernels and the α-robust fractional Grönwall inequality, we rigorously prove the α-robust unconditional stability of the developed fourth-order compact BCFD method whether for positive or negative reaction terms. Optimal sharp error estimates for both the primal variable and its flux are simultaneously derived and carefully analyzed. Finally, numerical examples are given to validate the efficiency and accuracy of the developed method.

AMS subject classifications: 35R11, 65M06, 65M12

Key words: Time-fractional reaction-diffusion equation, compact BCFD method, fast $L1$ method, α-robust unconditional stability, error estimates.

1. Introduction

Fractional differential equations have been widely used to describe challenging phenomena with long range time memory and spatial interactions due to their non-local nature [13, 30, 32, 45, 53, 57], and have drawn increasing attentions over the past several decades. In particular, time-fractional partial differential equations are typically

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used to model anomalous diffusion phenomenon. However, due to the nonlocal nature of fractional integral or differential operators, the analytical solutions are usually not available for such equations, and thus numerical modeling have been an efficient approach for studying the fractional differential models. So far, the time-fractional differential equations have been widely studied [6, 19, 21, 27, 32, 33, 43, 50, 54].

In this paper, we are interested in the following time-fractional reaction-diffusion problem:

$$\begin{cases} {}_0^C\mathcal{D}_t^\alpha p(x,t) - \partial_x\big(a(x)\partial_x p(x,t)\big) + cp(x,t) = f(x,t), & (x,t) \in I \times (0,T_f], \\ p(x,0) = p^o(x), & x \in \bar{I} \end{cases} \tag{1.1}$$

under periodic boundary conditions, where $I := (x_l, x_r) \subset \mathbb{R}$ and $\bar{I} := I \cup \{x_l, x_r\}$. Moreover, the time-fractional derivative ${}_0^C\mathcal{D}_t^\alpha p$ in (1.1) is given in the Caputo sense [32]

$${}_0^C\mathcal{D}_t^\alpha p(x,t) := \int_0^t \omega_{1-\alpha}(t-s)\partial_s p(x,s)ds, \quad 0 < \alpha < 1,$$

where the kernel function $\omega_\beta(t) := t^{\beta-1}/\Gamma(\beta),\ t > 0$.

Throughout the paper, we suppose f and p^o are two given sufficiently smooth source and initial functions and the following assumptions hold [41, 42]:

Assumption 1.1. Problem (1.1) has a unique solution $p(x,t)$, and there is a positive constant C_0 independent of α such that

$$\|p(\cdot,t)\|_{H^6} \le C_0, \tag{1.2}$$

and

$$\|\partial_t p(\cdot,t)\|_{H^5} \le C_0(1 + t^{\alpha-1}), \quad \|\partial_{tt} p(\cdot,t)\|_{H^1} \le C_0(1 + t^{\alpha-2}). \tag{1.3}$$

Assumption 1.2. Suppose that $a(x) \in C^1(\bar{I})$ is a periodic function, and there exist positive constants $a_* \le a^*$ such that $a_* \le a(x) \le a^*$. Besides, c is a constant that maybe positive or negative.

As pointed above, various methods have been presented to solve the time-fractional reaction-diffusion equation (1.1), see also [35, 56]. However, the papers mentioned above only considered the case where c is non-negative, and most papers have ignored the possible presence of an initial layer in the typical solution near the initial time $t = 0$, and have presented convergence analysis under the unrealistic assumption $p(x,\cdot) \in C^2[0,T_f]$ or even high-order assumption, e.g. $C^3[0,T_f]$. It is pointed and proved in [38, 42] that the typical solution of the α-th order time-fractional diffusion equation has weak singularity at $t = 0$, e.g. $\partial_t p \sim t^{\alpha-1}$. Thus, the forementioned theoretical analysis based on the assumption that the solution is smooth enough is not appropriate. To compensate for the weak singularity, an efficient strategy is to employ the graded meshes [4, 20, 44, 49], that is concentrating more mesh points around the (weak) singular points to catch the rapid variation of the solution and use large stepsize while the solution changes slowly. Rencently, the graded mesh strategy is also utilized to solve the time-fractional models. For example, a time-stepping discontinuous Petrov-Galerkin method on graded meshes was proposed and analyzed for time-fractional subdiffusion equations in [31]. Stynes and O'Riordan [42] considered the $L1$ method for the time-fractional diffusion equation, and they strictly proved the maximum error of the numerical solution is of order $N_t^{-\min\{2-\alpha,r\alpha\}}$, where N_t is the total number of temporal grids. Huang *et al.* [15] employed Alikhanov's $L2$-1_σ scheme on the graded meshes for the time-fractional Allen-Cahn equation, and furthermore they derived corresponding sharp $L^\infty(H^1)$ error estimate. Both theoretical analysis and numerical experiments show that the usage of graded mesh can effectively recover the convergence order, e.g., the $L1$ scheme can be recovered to order $2-\alpha$. Worthy of a special mention is that Liao *et al.* [24–26,36] put forward a novel analysis framework for time-fractional models discretized on general temporal meshes, in which discrete complementary convolution (DCC) kernels technique combined with a novel developed discrete fractional Grönwall inequality and a new concept named global consistency analysis are used. The DCC kernels $\{P_{m-j}^{(m)}\}$ are defined via the discrete convolution (DC) kernels $\{d_{m-j}^{(m)}\}$ arsing from approximation of the Caputo derivative, i.e., $\sum_{k=1}^m d_{m-k}^{(m)}\nabla_\tau v^k \approx {}_0^C\mathcal{D}_t^\alpha v^m$, and they satisfy the relation $\sum_{j=k}^m P_{m-j}^{(m)} d_{j-k}^{(j)} \equiv 1$ and possess good properties that are helpful for numerical analysis. In this paper, we shall employ Liao's approach for analysis of the developed high-order finite difference approximation of model (1.1), where the reaction term can be either positive or negative.

It is well known that when $\alpha \to 1^-$, the α-th order Caputo derivative ${}_0^C\mathcal{D}_t^\alpha p$ in (1.1) will degenerate to the first-order derivative $\partial_t p$, and correspondingly model (1.1) is reduced to the classical reaction-diffusion model

$$\partial_t p - \partial_x\big(a(x)\partial_x p\big) + cp = f(x,t) \quad \text{in } \ I \times (0, T_f]. \tag{1.4}$$

Thus, it is somewhat reasonable to demand that the numerical analysis of any reliable numerical methods for solving (1.1) should produce error bounds that remain valid as $\alpha \to 1^-$. Unfortunately, as pointed out and analyzed in [5] that the error analysis, see, [42, Lemma 3.2] and [24, Lemma 3.3], contain a factor $1-\alpha$ in its denominator, so the error bounds will blow up as $\alpha \to 1^-$. Recently, Chen *et al.* [5] proposed a robust error bound that do not blow up as $\alpha \to 1^-$, and thus can involve the error estimate for model (1.4). Recently, this technique was also applied to time-fractional biharmonic equation [14] and time-fractional Allen-Cahn equation [15]. However, to the best of our knowledge no robust and sharp analysis are presented for high-order finite difference approximation of model (1.1) even in one space dimension.

Due to the nonlocality of the Caputo fractional derivative in the model (1.1), traditional discretization approaches [1, 32, 43] unavoidably lead to a large amount of storage and CPU time consumption. To reduce the computational cost, various fast algorithms have been developed to solve the time-fractional models. For example, an approximate inversion method [29] and a divide-and-conquer strategy [18] are proposed for calculating the block lower triangular Toeplitz-like with tri-diagonal blocks system, which arising from the discretization of the time-fractional partial differential equation. Fu *et al.* [9] proposed a reduced-order model based on the proper orthogonal decomposition and the discrete empirical interpolation method for efficiently simulating the time-fractional diffusion equations. In [8, 47, 48], efficient parareal algorithms are respectively presented to reduce the computational cost due to the historical effect of the fractional operator. Specifically, Zhang *et al.* [16] present a fast $L1$ method for the evaluation of the Caputo derivative based on an efficient sum-of-exponentials (SOE) approximation for the kernel $t^{-1-\alpha}$ on the interval $[\hat{\tau}, T_f]$ with a uniform absolute error ϵ. Recently, the SOE technology combined with other spatial discretization methods is also adopted for modeling of various time-fractional models [11,12,26,28,55]. However, robust error analysis about the fast numerical schemes are still lack.

In this paper, we are concerned with analysis and implementation of a fast high-order compact difference method for model (1.1). In fact, compact difference operators which use few mesh points that can still gain high-order spatial accuracy, have been focused on promoting algorithm accuracy for the modeling of fractional differential equations [7, 10, 46]. But, the theoretical analysis there do not consider the inherent weak singularity of the time-fractional model. In addition, in practical applications, people are concerned not only with the primal unknown function itself, but also with its gradient or flux. Block-centered finite difference (BCFD) method, sometimes called cell-centered finite difference method [2], which is also thought as the lowest-order Raviart-Thomas mixed element method [34] with proper quadrature formula, is viewed as an effective mean for simultaneously approximating the primal variable and its flux to a same order of accuracy without any accuracy lost. Besides, the BCFD method can guarantee the mass conservation and result in a symmetric positive definite system, compared with a saddle-point system generated by the classical mixed element method [34]. Therefore, the BCFD method is more efficient and widely used for modeling of flow model [37, 58], convection-diffusion model [52], and even time-fractional model [17, 22, 23, 51]. Recently, Shi *et al.* [40] proposed a compact BCFD method for the elliptic and parabolic problems, which further improves the spatial accuracy from second-order to fourth-order. However, the analysis in [40] cannot be directly applied to the time-fractional reaction-diffusion equation (1.1), and up to now, there are indeed no work on high-order BCFD method for model (1.1), and error analysis of most available second-order BCFD methods [22, 23, 51] are based upon smoothness assumption of the solution.

In this work, we shall propose a compact BCFD scheme combined with fast SOE-based $L1$ time-stepping formula for model (1.1), where graded mesh is employed to compensate for the possible temporal accuracy lost caused by the singularity of the solution at $t = 0$. By defining new weighted norms $\|\cdot\|_{*,M}$ and $\|\cdot\|_{*,T}$, which are equivalent to norms $\|\cdot\|_M$ and $\|\cdot\|_T$, see Lemma 3.2, a rigorous prior estimate is proved no matter the reaction is positive or not (see Theorem 3.1). The main contributions of this paper can be summarized as follows:

- By introducing an auxiliary flux variable, a SOE-based fast fourth-order compact BCFD method is developed for the time-fractional reaction-diffusion equation with variably diffusion coefficient, which significantly reduces the memory requirement and computational cost.

- By introducing DCC kernals (see Lemma 2.2) and using an α-robust fractional Grönwall inequality (see Lemma 2.3), we bound robustly the local truncation errors in discrete convolution form, see Lemmas 3.5 and 3.6.

- α-robust stability and sharp error estimates for both primal p and flux u are derived simultaneously. In particular, the analysis for the flux is skillful, in which the fractional order operator has to be applied to the error equation of u to obtain a new error equation, and then by choosing special test functions, an α-robust stability and optimal error estimates are obtained.

The outline of the paper is as follows. In Section 2, a fast fourth-order compact BCFD method is proposed for the time-fractional model (1.1), and some preliminary lemmas are given. In Section 3, an α-robust unconditional stability and sharp error estimates for both primal variable and its flux are rigorously proved and carefully discussed. In Section 4, some numerical examples are carried out to validate the theoretical analysis. Finally, conclusions are given in the last section. Throughout the paper, we use C with or without subscript to represent a general α-robust positive constant, which is independent of the mesh stepsize and can be different under different circumstances.

2. A fast $L1$-compact BCFD method

In this section, we aim to develop a fast fourth-order compact BCFD method for the model problem (1.1) with periodic boundary conditions.

Let $u(x,t) = -a(x)\partial_x p(x,t)$, then (1.1) can be transformed into the following equivalent form:

$$\begin{cases} {}_0^C\mathcal{D}_t^\alpha p(x,t) + \partial_x u(x,t) + cp(x,t) = f(x,t) & \text{in } I \times (0,T_f], \\ \partial_x p(x,t) + a^{-1}(x)u(x,t) = 0 & \text{in } I \times (0,T_f], \\ p(x,0) = p^o(x) & \text{in } \bar{I}. \end{cases} \tag{2.1}$$

Subsequently, we shall present numerical approximations for (2.1) based on fast $L1$ discretization on graded temporal meshes and compact BCFD discretization on uniform staggered spatial meshes.

2.1. Fast $L1$ temporal discretization

Let N_t be a positive integer and $r \geq 1$, a user-defined mesh grading parameter. We set $t_m := (m/N_t)^r T_f,\ m = 0,1,\dots,N_t$, and $\tau_m := t_m - t_{m-1}$, $\tau := \max_{1\leq m\leq N_t} \tau_m$. To develop a fast $L1$ formula, we first give the following SOE approximation.

Lemma 2.1 ([16]). *For a given $\alpha \in (0,1)$, an absolute tolerance error ϵ, a cut-off time restriction $\hat{\tau}$, and a final time T_f, there are one positive integer N_o, positive quadrature points $\{s_i \mid i = 1,2,\dots,N_o\}$ and corresponding positive weights $\{\omega_i \mid i = 1,2,\dots,N_o\}$ such that*

$$\left|\omega_{1-\alpha}(t) - \sum_{i=1}^{N_o} \omega_i e^{-s_i t}\right| \le \epsilon, \quad t \in [\hat{\tau}, T_f],$$

where the number of quadrature nodes satisfies

$$N_o = \mathcal{O}\left(\log\frac{1}{\epsilon}\left(\log\log\frac{1}{\epsilon} + \log\frac{T_f}{\hat{\tau}}\right) + \log\frac{1}{\hat{\tau}}\left(\log\log\frac{1}{\epsilon} + \log\frac{1}{\hat{\tau}}\right)\right).$$

Based on Lemma 2.1, the fast version $L1$ approximation of the Caputo fractional derivative on the graded temporal meshes is drawn as a combination of history part and local part [16, 26]

$$\begin{aligned} {}^F\delta_t^\alpha p^m(x) &= \sum_{k=1}^{m-1}\left[\frac{1}{\tau_k}\int_{t_{k-1}}^{t_k}\sum_{i=1}^{N_o} w_i e^{-s_i(t_m-s)}ds\right]\nabla_\tau p^k(x) \\ &\quad + \left[\frac{1}{\tau_m}\int_{t_{m-1}}^{t_m}\omega_{1-\alpha}(t_m-s)ds\right]\nabla_\tau p^m(x) \\ &=: \sum_{k=1}^{m} d_{m-k}^{(m)}\nabla_\tau p^k(x), \end{aligned} \tag{2.2}$$

where the difference operator $\nabla_\tau p^k(x) := p^k(x) - p^{k-1}(x)$ and the DC kernels $\{d_{m-k}^{(m)}\}$ are defined on graded temporal meshes as follows:

$$d_{m-k}^{(m)} = \begin{cases} \dfrac{1}{\tau_k}\displaystyle\int_{t_{k-1}}^{t_k}\sum_{i=1}^{N_o}\omega_i e^{-s_i(t_m-s)}ds, & k = 1,2,\dots,m-1, \\ \dfrac{1}{\tau_m}\displaystyle\int_{t_{m-1}}^{t_m}\omega_{1-\alpha}(t_m-s)ds, & k = m. \end{cases} \tag{2.3}$$

Remark 2.1. Note that the formula (2.2) is only used for the subsequent numerical analysis. In practical computation, we denote the history part in (2.2) by

$$\mathcal{S}_i^{m-1} := \sum_{k=1}^{m-1}\left[\frac{1}{\tau_k}\int_{t_{k-1}}^{t_k} e^{-s_i(t_m-s)}ds\right]\nabla_\tau p^k.$$

Then, the fast version $L1$ formula can be computed via

$${}^F\delta_t^\alpha p^m = \sum_{i=1}^{N_o} w_i\mathcal{S}_i^{m-1} + d_0^{(m)}\nabla_\tau p^m, \tag{2.4}$$

where a direct calculus shows that $\mathcal{S}_i^0 = 0$ and $\mathcal{S}_i^{m-1}$ $(2 \le m \le N_t)$ satisfies the following recurrence relation:

$$\begin{aligned}\mathcal{S}_i^{m-1} &= \sum_{k=1}^{m-2}\left[\frac{1}{\tau_k}\int_{t_{k-1}}^{t_k} e^{-s_i(t_m-s)}ds\right]\nabla_\tau p^k + \left[\frac{1}{\tau_{m-1}}\int_{t_{m-2}}^{t_{m-1}} e^{-s_i(t_m-s)}ds\right]\nabla_\tau p^{m-1}\\ &= e^{-s_i\tau_m}\mathcal{S}_i^{m-2} + \frac{e^{-s_i\tau_m} - e^{-s_i(t_m-t_{m-2})}}{\tau_{m-1}s_i}\nabla_\tau p^{m-1}.\end{aligned} \tag{2.5}$$

Thus, compared with the classical $L1$ formula, the fast version $L1$ formula (2.4)-(2.5) has reduced the memory requirement from $\mathcal{O}(N_t)$ to $\mathcal{O}(N_o)$ and computational cost from $\mathcal{O}(N_t^2)$ to $\mathcal{O}(N_tN_o)$.

If the tolerance error ϵ of the SOE approximation satisfies $\epsilon \le \min\{\omega_{1-\alpha}(T_f)/3, \alpha\omega_{2-\alpha}(T_f)\}$, then the DC kernels $\{d_{m-k}^{(m)}\}$ defined by (2.3) are positive and satisfy [26]

$$d_0^{(m)} > d_1^{(m)} > \cdots > d_{m-1}^{(m)} > 0. \tag{2.6}$$

Moreover, with these DC kernels one can define a family of DCC kernels (cf. [25]) such that

$$\sum_{j=k}^{m} P_{m-j}^{(m)} d_{j-k}^{(j)} = 1, \quad 1 \le k \le m \le N_t. \tag{2.7}$$

The DCC kernels defined in (2.7) shall play an important role in the stability and convergence analysis of the presented numerical methods. We state the following key lemma.

Lemma 2.2 ([26]). *The DCC kernels $\{P_{m-j}^{(m)}\}$ are well defined with*

$$P_{m-k}^{(m)} = \frac{1}{d_0^{(k)}}\begin{cases}1, & k=m,\\ \sum\limits_{j=k+1}^{m}\left(d_{j-k-1}^{(j)} - d_{j-k}^{(j)}\right)P_{m-j}^{(m)}, & 1 \le k \le m-1,\end{cases}$$

and

$$0 < P_{m-k}^{(m)} \le \Gamma(2-\alpha)\tau_k^\alpha, \quad 1 \le k \le m \le N_t.$$

Furthermore, if $\epsilon \le \min\{\omega_{1-\alpha}(T_f)/3, \alpha\omega_{2-\alpha}(T_f)\}$, the following estimates hold for $m = 1, 2, \ldots, N_t$:

$$\sum_{k=1}^{m} P_{m-k}^{(m)} \le \frac{3t_m^\alpha}{2\Gamma(1+\alpha)}, \quad \frac{2\mu}{3}\sum_{k=1}^{m-1} P_{m-k}^{(m)} E_\alpha(\mu t_k^\alpha) \le E_\alpha(\mu t_m^\alpha) - 1,$$

where $\mu > 0$ is a constant, and $E_\alpha(z) := \sum_{k=0}^{\infty} z^k/\Gamma(1+k\alpha)$ denotes the single-parameter Mittag-Leffler function.

An α-robust discrete fractional Grönwall inequality based on the fast $L1$ formula is listed in the following lemma.

Lemma 2.3. *Let $\{\lambda_s\}$ be nonnegative constants with $0 < \sum_{s=0}^{m-1} \lambda_s \le \lambda$ for $m \ge 1$, where λ is some positive constant independent of m. Suppose $\epsilon \le \min\{\omega_{1-\alpha}(T_f)/3, \alpha\omega_{2-\alpha}(T_f)\}$ and the nonnegative grid functions $\{v^m \,|\, m \ge 0\}$ satisfy*

$$^F\delta_t^\alpha (v^m)^2 = \sum_{k=1}^{m} d_{m-k}^{(m)} \nabla_\tau (v^k)^2 \le \sum_{l=1}^{m} \lambda_{m-l} \, (v^l)^2 + v^m \xi^m + (\eta^m)^2, \quad m \ge 1,$$

where $\{\xi^m, \eta^m \mid m \ge 1\}$ are bounded nonnegative sequences. If the maximum time stepsize fulfills $\tau \le 1/\sqrt[\alpha]{3\Gamma(2-\alpha)\lambda}$, then

$$v^m \le 2E_\alpha(3\lambda t_m^\alpha) \left[v^0 + \max_{1\le k\le m} \sum_{j=1}^{k} P_{k-j}^{(k)} (\xi^j + \eta^j) + \max_{1\le k\le m} \eta^k \right], \quad 1 \le m \le N_t.$$

Proof. Note that the DC kernels $\{d_{m-k}^{(m)}\}$ defined in (2.3) are positive and monotone under condition $\epsilon \le \min\{\omega_{1-\alpha}(T_f)/3, \alpha\omega_{2-\alpha}(T_f)\}$, please refer to (2.6). Following the same line of proof of [15, Lemma 4.1], this conclusion can be derived immediately. □

Remark 2.2. If the given $\{\lambda_l\}_{l=0}^{N_t-1}$ are non-positive, it is shown in [14, Lemma 4.2] that the conclusion of Lemma 2.3 holds in a much simpler form

$$v^m \le v^0 + \sum_{j=1}^{m} P_{m-j}^{(m)} (\xi^j + \eta^j) + \max_{1\le k\le m} \eta^k, \quad 1 \le m \le N_t$$

without any restrictions on the time stepsize.

Now, a temporal semi-discrete approximation of model (2.1) is proposed as

$$^F\delta_t^\alpha p^m + u_x^m + cp^m = f^m + R_t^m[p], \quad p_x^m + a^{-1}u^m = 0 \quad \text{in } I, \tag{2.8}$$

where $R_t^m[p](x) := {}^F\delta_t^\alpha p^m(x) - {}_0^C\mathcal{D}_t^\alpha p^m(x)$ denotes the local truncation error of the fast $L1$ formula (2.2), and a robust (i.e., when $\alpha \to 1^-$, the estimate shall not blow up) global consistency error is stated below.

Lemma 2.4. *If $p(x,t)$ satisfies the condition (1.3) in Assumption 1.1, and moreover, $\epsilon \le \min\{\omega_{1-\alpha}(T_f)/3, \alpha\omega_{2-\alpha}(T_f)\}$ and $r \le 2(2-\alpha)/\alpha$, $N_t \ge 8$, it holds that*

$$\begin{aligned} \sum_{k=1}^{m} P_{m-k}^{(m)} |R_t^k[p]| \le C \Bigg(& \frac{e^r \Gamma(1+\gamma-\alpha)}{\Gamma(1+\gamma)} \left(1 + \frac{3\epsilon}{2\Gamma(1+\alpha)}\right) (T_f^\alpha + T_f^{2\alpha}) \\ & \times \left(\frac{t_m}{T_f}\right)^\gamma N_t^{-\min\{r\alpha, 2-\alpha\}} + \epsilon \frac{3t_m^\alpha (t_{m-1} + t_{m-1}^\alpha/\alpha)}{2\Gamma(1+\alpha)} \Bigg) \end{aligned} \tag{2.9}$$

for $1 \le m \le N_t$, where $\gamma = 1/\ln N_t + \alpha - \min\{\alpha, (2-\alpha)/r\}$.

Proof. For simplicity, below we denote $R_t^k[p]$ as R_t^k whenever no confusion caused. By triangle inequality, we get

$$\sum_{k=1}^{m} P_{m-k}^{(m)} |R_t^k| \le \sum_{k=1}^{m} P_{m-k}^{(m)} |\delta_t^\alpha p^k - {}_0^C\mathcal{D}_t^\alpha p^k| + \sum_{k=2}^{m} P_{m-k}^{(m)} |\delta_t^\alpha p^k - {}^F\delta_t^\alpha p^k|, \tag{2.10}$$

where δ_t^α denote classical $L1$ discrete fractional operator [27,43], that is

$$\delta_t^\alpha p^m(x) = \sum_{k=1}^{m} \left[\frac{1}{\tau_k} \int_{t_{k-1}}^{t_k} \omega_{1-\alpha}(t_m - s) ds \right] \nabla_\tau p^k(x) =: \sum_{k=1}^{m} \hat{d}_{m-k}^{(m)} \nabla_\tau p^k(x). \tag{2.11}$$

It follows from [42, Remark 5.5] that

$$\left|\delta_t^\alpha p^k - {}_0^C\mathcal{D}_t^\alpha p^k\right| \le C k^{-\alpha_*} \le C k^{r/\ln N_t - \alpha_*}$$

with $\alpha_* := \min\{r\alpha, 2-\alpha\}$, and thus we have $r/\ln N_t - \alpha_* = r(\gamma - \alpha)$ and

$$\sum_{k=1}^{m} P_{m-k}^{(m)} |\delta_t^\alpha p^k - {}_0^C\mathcal{D}_t^\alpha p^k| \le C T_f^{\alpha-\gamma} N_t^{\frac{r}{\ln N_t} - \alpha_*} \sum_{k=1}^{m} P_{m-k}^{(m)} t_k^{\gamma-\alpha},$$

which, together with [5, Lemma 5.3], gives

$$\begin{aligned}
&\sum_{k=1}^{m} P_{m-k}^{(m)} |\delta_t^\alpha p^k - {}_0^C\mathcal{D}_t^\alpha p^k| \\
&\le C \frac{\Gamma(1+\gamma-\alpha)}{\Gamma(1+\gamma)} T_f^{\alpha-\gamma} N_t^{\frac{r}{\ln N_t} - \alpha_*} \sum_{k=1}^{m} P_{m-k}^{(m)} \sum_{j=1}^{k} \hat{d}_{k-j}^{(k)} \left[(t_j)^\gamma - (t_{j-1})^\gamma \right] \\
&\le C \frac{\Gamma(1+\gamma-\alpha)}{\Gamma(1+\gamma)} T_f^{\alpha-\gamma} N_t^{\frac{r}{\ln N_t} - \alpha_*} \sum_{k=1}^{m} P_{m-k}^{(m)} \sum_{j=1}^{k} \left[d_{k-j}^{(k)} + \left| \hat{d}_{k-j}^{(k)} - d_{k-j}^{(k)} \right| \right] \left[(t_j)^\gamma - (t_{j-1})^\gamma \right].
\end{aligned}$$

By using Lemma 2.1 and exchanging the order of summation, the above inequality leads to

$$\begin{aligned}
&\sum_{k=1}^{m} P_{m-k}^{(m)} |\delta_t^\alpha p^k - {}_0^C\mathcal{D}_t^\alpha p^k| \\
&\le C \frac{\Gamma(1+\gamma-\alpha)}{\Gamma(1+\gamma)} T_f^{\alpha-\gamma} N_t^{\frac{r}{\ln N_t} - \alpha_*} \left(\sum_{j=1}^{m} \left[(t_j)^\gamma - (t_{j-1})^\gamma \right] + \epsilon \sum_{k=1}^{m} P_{m-k}^{(m)} (t_k)^\gamma \right) \\
&\le C \frac{\Gamma(1+\gamma-\alpha)}{\Gamma(1+\gamma)} \left(1 + \frac{3\epsilon}{2\Gamma(1+\alpha)} \right) \left(T_f^\alpha + T_f^{2\alpha} \right) \left(\frac{t_m}{T_f} \right)^\gamma N_t^{\frac{r}{\ln N_t} - \alpha_*},
\end{aligned}$$

where Lemma 2.2 is used in the last inequality. Furthermore, due to $1 \le N_t^{r/\ln N_t} \le e^r$, we derive

$$\begin{aligned}&\sum_{k=1}^{m}P_{m-k}^{(m)}\left|\delta_t^\alpha p^k-{}_0^C\mathcal{D}_t^\alpha p^k\right|\\ \le\,& C\frac{e^r\Gamma(1+\gamma-\alpha)}{\Gamma(1+\gamma)}\left(1+\frac{3\epsilon}{2\gamma(1+\alpha)}\right)\left(T_f^\alpha+T_f^{2\alpha}\right)\left(\frac{t_m}{T_f}\right)^\gamma N_t^{-\alpha_*}.\end{aligned}\tag{2.12}$$

Moreover, by (1.3) in Assumption 1.1, (2.11) and Lemma 2.2, the second term on the right-hand side of (2.10) can be bounded by

$$\begin{aligned}&\sum_{k=2}^{m}P_{m-k}^{(m)}\left|\delta_t^\alpha p^k-{}^F\delta_t^\alpha p^k\right|\\ \le\,&\sum_{k=2}^{m}P_{m-k}^{(m)}\sum_{j=1}^{k}|\nabla_\tau p^j|\left[\frac{1}{\tau_j}\int_{t_{j-1}}^{t_j}\left|\omega_{1-\alpha}(t_m-s)^{-\alpha}-\sum_{i=1}^{N_o}\omega_i e^{-s_i(t_m-s)}\right|ds\right]\\ \le\,& C\epsilon\sum_{k=2}^{m}P_{m-k}^{(m)}\int_{t_0}^{t_{k-1}}|p'(s)|ds\le C\epsilon\frac{3t_m^\alpha(t_{m-1}+t_{m-1}^\alpha/\alpha)}{2\Gamma(1+\alpha)}.\end{aligned}\tag{2.13}$$

Therefore, based on (2.12)-(2.13), we can obtain (2.9). □

2.2. Compact BCFD method

Let N be a positive integer. Define two sets of staggered spatial grids by

$$\begin{aligned}&\Pi_h: x_{-1/2}=x_l-h,\quad x_{i+1/2}=x_l+ih,\qquad\quad i=0,1,\ldots,N,\quad x_{N+3/2}=x_r+h,\\ &\Pi_h^*: x_0=x_l-h/2,\quad x_i=(x_{i+1/2}+x_{i-1/2})/2,\quad i=1,\ldots,N,\qquad x_{N+1}=x_r+h/2\end{aligned}$$

with spatial mesh size $h=(x_r-x_l)/N$. Furthermore, define the spatial difference operators $\delta_x w_\kappa=(w_{\kappa+1/2}-w_{\kappa-1/2})/h$ and $\delta_x^2 w_\kappa=(w_{\kappa+1}-2w_\kappa+w_{\kappa-1})/h^2$ for $\kappa=i,i+1/2$.

In addition, define the spaces of grid functions with periodic boundary conditions

$$\begin{aligned}&\mathcal{U}_h:=\left\{v\mid v=\{v_{i+1/2}\},\ i=0,\ldots,N,\ \text{ and }\ v_{i+1/2}=v_{N+i+1/2}\right\},\\ &\mathcal{P}_h:=\left\{w\mid w=\{w_i\},\ i=1,\ldots,N,\ \text{ and }\ w_i=w_{N+i}\right\},\end{aligned}$$

respectively on Π_h and Π_h^*. Besides, define the discrete inner products and norms on $\mathcal{P}_h$ and $\mathcal{U}_h$ as follows:

$$\begin{aligned}&\langle v,w\rangle=h\sum_{i=1}^{N}v_iw_i, &&\|v\|_M=\sqrt{\langle v,v\rangle},\\ &(v,w)=h\sum_{i=0}^{N-1}v_{i+1/2}w_{i+1/2}, &&\|v\|_T=\sqrt{(v,v)}.\end{aligned}$$

Let $\partial_x v = g$ and define the compact operator $\mathcal{L}_x := 1 + h^2\delta_x^2/24$. We conclude that for $g(x) \in H^4(I)$

$$\delta_x v_i = \mathcal{L}_x g_i + R_{1,i}[v], \quad \delta_x v_{i+1/2} = \mathcal{L}_x g_{i+1/2} + R_{2,i+1/2}[v], \tag{2.14}$$

and by the well-known Bramble-Hilbert Lemma [3], the truncation errors can be estimated as below

$$\|R_1[v]\|_M + \|R_2[v]\|_T \le Ch^4\|v\|_{H^5(I)}. \tag{2.15}$$

Besides, it is easy to show that $\mathcal{L}_x$ is symmetric and positive definite on $\mathcal{P}_h$ and $\mathcal{U}_h$ (see [39]). Thus, we can define the discrete norms $\|w\|_{*,M}^2 = \langle \mathcal{L}_x w, \mathcal{L}_x w\rangle$ and $\|v\|_{*,T}^2 = (\mathcal{L}_x v, \mathcal{L}_x v)$, respectively, for $w \in \mathcal{P}_h$ and $v \in \mathcal{U}_h$.

Now, applying operator $\mathcal{L}_x$ on both sides of the semi-discrete formulation (2.8), and utilizing (2.14) we see

$$\begin{cases} \mathcal{L}_x \left({}^F\delta_t^\alpha p_i^m + cp_i^m\right) + \delta_x u_i^m = \mathcal{L}_x f_i^m + \mathcal{L}_x R_{t,i}^m[p] + R_{1,i}^m[u], \\ \delta_x p_{i+1/2}^m + \mathcal{L}_x \left(a^{-1}u\right)_{i+1/2}^m = R_{2,i+1/2}^m[p] \end{cases} \tag{2.16}$$

for $i = 1, \ldots, N$. Omitting the local truncation errors and letting $\{P_i^m,\ U_{i+1/2}^m\}$ denote the approximations to the exact solution $\{p(x_i, t_m), u(x_{i+1/2}, t_m)\}$, a fully-discrete compact BCFD scheme can be proposed as follows ($m \ge 1$):

$$\begin{cases} \mathcal{L}_x \left({}^F\delta_t^\alpha P_i^m + cP_i^m\right) + \delta_x U_i^m = \mathcal{L}_x f_i^m, & i = 1, \ldots, N, \\ \delta_x P_{i+1/2}^m + \mathcal{L}_x \left(a^{-1}U\right)_{i+1/2}^m = 0, & i = 1, \ldots, N, \\ P_i^0 = p^o(x_i), \ \ U_{i-1/2}^0 = -a(x_{i-1/2})p^o(x_{i-1/2}), & i = 1, \ldots, N, \end{cases} \tag{2.17}$$

enclosed with periodic boundary conditions

$$U_{1/2}^m = U_{N+1/2}^m, \quad P_0^m = P_N^m, \quad P_{N+1}^m = P_1^m. \tag{2.18}$$

3. α-robust unconditional stability and error analysis

In this section, we shall discuss the stability and convergence of the scheme (2.17)-(2.18) for the time-fractional reaction-diffusion model (2.1).

First, some useful lemmas are given for the subsequent analysis.

Lemma 3.1 ([1]). *Let ${}^F\delta_t^\alpha$ be the discrete fractional operator defined by* (2.2)*. Suppose the tolerance error ϵ of the SOE approximation satisfies $\epsilon \le \min\{\omega_{1-\alpha}(T_f)/3, \alpha\omega_{2-\alpha}(T_f)\}$. Then for any grid functions $\{v^m | \, m \ge 0\}$, it holds*

$$2v^m\left({}^F\delta_t^\alpha v^m\right) \ge {}^F\delta_t^\alpha |v^m|^2 + \frac{\left({}^F\delta_t^\alpha v^m\right)^2}{d_0^{(m)}} \quad \textit{for} \quad 1 \le m \le N_t.$$

Lemma 3.2 ([39]). *For any $w \in \mathcal{P}_h$ and $v \in \mathcal{U}_h$, we have*

$$\frac{11}{16}\|w\|_M^2 \le \|w\|_{*,M}^2 \le \|w\|_M^2, \quad \frac{11}{16}\|v\|_T^2 \le \|v\|_{*,T}^2 \le \|v\|_T^2.$$

Lemma 3.3. *Let $w \in \mathcal{P}_h$ and $v \in \mathcal{U}_h$. Then we have*

$$\langle \delta_x v, \mathcal{L}_x w\rangle = -(\mathcal{L}_x v, \delta_x w).$$

Proof. By definition of the discrete inner products, we see

$$\begin{aligned}\langle \delta_x v, \mathcal{L}_x w\rangle &= \frac{h}{24}\sum_{i=1}^{N}\delta_x v_i\,(w_{i-1}+22w_i+w_{i+1})\\ &= -\frac{h}{24}\sum_{i=0}^{N-1}\big(v_{i+1/2}\delta_x w_{i-1/2}+22v_{i+1/2}\delta_x w_{i+1/2}+v_{i+1/2}\delta_x w_{i+3/2}\big)\\ &= -(\mathcal{L}_x v, \delta_x w),\end{aligned}$$

where periodic conditions are used in the last step. □

Lemma 3.4. *If Assumption* 1.2 *holds and $v \in \mathcal{U}_h$, then we have*

$$\big(\mathcal{L}_x v, \mathcal{L}_x(a^{-1}v)\big) \ge \left(\frac{11}{16a^*} - C_a h\right)\|v\|_T^2,$$

where $C_a = 23\|\partial_x a\|_\infty/288a_*^2$.

Proof. By definition of the discrete inner product and Cauchy-Schwarz inequality, we see for $v \in \mathcal{U}_h$

$$\begin{aligned}&\big(\mathcal{L}_x v, \mathcal{L}_x(a^{-1}v)\big)\\ =&\ \frac{h}{24^2}\sum_{i=0}^{N-1}(v_{i-1/2}+22v_{i+1/2}+v_{i+3/2})\Big(a_{i-1/2}^{-1}v_{i-1/2}+22a_{i+1/2}^{-1}v_{i+1/2}+a_{i+3/2}^{-1}v_{i+3/2}\Big)\\ \ge&\ \frac{h}{24^2}\sum_{i=0}^{N-1}\left(441a_{i+1/2}^{-1}-22a_{i-1/2}^{-1}-22a_{i+3/2}^{-1}-\frac{1}{2}a_{i-3/2}^{-1}-\frac{1}{2}a_{i+5/2}^{-1}\right)v_{i+1/2}^2,\end{aligned}$$

where periodic conditions are used in the last step. Thus, we have

$$\begin{aligned}\big(\mathcal{L}_x v, \mathcal{L}_x(a^{-1}v)\big) &\ge \frac{11}{16a^*}\|v\|_T^2 - \frac{23}{288}h\big\|\delta_x a^{-1}\big\|_\infty\|v\|_T^2\\ &\ge \left(\frac{11}{16a^*} - \frac{23\|\partial_x a\|_\infty}{288a_*^2}h\right)\|v\|_T^2,\end{aligned}$$

which proves the conclusion. □

The a prior estimate below plays a critical role in the following α-robust unconditional stability and error analysis of the fast BCFD method.

Theorem 3.1. *Let $W^m = \{W_i^m\} \in \mathcal{P}_h$ and $V^m = \{V_{i+1/2}^m\} \in \mathcal{U}_h$ be the solution of the following compact BCFD scheme:*

$$\begin{cases} \mathcal{L}_x \left({}^F\delta_t^\alpha W_i^m + cW_i^m \right) + \delta_x V_i^m = H_i^m + Q_{1,i}^m, & i = 1, \ldots, N, \\ \delta_x W_{i+1/2}^m + \mathcal{L}_x \left(a^{-1} V \right)_{i+1/2}^m = Q_{2,i+1/2}^m, & i = 1, \ldots, N \end{cases} \tag{3.1}$$

enclosed with periodic boundary conditions

$$V_{1/2}^m = V_{N+1/2}^m, \quad W_0^m = W_N^m, \quad W_{N+1}^m = W_1^m, \tag{3.2}$$

where $H^m := \{H_i^m\}$, $Q_1^m := \{Q_{1,i}^m\} \in \mathcal{P}_h$, $Q_2^m := \{Q_{2,i+1/2}^m\} \in \mathcal{U}_h$.

If Assumption 1.2 holds and the SOE approximation error $\epsilon \le \min\{\omega_{1-\alpha}(T_f)/3, \alpha\omega_{2-\alpha}(T_f)\}$, then there exists a positive constant $h_0 := 144a_*^2/23a^*\|a'\|_\infty$, such that for $h \le h_0$, the following estimates for W^m hold:

Case I. *If $c \ge 0$, we have*

$$\|W^m\|_M \le \frac{4}{\sqrt{11}} \left(\|W^0\|_M + 2\sum_{j=1}^m P_{m-j}^{(m)} \|H^j\|_M + C_p \max_{1\le k\le m} \left(\|Q_1^k\|_M + \|Q_2^k\|_T \right) \right), \tag{3.3}$$

where C_p is an α-robust positive constant defined by (3.15).

Case II. *If $c < 0$ and the maximum time stepsize $\tau \le 1/\sqrt[\alpha]{-6c\Gamma(2-\alpha)}$, we have*

$$\begin{aligned} \|W^m\|_M \le \frac{8}{\sqrt{11}} E_\alpha(-6c\, t_m^\alpha) \Bigg(& \|W^0\|_M + 2 \max_{1\le k\le m} \sum_{j=1}^k P_{k-j}^{(k)} \|H^j\|_M \\ & + C_p \max_{1\le k\le m} \left(\|Q_1^k\|_M + \|Q_2^k\|_T \right) \Bigg). \end{aligned} \tag{3.4}$$

Furthermore, the following estimates for V^m hold:

Case I. *If $c \ge 0$ and the maximum time stepsize $\tau \le 1/\sqrt[\alpha]{3\Gamma(2-\alpha)\lambda_*}$, we have*

$$\begin{aligned} \|V^m\|_T \le \frac{8a^*}{\sqrt{11}} E_\alpha(3\lambda_* t_m^\alpha) \Bigg(& \frac{1}{a_*}\|V^0\|_T + \max_{1\le k\le m} \sum_{j=1}^k P_{k-j}^{(k)} \left(\frac{8}{\sqrt{11}} \|\delta_x H^j\|_T + 2\left\| {}^F\delta_t^\alpha Q_2^j \right\|_T \right) \\ & + C_u \max_{1\le k\le m} \sqrt{\|Q_1^k\|_M^2 + \|Q_2^k\|_T^2} \Bigg), \end{aligned} \tag{3.5}$$

where λ_* and C_u are α-independent constants defined by (3.27) and (3.29).

Case II. *If $c<0$ and the maximum time stepsize $\tau \le 1/\sqrt[\alpha]{3\Gamma(2-\alpha)\hat{\lambda}_*}$, we have*

$$\|V^m\|_T \le \frac{8a^*}{\sqrt{11}} E_\alpha(3\hat{\lambda}_* t_m^\alpha) \Bigg(\frac{1}{a_*}\|V^0\|_T + \max_{1\le k\le m} \sum_{j=1}^{k} P_{k-j}^{(k)} \Big(\frac{8}{\sqrt{11}} \|\delta_x H^j\|_T + 2\big\|{}^F\delta_t^\alpha Q_2^j\big\|_T \Big) + \hat{C}_u \max_{1\le k\le m} \sqrt{\|Q_1^k\|_M^2 + \|Q_2^k\|_T^2} \Bigg), \tag{3.6}$$

where $\hat{\lambda}_*$ and $\hat{C}_u$ are α-independent constants defined by (3.31) *and* (3.33).

Proof. The proof is split into two parts.

Part I. Estimate for W^m. Taking inner products on both sides of (3.1) with $\mathcal{L}_x W^m$ and $\mathcal{L}_x V^m$, respectively, for the first and second equations, we obtain

$$\big\langle \mathcal{L}_x \big({}^F\delta_t^\alpha W^m + cW^m\big), \mathcal{L}_x W^m \big\rangle + \langle \delta_x V^m, \mathcal{L}_x W^m\rangle = \langle H^m + Q_1^m, \mathcal{L}_x W^m\rangle, \tag{3.7}$$

$$(\delta_x W^m, \mathcal{L}_x V^m) + \big(\mathcal{L}_x(a^{-1}V^m), \mathcal{L}_x V^m\big) = (Q_2^m, \mathcal{L}_x V^m). \tag{3.8}$$

Note that Lemma 3.3 shows that

$$\langle \delta_x V^m, \mathcal{L}_x W^m\rangle = -(\mathcal{L}_x V^m, \delta_x W^m).$$

We then sum the two equations (3.7) and (3.8) together to obtain

$$\begin{aligned} &\big\langle \mathcal{L}_x \big({}^F\delta_t^\alpha W^m + cW^m\big), \mathcal{L}_x W^m \big\rangle + \big(\mathcal{L}_x(a^{-1}V^m), \mathcal{L}_x V^m\big) \\ =& \langle H^m + Q_1^m, \mathcal{L}_x W^m\rangle + (Q_2^m, \mathcal{L}_x V^m). \end{aligned} \tag{3.9}$$

Below we shall give estimates for (3.9) term by term. By Lemma 3.1, the first term on the left-hand side of (3.9) can be bounded below by

$$\big\langle \mathcal{L}_x \big({}^F\delta_t^\alpha W^m + cW^m\big), \mathcal{L}_x W^m \big\rangle \ge \frac{1}{2}{}^F\delta_t^\alpha \|W^m\|_{*,M}^2 + c\,\|W^m\|_{*,M}^2. \tag{3.10}$$

While, Lemma 3.4 shows that the second term on the left-hand side of (3.9) can be bounded below by

$$\big(\mathcal{L}_x(a^{-1}V^m), \mathcal{L}_x V^m\big) \ge \left(\frac{11}{16a^*} - C_a h\right)\|V^m\|_T^2. \tag{3.11}$$

Next, for the right-hand side of (3.9), a direct application of Cauchy-Schwarz inequality and Lemma 3.2 shows that

$$(Q_2^m, \mathcal{L}_x V^m) \le \frac{4a^*}{3}\|Q_2^m\|_T^2 + \frac{3}{16a^*}\|V^m\|_T^2, \tag{3.12}$$

and

$$\langle H^m + Q_1^m, \mathcal{L}_x W^m\rangle \le \|W^m\|_{*,M}\|H^m + Q_1^m\|_M. \tag{3.13}$$

Now, we invoke the above estimates (3.10)-(3.13) into (3.9), then taking h sufficiently small so that $1/a^* - 2C_a h \geq 0$, i.e, $h \leq h_0 := 144a_*^2/(23a^*\|a'\|_\infty)$, we have

$$ {}^F\delta_t^\alpha \|W^m\|_{*,M}^2 \leq -2c\,\|W^m\|_{*,M}^2 + 2\|W^m\|_{*,M}\|H^m + Q_1^m\|_M + \frac{8a^*}{3}\|Q_2^m\|_T^2. \tag{3.14} $$

Therefore, if $c \geq 0$, the first term on the right-hand side of (3.14) can be cancelled. Then, applying the discrete fractional Grönwall inequality of Remark 2.2 to (3.14), we see from Lemmas 3.2 and 2.2 that

$$ \begin{aligned} &\frac{\sqrt{11}}{4}\|W^m\|_M \leq \|W^m\|_{*,M} \\ &\leq \|W^0\|_{*,M} + \sum_{j=1}^m P_{m-j}^{(m)}\left(2\|H^j + Q_1^j\|_M + \frac{2\sqrt{6a^*}}{3}\|Q_2^j\|_T\right) + \frac{2\sqrt{6a^*}}{3}\max_{1\leq k\leq m}\|Q_2^k\|_T \\ &\leq \|W^0\|_M + 2\sum_{j=1}^m P_{m-j}^{(m)}\|H^j\|_M + C_p\max_{1\leq k\leq m}\left(\|Q_1^k\|_M + \|Q_2^k\|_T\right), \end{aligned} $$

which implies (3.3), where the constant

$$ C_p := \max\left\{\frac{3t_m^\alpha}{\Gamma(1+\alpha)}, \frac{2\sqrt{6a^*}}{3}\left(1 + \frac{3t_m^\alpha}{2\Gamma(1+\alpha)}\right)\right\} \tag{3.15} $$

is always bounded and independent of the solution. In fact, it is only related to the upper bound of the diffusion coefficient, the fractional order α and time instant t_m. In particular, it is robust with respect to α, i.e., when $\alpha \to 1^-$, the bound shall not blow up.

However, if $c < 0$, we have to apply the discrete fractional Grönwall inequality in Lemma 2.3 to (3.14) and also use the estimates in Lemmas 3.2 and 2.2 to obtain

$$ \begin{aligned} &\frac{\sqrt{11}}{4}\|W^m\|_M \leq \|W^m\|_{*,M} \\ &\leq 2E_\alpha(-6c\,t_m^\alpha)\Bigg(\|W^0\|_{*,M} + \max_{1\leq k\leq m}\sum_{j=1}^k P_{k-j}^{(k)}\left(2\|H^j + Q_1^j\|_M + \frac{2\sqrt{6a^*}}{3}\|Q_2^j\|_T\right) \\ &\qquad\qquad + \frac{2\sqrt{6a^*}}{3}\max_{1\leq k\leq m}\|Q_2^k\|_T\Bigg) \\ &\leq 2E_\alpha(-6c\,t_m^\alpha)\left(\|W^0\|_M + 2\max_{1\leq k\leq m}\sum_{j=1}^k P_{k-j}^{(k)}\|H^j\|_M + C_p\max_{1\leq k\leq m}\left(\|Q_1^k\|_M + \|Q_2^k\|_T\right)\right), \end{aligned} $$

which implies (3.4).

Part II. Estimate for V^m. Applying the discrete fractional operator ${}^F\delta_t^\alpha$ on both sides of the second equation in (3.1), we obtain

$$\delta_x {}^F\delta_t^\alpha W_{i+1/2}^m + \mathcal{L}_x {}^F\delta_t^\alpha (a^{-1}V)_{i+1/2}^m = {}^F\delta_t^\alpha Q_{2,i+1/2}^m, \quad i=1,\ldots,N. \tag{3.16}$$

Then, taking inner products on both sides of the first equation of (3.1) and (3.16) with $\delta_x(a^{-1}V^m)$ and $\mathcal{L}_x(a^{-1}V^m)$, respectively, we have

$$\begin{aligned}&\big\langle \mathcal{L}_x({}^F\delta_t^\alpha W^m + cW^m), \delta_x(a^{-1}V^m)\big\rangle + \big\langle \delta_x V^m, \delta_x(a^{-1}V^m)\big\rangle \\ &= \big\langle H^m + Q_1^m, \delta_x(a^{-1}V^m)\big\rangle,\end{aligned} \tag{3.17}$$

$$\begin{aligned}&\Big(\delta_x {}^F\delta_t^\alpha W^m, \mathcal{L}_x(a^{-1}V^m)\Big) + \Big(\mathcal{L}_x {}^F\delta_t^\alpha (a^{-1}V^m), \mathcal{L}_x(a^{-1}V^m)\Big) \\ &= \Big({}^F\delta_t^\alpha Q_2^m, \mathcal{L}_x(a^{-1}V^m)\Big).\end{aligned} \tag{3.18}$$

Similar as the proof in Part I, by Lemma 3.3 we get

$$\big\langle \mathcal{L}_x {}^F\delta_t^\alpha W^m, \delta_x(a^{-1}V^m)\big\rangle = -\big(\delta_x {}^F\delta_t^\alpha W^m, \mathcal{L}_x(a^{-1}V^m)\big),$$

and then utilizing this relation and summing (3.17) and (3.18) together yields

$$\begin{aligned}&\Big(\mathcal{L}_x {}^F\delta_t^\alpha (a^{-1}V^m), \mathcal{L}_x(a^{-1}V^m)\Big) + c\big\langle \mathcal{L}_x W^m, \delta_x(a^{-1}V^m)\big\rangle + \big\langle \delta_x V^m, \delta_x(a^{-1}V^m)\big\rangle \\ &= \big\langle H^m + Q_1^m, \delta_x(a^{-1}V^m)\big\rangle + \Big({}^F\delta_t^\alpha Q_2^m, \mathcal{L}_x(a^{-1}V^m)\Big).\end{aligned} \tag{3.19}$$

Now, we estimate the left-hand side of (3.19) term by term. Using Lemma 3.1, the first term can be bounded below by

$$\Big(\mathcal{L}_x {}^F\delta_t^\alpha (a^{-1}V^m), \mathcal{L}_x(a^{-1}V^m)\Big) \geq \frac{1}{2} {}^F\delta_t^\alpha \|a^{-1}V^m\|_{*,T}^2. \tag{3.20}$$

And for the second term, by Lemma 3.3 and using the second equation in (3.1), we see

$$\begin{aligned}\big\langle \mathcal{L}_x W^m, \delta_x(a^{-1}V^m)\big\rangle &= -\big(\mathcal{L}_x(a^{-1}V^m), \delta_x W^m\big) \\ &= \|a^{-1}V^m\|_{*,T}^2 - \big(\mathcal{L}_x(a^{-1}V^m), Q_2^m\big).\end{aligned}$$

Then, by Young's inequality, we can easily prove that

$$\big\langle \mathcal{L}_x W^m, \delta_x(a^{-1}V^m)\big\rangle \geq \frac{1}{2}\|a^{-1}V^m\|_{*,T}^2 - \frac{1}{2}\|Q_2^m\|_T^2, \tag{3.21}$$

and also

$$\big\langle \mathcal{L}_x W^m, \delta_x(a^{-1}V^m)\big\rangle \leq \frac{3}{2}\|a^{-1}V^m\|_{*,T}^2 + \frac{1}{2}\|Q_2^m\|_T^2. \tag{3.22}$$

For the third term, by Cauchy-Schwarz inequality, we have

$$\begin{aligned}\big\langle \delta_x V^m, \delta_x(a^{-1}V^m)\big\rangle &= h\sum_{i=1}^{N}(\delta_x V_i^m)\Big(a_{i+1/2}^{-1}\delta_x V_i^m + V_{i-1/2}^m(\delta_x a_i^{-1})\Big) \\ &\geq \frac{1}{a^*}\|\delta_x V^m\|_M^2 - \|\delta_x a^{-1}\|_\infty \|\delta_x V^m\|_M \|V^m\|_T \\ &\geq \frac{1}{2a^*}\|\delta_x V^m\|_M^2 - \frac{(a^*)^3\|a'\|_\infty^2}{2a_*^4}\|a^{-1}V^m\|_T^2.\end{aligned} \tag{3.23}$$

Next, we estimate the right-hand side of (3.19). For the first term, by using Lemma 3.3, the fact that

$$\delta_x(a^{-1}V)_i^m = a_{i+1/2}^{-1}\delta_x V_i^m + V_{i-1/2}^m(\delta_x a_i^{-1}),$$

and Cauchy-Schwarz inequality, we have

$$\begin{aligned}\big\langle H^m + Q_1^m, \delta_x(a^{-1}V^m)\big\rangle &= -\big(\delta_x H^m, a^{-1}V^m\big) + \big\langle Q_1^m, \delta_x(a^{-1}V^m)\big\rangle \\ &\le \|\delta_x H^m\|_T \|a^{-1}V^m\|_T + \left(\frac{a^*}{2a_*^2} + \frac{1}{2a^*}\right)\|Q_1^m\|_M^2 \\ &\quad + \frac{1}{2a^*}\|\delta_x V^m\|_M^2 + \frac{(a^*)^3\|a'\|_\infty^2}{2a_*^4}\|a^{-1}V^m\|_T^2. \end{aligned} \tag{3.24}$$

Meanwhile, the second term can be bounded by

$$\Big({}^F\delta_t^\alpha Q_2^m, \mathcal{L}_x(a^{-1}V^m)\Big) \le \big\|{}^F\delta_t^\alpha Q_2^m\big\|_T \big\|a^{-1}V^m\big\|_{*,T}. \tag{3.25}$$

Therefore, if $c \ge 0$, inserting the estimates (3.20)-(3.21), (3.23)-(3.25) into (3.19), and utilizing Lemma 3.2, we have

$$\begin{aligned}{}^F\delta_t^\alpha \big\|a^{-1}V^m\big\|_{*,T}^2 &\le \lambda_* \big\|a^{-1}V^m\big\|_{*,T}^2 + \left(\frac{8}{\sqrt{11}}\|\delta_x H^m\|_T + 2\big\|{}^F\delta_t^\alpha Q_2^m\big\|_T\right)\big\|a^{-1}V^m\big\|_{*,T} \\ &\quad + C_*^2\Big(\|Q_1^m\|_M^2 + \|Q_2^m\|_T^2\Big), \end{aligned} \tag{3.26}$$

where the constants

$$\lambda_* := \frac{32(a^*)^3\|a'\|_\infty^2}{11a_*^4}, \quad C_*^2 := \max\left\{c, \frac{a^*}{a_*^2} + \frac{1}{a^*}\right\} \tag{3.27}$$

are both bounded, independent of the solution, and only related to the coefficients in model (2.1).

Thus, applying the discrete fractional Grönwall inequality in Lemma 2.3 to (3.26) and using the estimate in Lemma 2.2, we have

$$\begin{aligned} &\|a^{-1}V^m\|_{*,T} \le 2E_\alpha(3\lambda_* t_m^\alpha) \\ &\times \left(\|a^{-1}V^0\|_{*,T} + \max_{1\le k\le m}\sum_{j=1}^k P_{k-j}^{(k)}\left(\frac{8}{\sqrt{11}}\|\delta_x H^j\|_T + 2\|{}^F\delta_t^\alpha Q_2^j\|_T + C_*\sqrt{\|Q_1^j\|_M^2 + \|Q_2^j\|_T^2}\right)\right. \\ &\qquad\qquad \left. + C_* \max_{1\le k\le m}\sqrt{\|Q_1^k\|_M^2 + \|Q_2^k\|_T^2}\right) \\ &\le 2E_\alpha(3\lambda_* t_m^\alpha)\left(\|a^{-1}V^0\|_{*,T} + \max_{1\le k\le m}\sum_{j=1}^k P_{k-j}^{(k)}\left(\frac{8}{\sqrt{11}}\|\delta_x H^j\|_T + 2\|{}^F\delta_t^\alpha Q_2^j\|_T\right)\right. \\ &\qquad\qquad \left. + C_u \max_{1\le k\le m}\sqrt{\|Q_1^k\|_M^2 + \|Q_2^k\|_T^2}\right), \end{aligned} \tag{3.28}$$

where the constant

$$C_u = C_* + \frac{3t_m^\alpha C_*}{2\Gamma(1+\alpha)} \tag{3.29}$$

is also bounded and α-robust.

However, if $c < 0$, we insert the estimates (3.20), (3.22), (3.23)-(3.25) into (3.19), and also utilize Lemma 3.2 to get

$$\begin{aligned} {}^F\delta_t^\alpha \|a^{-1}V^m\|_{*,T}^2 \le \hat{\lambda}_* \|a^{-1}V^m\|_{*,T}^2 + \left(\frac{8}{\sqrt{11}}\|\delta_x H^m\|_T + 2\|{}^F\delta_t^\alpha Q_2^m\|_T\right)\|a^{-1}V^m\|_{*,T} \\ + \hat{C}_*^2\Big(\|Q_1^m\|_M^2 + \|Q_2^m\|_T^2\Big), \end{aligned} \tag{3.30}$$

where the constants

$$\hat{\lambda}_* := \frac{32(a^*)^3\|a'\|_\infty^2}{11a_*^4} - 3c, \quad \hat{C}_*^2 := \max\left\{-c, \frac{a^*}{a_*^2} + \frac{1}{a^*}\right\} \tag{3.31}$$

are also bounded, independent of the solution, and only related to the coefficients in model (2.1).

Similarly, applying the discrete fractional Grönwall inequality in Lemma 2.3 to (3.30) and using the estimate in Lemma 2.2, we have

$$\begin{aligned} &\|a^{-1}V^m\|_{*,T} \le 2E_\alpha(3\hat{\lambda}_* t_m^\alpha) \\ &\times \Bigg(\|a^{-1}V^0\|_{*,T} + \max_{1\le k\le m}\sum_{j=1}^k P_{k-j}^{(k)}\Big(\frac{8}{\sqrt{11}}\|\delta_x H^j\|_T + 2\big\|{}^F\delta_t^\alpha Q_2^j\big\|_T + \hat{C}_*\sqrt{\|Q_1^j\|_M^2 + \|Q_2^j\|_T^2}\Big) \\ &\qquad + \hat{C}_* \max_{1\le k\le m}\sqrt{\|Q_1^k\|_M^2 + \|Q_2^k\|_T^2}\Bigg) \\ &\le 2E_\alpha(3\hat{\lambda}_* t_m^\alpha)\Bigg(\|a^{-1}V^0\|_{*,T} + \max_{1\le k\le m}\sum_{j=1}^k P_{k-j}^{(k)}\Big(\frac{8}{\sqrt{11}}\|\delta_x H^j\|_T + 2\big\|{}^F\delta_t^\alpha Q_2^j\big\|_T\Big) \\ &\qquad + \hat{C}_u \max_{1\le k\le m}\sqrt{\|Q_1^k\|_M^2 + \|Q_2^k\|_T^2}\Bigg), \end{aligned} \tag{3.32}$$

where the constant

$$\hat{C}_u = \hat{C}_* + \frac{3t_m^\alpha \hat{C}_*}{2\Gamma(1+\alpha)} \tag{3.33}$$

is also bounded and α-robust.

Finally, note that

$$\|V^m\|_{*,T} \le a^*\|a^{-1}V^m\|_{*,T}, \quad \|a^{-1}V^0\|_{*,T} \le a_*^{-1}\|V^0\|_{*,T}. \tag{3.34}$$

Therefore, combinations of (3.28), (3.32) with (3.34), and using Lemma 3.2 directly concludes the estimates (3.5) and (3.6), respectively. $\square$

Theorem 3.2 (Stability). *Let $P^m = \{P_i^m\} \in \mathcal{P}_h$ and $U^m = \{U_{i+1/2}^m\} \in \mathcal{U}_h$ be the solution of the fast compact BCFD scheme* (2.17)-(2.18). *Suppose Assumption* 1.2 *hold and the SOE approximation error satisfies $\epsilon \leq \min\{\omega_{1-\alpha}(T_f)/3, \alpha\omega_{2-\alpha}(T_f)\}$, then there exist positive constant h_0, such that for $h \leq h_0$, if $c \geq 0$, the following stability estimate holds:*

$$\|P^m\|_M \leq C_1 \left(\|P^0\|_M + \max_{1\leq k\leq m} \|f^k\|_M \right), \tag{3.35}$$

otherwise, if $c < 0$ and the maximum time stepsize $\tau \leq 1/\sqrt[\alpha]{-6c\,\Gamma(2-\alpha)}$, we have

$$\|P^m\|_M \leq \hat{C}_1 \left(\|P^0\|_M + \max_{1\leq k\leq m} \|f^k\|_M \right), \tag{3.36}$$

where C_1 and $\hat{C}_1$ are two α-robust positive constants that related to C_p.

Furthermore, if $c \geq 0$ and the maximum time stepsize $\tau \leq 1/\sqrt[\alpha]{3\Gamma(2-\alpha)\lambda_*}$, we have

$$\|U^m\|_T \leq C_2 \left(\|U^0\|_T + \max_{1\leq k\leq m} \|f^k\|_M \right), \tag{3.37}$$

otherwise, if $c < 0$ and the maximum time stepsize $\tau \leq 1/\sqrt[\alpha]{3\,\Gamma(2-\alpha)\hat{\lambda}_*}$, we have

$$\|U^m\|_T \leq \hat{C}_2 \left(\|U^0\|_T + \max_{1\leq k\leq m} \|f^k\|_M \right), \tag{3.38}$$

where C_2 and $\hat{C}_2$ are two α-robust positive constants that related to C_u and $\hat{C}_u$, respectively.

Proof. The conclusion is a direct result of Theorem 3.1. In fact, the solution pair (P^m, U^m) of (2.17)-(2.18) can be viewed as (W^m, V^m) of (3.1) with $Q_1^m = \mathcal{L}_x f^m$ and $H^m = Q_2^m = 0$. Therefore, under suitable conditions, if $c \geq 0$, we have

$$\begin{aligned}
\|P^m\|_M &\leq \frac{4}{\sqrt{11}} \left(\|P^0\|_M + C_p \max_{1\leq k\leq m} \|f^k\|_{*,M} \right), \\
\|U^m\|_T &\leq \frac{8a^*}{\sqrt{11}a_*} E_\alpha(3\lambda_* t_m^\alpha) \left(\|U^0\|_T + a_*\, C_u \max_{1\leq k\leq m} \|f^k\|_{*,M} \right),
\end{aligned}$$

which together with Lemma 3.2 proves (3.35) and (3.37). Otherwise, if $c < 0$, we have

$$\begin{aligned}
\|P^m\|_M &\leq \frac{8}{\sqrt{11}} E_\alpha(-6c\, t_m^\alpha) \left(\|P^0\|_M + \hat{C}_p \max_{1\leq k\leq m} \|f^k\|_{*,M} \right), \\
\|U^m\|_T &\leq \frac{8a^*}{\sqrt{11}a_*} E_\alpha(3\hat{\lambda}_* t_m^\alpha) \left(\|U^0\|_T + a_*\, \hat{C}_u \max_{1\leq k\leq m} \|f^k\|_{*,M} \right),
\end{aligned}$$

which together with Lemma 3.2 proves (3.36) and (3.38). □

Before deriving the error estimate for the fast compact BCFD scheme (2.17)-(2.18), we first prove the following two lemmas which are useful in our analysis.

Lemma 3.5. *If $p(x,t)$ satisfies the condition* (1.3) *in Assumption* 1.1*, then there exists a positive constant C_3 independent of α, such that for $R_2 = R_2[p]$ defined by* (2.14) *we have*

$$\sum_{k=1}^{m} P_{m-k}^{(m)} \|{}^F\delta_t^\alpha R_2^k\|_T \leq C_3(t_m + t_m^\alpha/\alpha)h^4. \tag{3.39}$$

Proof. By definition of the fractional operator ${}^F\delta_t^\alpha$, and considering the positive properties of the DC and DCC kernels $\{d_{m-k}^{(m)}\}$ and $\{P_{m-j}^{(m)}\}$, by Lemma 2.2, we obtain

$$\begin{aligned}\sum_{k=1}^{m} P_{m-k}^{(m)} \|{}^F\delta_t^\alpha R_2^k\|_T &= \sum_{k=1}^{m} P_{m-k}^{(m)} \left\| \sum_{j=1}^{k} d_{k-j}^{(k)} \nabla_\tau R_2^j \right\|_T \\ &\leq \sum_{k=1}^{m} P_{m-k}^{(m)} \sum_{j=1}^{k} d_{k-j}^{(k)} \|R_2^j - R_2^{j-1}\|_T.\end{aligned}$$

Then, we change the order of summations and apply the identity (2.7) to obtain

$$\begin{aligned}\sum_{k=1}^{m} P_{m-k}^{(m)} \|{}^F\delta_t^\alpha R_2^k\|_T &\leq \sum_{k=1}^{m} \|R_2^k - R_2^{k-1}\|_T = \sum_{k=1}^{m} \left\| \int_{t_{k-1}}^{t_k} \partial_s R_2(s) ds \right\|_T \\ &\leq \int_{t_0}^{t_m} \|\partial_s R_2(s)\|_T ds.\end{aligned}$$

Based on the estimate (2.15) and the assumption $\|\partial_t p\|_{H^5(I)} \leq C(1+t^{\alpha-1})$, we immediately get the conclusion (3.39). □

Lemma 3.6. *If $p(x,t)$ satisfies the condition* (1.3) *in Assumption* 1.1*, then there exist an α-robust positive constant C_4, such that for $R_t = R_t[p]$ we have*

$$\sum_{k=1}^{m} P_{m-k}^{(m)} \|\delta_x R_t^k\|_T \leq C_4 \left(N_t^{-\min\{2-\alpha, r\alpha\}} + \epsilon \right). \tag{3.40}$$

Proof. By definition of the discrete norm and Cauchy-Schwarz inequality, we have

$$\begin{aligned}\sum_{k=1}^{m} P_{m-k}^{(m)} \|\delta_x R_t^k\|_T &= \sum_{k=1}^{m} P_{m-k}^{(m)} \sqrt{h \sum_{i=0}^{N-1} \left(\frac{1}{h} \int_{x_i}^{x_{i+1}} (R_t^k)_x dx \right)^2} \\ &\leq \sum_{k=1}^{m} P_{m-k}^{(m)} \sqrt{\frac{1}{h} \sum_{i=0}^{N-1} \left(\int_{x_i}^{x_{i+1}} 1^2 dx \int_{x_i}^{x_{i+1}} (R_t^k)_x^2 dx \right)} \\ &\leq \sum_{k=1}^{m} P_{m-k}^{(m)} \|(R_t^k)_x\|_{L^2(I)}.\end{aligned}$$

Based on the estimate in Lemma 2.4 and using the assumption

$$\|\partial_t p\|_{H^1(I)} + t\|\partial_{tt} p\|_{H^1(I)} \le C(1 + t^{\alpha-1}),$$

we immediately get (3.40). □

Theorem 3.3 (Convergence). *Let* (p^m, u^m) *be the exact solution pair of model* (2.1)*, and* (P^m, U^m) *be the numerical solution pair of the fast compact BCFD scheme* (2.17)-(2.18). *Suppose Assumptions* 1.1-1.2 *hold, and the SOE tolerance error* $\epsilon \le \min\{\omega_{1-\alpha}(T_f)/3, \alpha\omega_{2-\alpha}(T_f)\}$. *Moreover, assume that* $a(x) \in C^4(I)$. *Then, if* $c \ge 0$, *we have*

$$\|p^m - P^m\|_M \le C_5 \left(N_t^{-\min\{2-\alpha, r\alpha\}} + h^4 + \epsilon \right) \quad \text{for} \quad h \le h_0, \tag{3.41}$$

otherwise, if $c < 0$ *and the maximum time stepsize* $\tau \le 1/\sqrt[\alpha]{-6c\Gamma(2-\alpha)}$, *we have*

$$\|p^m - P^m\|_M \le \hat{C}_5 \left(N_t^{-\min\{2-\alpha, r\alpha\}} + h^4 + \epsilon \right) \quad \text{for} \quad h \le h_0, \tag{3.42}$$

where C_5 *and* $\hat{C}_5$ *are two* α*-robust positive constants that related to* C_p.

Furthermore, if $c \ge 0$ *and the maximum time stepsize* $\tau \le 1/\sqrt[\alpha]{3\Gamma(2-\alpha)\lambda_*}$,

$$\|u^m - U^m\|_T \le C_6 \left(N_t^{-\min\{2-\alpha, r\alpha\}} + h^4 + \epsilon \right), \tag{3.43}$$

if $c < 0$ *and the maximum time stepsize* $\tau \le 1/\sqrt[\alpha]{3\Gamma(2-\alpha)\hat{\lambda}_*}$, *we have*

$$\|u^m - U^m\|_T \le \hat{C}_6 \left(N_t^{-\min\{2-\alpha, r\alpha\}} + h^4 + \epsilon \right), \tag{3.44}$$

where C_6 *and* $\hat{C}_6$ *are two* α*-robust positive constants that related to* C_u *and* $\hat{C}_u$*, respectively.*

Proof. Let $\xi_i^m := p(x_i, t_m) - P_i^m$ and $\eta_{i+1/2}^m := u(x_{i+1/2}, t_m) - U_{i+1/2}^m$ with $\xi_i^0 = \eta_{i+1/2}^0 = 0$. By subtracting the equivalent formulation (2.16) of model (2.1) from (2.17)-(2.18), we obtain the following error equations:

$$\begin{cases} \mathcal{L}_x \left({}^F\delta_t^\alpha \xi_i^m + c\xi_i^m \right) + \delta_x \eta_i^m = \mathcal{L}_x(R_{t,i}^m) + R_{1,i}^m[u], & i = 1, \dots, N, \\ \delta_x \xi_{i+1/2}^m + \mathcal{L}_x \left(a^{-1}\eta \right)_{i+1/2}^m = R_{2,i+1/2}^m[p], & i = 1, \dots, N \end{cases} \tag{3.45}$$

for $m = 1, \dots, N_t$.

It is clear that, with $H^m = \mathcal{L}_x({}^F R_t^m)$, $Q_1^m = R_1^m[u]$ and $Q_2^m = R_2^m[p]$ in (3.1), we have $W^m = \xi^m$ and $V^m = \eta^m$. Therefore, if $c \ge 0$, we conclude from Theorem 3.1 together with Lemma 3.2 that

$$\|\xi^m\|_M \le \frac{4}{\sqrt{11}} \left(2\sum_{j=1}^m P_{m-j}^{(m)} \|R_t^j\|_M + C_p \max_{1\le k\le m} \left(\|R_1^k\|_M + \|R_2^k\|_T \right) \right) \quad \text{for} \quad h \le h_0.$$

Furthermore, if the maximum time stepsize $\tau \le 1/\sqrt[\alpha]{3\Gamma(2-\alpha)\lambda_*}$, we also have

$$\|\eta^m\|_T \le \frac{8a^*}{\sqrt{11}} E_\alpha(3\lambda_* t_m^\alpha)\Bigg(\max_{1\le k\le m} \sum_{j=1}^{k} P_{k-j}^{(k)} \left(\frac{8}{\sqrt{11}} \big\|\delta_x(R_t^j)\big\|_T + 2\big\|{}^F\delta_t^\alpha R_2^j\big\|_T \right) + C_u \max_{1\le k\le m} \sqrt{\|R_1^k\|_M^2 + \|R_2^k\|_T^2} \Bigg).$$

Therefore, the conclusions (3.41) and (3.43) follows from (2.15), Lemma 2.4 and similar results of Lemmas 3.5 and 3.6.

Otherwise, for $c<0$, if the maximum time stepsize $\tau \le 1/\sqrt[\alpha]{-6c\,\Gamma(2-\alpha)}$, we have

$$\|\xi^m\|_M \le \frac{8}{\sqrt{11}} E_\alpha(-6c\,t_m^\alpha)\Bigg(2\max_{1\le k\le m} \sum_{j=1}^{k} P_{k-j}^{(k)} \big\|R_t^j\big\|_M + C_p \max_{1\le k\le m} \Big(\|R_1^k\|_M + \|R_2^k\|_T \Big) \Bigg) \quad \text{for} \quad h \le h_0.$$

Besides, if the maximum time stepsize $\tau \le 1/\sqrt[\alpha]{3\,\Gamma(2-\alpha)\hat{\lambda}_*}$, we have

$$\|\eta^m\|_T \le \frac{8a^*}{\sqrt{11}} E_\alpha(3\hat{\lambda}_* t_m^\alpha)\Bigg(\max_{1\le k\le m} \sum_{j=1}^{k} P_{k-j}^{(k)} \left(\frac{8}{\sqrt{11}} \big\|\delta_x R_t^j\big\|_T + 2\big\|{}^F\delta_t^\alpha R_2^m\big\|_T \right) + \hat{C}_u \max_{1\le k\le m} \sqrt{\|Q_1^k\|_M^2 + \|Q_2^k\|_T^2} \Bigg).$$

Thus, we can immediately get the estimates (3.42) and (3.44) for $c<0$. □

4. Numerical results

In this section, we shall numerically show the performance of the proposed fast fourth-order compact BCFD scheme for solving model (1.1). All the numerical experiments are performed in Matlab R2019b on a laptop with the configuration: 11th Gen Intel(R) Core (TM) i7-11700 @ 2.50GHz 2.50 GHz and 16.00 GB RAM. In most tests, without special statement the mesh grading parameter is chosen as $r=(2-\alpha)/\alpha$ to ensure the optimal $(2-\alpha)$-th order temporal convergence. Besides, we choose the tolerance $\epsilon = 10^{-12}$ in order to maintain the convergence rates of the fast version BCFD scheme. In all tests, we use Error_p and Error_u to denote the errors for p and u in the discrete L^2-norm at time $t=T_f$, i.e., $\mathrm{Error}_p := \|p^{N_t} - P^{N_t}\|_M$ and $\mathrm{Error}_u := \|u^{N_t} - U^{N_t}\|_T$.

Example 4.1. Let $I=(0,1)$ and $T_f=1$. We consider an example of the time-fractional model (1.1) with periodic boundary conditions. Set $a(x)=2+\cos(2\pi x)$ and $c=1/2$ (positive) or $c=-1/2$ (negative). Given the exact solutions

$$p(x,t) = t^\alpha \cos(2\pi x), \quad u(x,t) = 2\pi t^\alpha \big(2+\cos(2\pi x)\big)\sin(2\pi x),$$

such that the source function $f(x,t)$ can be computed accordingly.

First, we set $N_t = 5000$ to investigate the spatial accuracy of the fast $L1$-compact BCFD scheme (2.17)-(2.18). Numerical results with $\alpha = 0.3, 0.5, 0.7$ are presented in Tables 1-2, where both positive and negative reaction terms are tested. The results indicate that the fast compact BCFD method (2.17)-(2.18) is actually fourth-order accurate in space for both the primal variable p and its flux u, whether the reaction is positive or not. Therefore, the numerical findings are well in agreement with the theoretical analysis.

Second, we set $N = [10N_t^{(2-\alpha)/4}]$ to show the temporal convergence rate of the fast $L1$-compact BCFD scheme (2.17). Numerical errors and convergence rates for both p and u are listed in Tables 3-4, in which $(2-\alpha)$-th order accurate in time is observed when the mesh grading parameter $r = (2-\alpha)/\alpha$, regardless of positive

Table 1: Errors and spatial convergence rates of (2.17) for Example 4.1 with $c = 1/2$.

α	N	Error$_p$	Error$_u$	Rate$_p$	Rate$_u$	
0.3	6	5.9369e-03	1.3203e-01	—	—	
	12	3.7320e-04	8.1161e-03	3.9916	4.0240	
	24	2.3228e-05	5.0610e-04	4.0059	4.0032	≈ 4
	48	1.4949e-06	3.2056e-05	3.9577	3.9807	
0.5	6	5.9371e-03	1.3204e-01	—	—	
	12	3.7317e-04	8.1159e-03	3.9918	4.0241	
	24	2.3187e-05	5.0567e-04	4.0084	4.0044	≈ 4
	48	1.4461e-06	3.1560e-05	4.0031	4.0020	
0.7	6	5.9365e-03	1.3203e-01	—	—	
	12	3.7314e-04	8.1152e-03	3.9918	4.0241	
	24	2.3181e-05	5.0558e-04	4.0086	4.0046	≈ 4
	48	1.4422e-06	3.1517e-05	4.0065	4.0037	

Table 2: Errors and spatial convergence rates of (2.17) for Example 4.1 with $c = -1/2$.

α	N	Error$_p$	Error$_u$	Rate$_p$	Rate$_u$	
0.3	6	5.9611e-03	1.3257e-01	—	—	
	12	3.7467e-04	8.1478e-03	3.9918	4.0242	
	24	2.3320e-05	5.0807e-04	4.0059	4.0032	≈ 4
	48	1.5015e-06	3.2187e-05	3.9570	3.9804	
0.5	6	5.9613e-03	1.3257e-01	—	—	
	12	3.7465e-04	8.1476e-03	3.9920	4.0243	
	24	2.3279e-05	5.0763e-04	4.0084	4.0045	≈ 4
	48	1.4517e-06	3.1682e-05	4.0031	4.0019	
0.7	6	5.9608e-03	1.3256e-01	—	—	
	12	3.7461e-04	8.1468e-03	3.9920	4.0243	
	24	2.3273e-05	5.0754e-04	4.0086	4.0046	≈ 4
	48	1.4478e-06	3.1638e-05	4.0066	4.0037	

Table 3: Errors and temporal convergence rates of (2.17) for Example 4.1 with $c = 1/2$.

α	N_t	Error$_p$	Error$_u$	Rate$_p$	Rate$_u$	
0.3	48	6.6586e-06	4.3784e-05	–	–	
	96	2.1412e-06	1.7753e-05	1.6368	1.6390	
	192	6.8204e-07	7.2040e-06	1.6504	1.6502	≈ 1.7
	384	2.1119e-07	2.9244e-06	1.6912	1.6916	
0.5	48	8.2408e-06	9.9446e-05	–	–	
	96	2.9393e-06	3.5684e-05	1.4873	1.4786	
	192	1.0459e-06	1.2625e-05	1.4906	1.4989	≈ 1.5
	384	3.7162e-07	4.4920e-06	1.4928	1.4908	
0.7	48	9.8380e-06	1.3924e-04	–	–	
	96	3.9765e-06	5.6064e-05	1.3068	1.3124	
	192	1.6144e-06	2.2842e-05	1.3004	1.2953	≈ 1.3
	384	6.5461e-07	9.2453e-06	1.3023	1.3048	

Table 4: Errors and temporal convergence rates of (2.17) for Example 4.1 with $c = -1/2$.

α	N_t	Error$_p$	Error$_u$	Rate$_p$	Rate$_u$	
0.3	48	6.7504e-06	7.8391e-05	–	–	
	96	2.1707e-06	2.5170e-05	1.6367	1.6389	
	192	6.9145e-07	8.0192e-06	1.6504	1.6502	≈ 1.7
	384	2.1410e-07	2.4824e-06	1.6912	1.6916	
0.5	48	8.3522e-06	1.0063e-04	–	–	
	96	2.9788e-06	3.6105e-05	1.4874	1.4788	
	192	1.0600e-06	1.2776e-05	1.4906	1.4987	≈ 1.5
	384	3.7664e-07	4.5455e-06	1.4928	1.4909	
0.7	48	9.9551e-06	1.4040e-04	–	–	
	96	4.0240e-06	5.6537e-05	1.3067	1.3123	
	192	1.6336e-06	2.3033e-05	1.3005	1.2954	≈ 1.3
	384	6.6242e-07	9.3230e-06	1.3022	1.3048	

or negative reaction. Besides, we also assess the significant impact of various mesh grading parameter values r on the temporal accuracy for fixed $\alpha = 0.5$. The results in Table 5 demonstrate clearly that the temporal accuracy is of order $\min\{2-\alpha, r\alpha\}$, which shows that Theorem 3.3 gives a sharp temporal error bound for the computed BCFD solution.

For comparison, the classical $L1$ compact BCFD formula is also proposed as follows:

$$\begin{cases} \mathcal{L}_x(\delta_t^\alpha P_i^m + cP_i^m) + \delta_x U_i^m = \mathcal{L}_x f_i^m, & i = 1, \dots, N, \\ \delta_x P_{i+1/2}^m + \mathcal{L}_x(a^{-1}U)_{i+1/2}^m = 0, & i = 1, \dots, N, \\ P_i^0 = p^o(x_i), \ \ U_{i-1/2}^0 = -a(x_{i-1/2})p^o(x_{i-1/2}), & i = 1, \dots, N, \end{cases} \tag{4.1}$$

enclosed with periodic boundary conditions (2.18), where δ_t^α is defined by (2.11).

Table 5: Errors and temporal convergence rates of (2.17) for Example 4.1 with $\alpha = 0.5$, $c = 1/2$.

r	N_t	Error_p	Error_u	Rate_p	Rate_u	
1	300	4.8047e-05	1.0483e-03	—	—	≈ 0.5
	600	3.2825e-05	7.1607e-04	0.5496	0.5499	
	1200	2.3179e-05	5.0559e-04	0.5019	0.5021	
	2400	1.6829e-05	3.6705e-04	0.4618	0.4619	
$\frac{2-\alpha}{2\alpha}$	300	1.0821e-05	2.3656e-04	—	—	≈ 0.75
	600	6.4653e-06	1.4120e-04	0.7430	0.7444	
	1200	3.6807e-06	8.0344e-05	0.8127	0.8135	
	2400	2.2464e-06	4.9017e-05	0.7123	0.7129	
$\frac{2-\alpha}{\alpha}$	300	5.3730e-07	6.4874e-06	—	—	≈ 1.5
	600	1.9073e-07	2.3042e-06	1.4941	1.4933	
	1200	6.7631e-08	8.1575e-07	1.4958	1.4980	
	2400	2.3962e-08	2.8929e-07	1.4969	1.4955	
$\frac{2(2-\alpha)}{\alpha}$	300	1.5039e-06	1.7349e-05	—	—	≈ 1.5
	600	5.3700e-07	6.1947e-06	1.4857	1.4857	
	1200	1.9143e-07	2.2081e-06	1.4880	1.4881	
	2400	6.7897e-08	7.8320e-07	1.4954	1.4953	

Third, comparisons of the compact BCFD scheme (4.1) and its fast version (2.17) for $\alpha = 0.4, 0.6$ are tested. It is observed from Table 6 that the fast version $L1$-compact BCFD scheme can keep almost the same accuracy as the conventional $L1$-compact BCFD scheme, but it costs less CPU running time. For example, when $\alpha = 0.4$ and $(N_t, N) = (3500, 50)$, with getting almost the same errors respectively for the approxi-

Table 6: Comparisons of (2.17) and (4.1) for Example 4.1.

	α	N_t	N	Error_p	Error_u	CPU (s)
Method (4.1)	0.4	1000	20	4.8098e-05	1.0488e-03	2min 9s
		2000	100	7.4274e-08	1.6347e-06	58min 57s
		3500	50	1.2288e-06	2.6812e-05	8h 12min
	0.6	1000	100	1.3573e-07	1.9912e-06	2min 57s
		2000	30	9.5273e-06	2.0719e-04	49min 26s
		3500	25	1.9685e-05	4.2940e-04	7h 50min
Method (2.17)	0.4	1000	20	4.8098e-05	1.0488e-03	3s
		2000	100	7.4267e-08	1.6352e-06	34s
		3500	50	1.2288e-06	2.6812e-05	41s
	0.6	1000	100	1.3572e-07	1.9912e-06	7s
		2000	30	9.5273e-06	2.0719e-04	7s
		3500	25	1.9685e-05	4.2940e-04	12s

Table 7: α-robustness of (2.17) for Example 4.1 with $c = \pm 1/2$.

c		$\alpha = 0.9$	$\alpha = 0.95$	$\alpha = 0.99$	$\alpha = 0.995$
1/2	Error_p	1.4210e-06	9.2922e-07	2.3014e-07	1.1773e-07
	Error_u	1.6494e-05	1.0782e-05	2.6639e-06	1.3595e-06
-1/2	Error_p	1.4413e-06	9.4245e-07	2.3342e-07	1.1940e-07
	Error_u	1.6724e-05	1.0932e-05	2.7010e-06	1.3784e-06

mations of p and u, the scheme (4.1) takes more than 8 hours, while the fast algorithm (2.17)-(2.18) costs only 41 seconds! This indeed shows that the SOE-based fast algorithm has a great advantage in long-term or small temporal stepsize modeling and simulations.

Finally, to test the α-robustness of the estimates in Theorem 3.3, we respectively choose $\alpha = 0.9, 0.95, 0.99, 0.995$ which approaches to 1 to observe the convergence results. As seen from Table 7 that, for fixed $N_t = N = 200$, all the errors change slightly as $\alpha \to 1^-$, not only for positive reaction but also for negative one, which shows the method is α-robust–as our convergence analysis has already predicted.

Example 4.2. Let $I = (0,1)$ and $T_f = 1$. Here we consider a time-fractional model (1.1) with Neumann boundary conditions $a(x)p_x(x,t) = 0$ for $x = \{0,1\}$. Set $a(x) = 1 + x^2$ and $c = 1/4$ or $c = -1/4$. Given the exact solutions

$$p(x,t) = (t^\alpha + t)\cos(2\pi x), \quad u(x,t) = 2\pi(t^\alpha + t)(1 + x^2)\sin(2\pi x),$$

such that the source function $f(x,t)$ can be computed accordingly.

In the context of Neumann boundary conditions, the classical $L1$-compact BCFD scheme and its fast version are respectively proposed as follows:

$$\begin{cases} \hat{\mathcal{L}}_x\left(\delta_t^\alpha P_i^m + cP_i^m\right) + \delta_x U_i^m = \tilde{\mathcal{L}}_x f_i^m, & i = 1,\dots,N, \\ \delta_x P_{i+1/2}^m + \mathcal{L}_x\left(a^{-1}U\right)_{i+1/2}^m = 0, & i = 1,\dots,N, \\ P_i^0 = p^o(x_i), & i = 1,\dots,N, \end{cases} \tag{4.2}$$

$$\begin{cases} \hat{\mathcal{L}}_x\left({}^F\delta_t^\alpha P_i^m + cP_i^m\right) + \delta_x U_i^m = \tilde{\mathcal{L}}_x f_i^m, & i = 1,\dots,N, \\ \delta_x P_{i+1/2}^m + \mathcal{L}_x\left(a^{-1}U\right)_{i+1/2}^m = 0, & i = 1,\dots,N, \\ P_i^0 = p^o(x_i), & i = 1,\dots,N, \end{cases} \tag{4.3}$$

enclosed with boundary conditions

$$U_{1/2}^m = U_{N+1/2}^m = 0, \tag{4.4}$$

where

$$\hat{\mathcal{L}}_x g_i = \begin{cases} \dfrac{26g_1 - 5g_2 + 4g_3 - g_4}{24}, & i = 1, \\ \mathcal{L}_x g_i, & i = 2,\dots,N-1, \\ \dfrac{26g_N - 5g_{N-1} + 4g_{N-2} - g_{N-3}}{24}, & i = N, \end{cases} \tag{4.5}$$

$$\tilde{\mathcal{L}}_x g_i = \begin{cases} \dfrac{g_{1/2}+4g_1+g_{3/2}}{6}, & i=1, \\ \mathcal{L}_x g_i, & i=2,\ldots,N-1, \\ \dfrac{g_{N-1/2}+4g_N+g_{N+1/2}}{6}, & i=N. \end{cases} \tag{4.6}$$

The same tests as those in Example 4.1 for the fast compact BCFD scheme (4.3)-(4.4) are given in this example. We basically have the following observations:

(i) Although only the fast compact BCFD scheme (2.17)-(2.18) for the time-fractional model (1.1) with periodic boundary conditions is analyzed, it is experimentally demonstrated that the above algorithm can also guarantee convergence of fourth-order accurate in space (see, Tables 8-9 for fixed $N_t = 3000$) and $(2-\alpha)$-th order accurate in time (see, Tables 10-11 for $N = 1000$), in which both positive and negative reaction $c = \pm 1/4$ are tested.

(ii) As anticipated, when $\alpha \to 1^-$, α-robustness of the compact BCFD scheme (4.3)-(4.4) is also verified, see Table 13 for fixed $N_t = N = 200$ and reaction coefficient $c = \pm 1/4$.

(iii) From Table 12, we see that the fast version $L1$-compact BCFD scheme (4.3)-(4.4) has the same accuracy as the conventional $L1$-compact BCFD scheme (4.2) and (4.4), but is much faster than the latter one.

For comparison, we also test the second-order in space fast $L1$-BCFD method for model (1.1)

$$\begin{cases} {}^F\delta_t^\alpha P_i^m + cP_i^m + \delta_x U_i^m = f_i^m, & i=1,\ldots,N, \\ (a\delta_x P)_{i+1/2}^m + U_{i+1/2}^m = 0, & i=1,\ldots,N, \\ P_i^0 = p^o(x_i), & i=1,\ldots,N, \end{cases} \tag{4.7}$$

Table 8: Errors and spatial convergence rates of (4.3) for Example 4.2 with $c = 1/4$.

α	N	Error$_p$	Error$_u$	Rate$_p$	Rate$_u$	
0.3	6	7.5338e-02	3.0875e-02	—	—	
	12	3.5678e-03	2.2579e-03	4.4002	3.7733	
	24	1.9913e-04	1.4269e-04	4.1632	3.9840	≈ 4
	48	1.1743e-05	9.2714e-06	4.0837	3.9439	
0.5	6	6.8840e-02	3.0833e-02	—	—	
	12	3.2034e-03	2.2553e-03	4.4255	3.7730	
	24	1.7967e-04	1.4203e-04	4.1561	3.9890	≈ 4
	48	1.0639e-05	8.6912e-06	4.0778	4.0305	
0.7	6	6.0487e-02	3.0783e-02	—	—	
	12	2.7370e-03	2.2525e-03	4.4659	3.7725	
	24	1.5482e-04	1.4168e-04	4.1439	3.9907	≈ 4
	48	9.2510e-06	8.5096e-06	4.0648	4.0574	

Table 9: Errors and spatial convergence rates of (4.3) for Example 4.2 with $c = -1/4$.

α	N	Error$_p$	Error$_u$	Rate$_p$	Rate$_u$	
0.3	6	1.1592e-01	3.2000e-02	—	—	
	12	5.8574e-03	2.3052e-05	4.3068	3.7950	
	24	3.2151e-04	1.4597e-04	4.1873	3.9811	≈ 4
	48	1.8530e-05	9.5076e-06	4.1169	3.9405	
0.5	6	9.7266e-02	3.1952e-02	—	—	
	12	4.8029e-03	2.3023e-03	4.3399	3.7947	
	24	2.6511e-04	1.4530e-04	4.1792	3.9859	≈ 4
	48	1.5383e-05	8.9268e-06	4.1071	4.0247	
0.7	6	7.8479e-02	3.1897e-02	—	—	
	12	3.7444e-03	2.2990e-03	4.3895	3.7943	
	24	2.0857e-04	1.4493e-04	4.1660	3.9875	≈ 4
	48	1.0455e-05	8.0255e-05	4.0910	4.0510	

Table 10: Errors and temporal convergence rates of (4.3) for Example 4.2 with $c = 1/4$.

α	N_t	Error$_p$	Error$_u$	Rate$_p$	Rate$_u$	
0.3	100	3.1007e-06	2.3876e-05	—	—	
	200	9.8753e-07	7.6042e-06	1.6507	1.6507	
	400	3.1241e-07	2.4056e-06	1.6603	1.6603	≈ 1.7
	800	9.6300e-08	7.5698e-07	1.6978	1.7067	
0.5	100	4.3413e-06	3.3310e-05	—	—	
	200	1.5449e-06	1.1854e-05	1.4905	1.4905	
	400	5.4887e-07	4.2114e-06	1.4930	1.4930	≈ 1.5
	800	1.9473e-07	1.4942e-06	1.4949	1.4948	
0.7	100	5.7776e-06	4.4079e-05	—	—	
	200	2.3429e-06	1.7874e-05	1.3021	1.3022	
	400	9.5082e-07	7.2533e-06	1.3010	1.3011	≈ 1.3
	800	3.8600e-07	2.9445e-06	1.3005	1.3006	

Table 11: Errors and temporal convergence rates of (4.3) for Example 4.2 with $c = -1/4$.

α	N_t	Error$_p$	Error$_u$	Rate$_p$	Rate$_u$	
0.3	100	3.1412e-06	2.4112e-05	—	—	
	200	1.0004e-06	7.6793e-06	1.6507	1.6507	
	400	3.1649e-07	2.4294e-06	1.6603	1.6603	≈ 1.7
	800	9.7485e-08	7.5245e-07	1.6989	1.6909	
0.5	100	4.4002e-06	3.3644e-05	—	—	
	200	1.5659e-06	1.1972e-05	1.4905	1.4905	
	400	5.5631e-07	4.2536e-06	1.4930	1.4930	≈ 1.5
	800	1.9738e-07	1.5092e-06	1.4949	1.4948	
0.7	100	5.8608e-06	4.4532e-05	—	—	
	200	2.3767e-06	1.8057e-05	1.3021	1.3022	
	400	9.6453e-07	7.3279e-06	1.3010	1.3011	≈ 1.3
	800	3.9156e-07	2.9748e-06	1.3005	1.3006	

enclosed with boundary conditions (4.4). This scheme can be derived similarly as [51] for the time-fractional diffusion equation or [23] for the time-fractional Cattaneo equation.

We display the results of the fast $L1$-BCFD scheme for $\alpha = 0.3, 0.5, 0.7$ in Table 14, where $N_t = 3000$ is still taken so that the spatial error dominates the temporal error. Compared with the results in Table 8, it can be seen that the proposed fast $L1$-compact BCFD method can greatly improve the convergence rates, and thus the errors decrase much more faster than the $L1$-BCFD method when the number of spatial grids N increase. For example, when $\alpha = 0.5$, the error order of magnitude for the fast $L1$-BCFD scheme is about 10^{-4} for $N = 192$, while that for the fast $L1$-compact BCFD scheme can reach 10^{-4} for only $N = 24$. Moreover, compared to the fast $L1$-BCFD method, we see from Table 15 that the fast $L1$-compact BCFD method takes less CPU time when the same error accuracy is achieved. For example, when $\alpha = 0.4$, the fast $L1$-compact BCFD scheme (4.3) costs only 9 seconds to get the error order of magnitude 10^{-7}, while the $L1$-BCFD scheme (4.7) runs more than 10 minutes for the same error. The comparison is believed to be more obviously for large-scale modeling and simulations or for high-dimensional model problem.

Table 12: Comparisons of (4.2) and (4.3) for Example 4.2 with $c = 1/4$.

	α	N_t	N	Error_p	Error_u	CPU (s)
Method (4.2)	0.3	1500	40	2.4613e-05	1.8174e-05	23min 6s
		3000	10	7.7798e-03	4.6070e-03	4h 23min
		5000	30	7.9710e-05	5.8093e-05	43h 36min
	0.7	2000	30	6.2330e-05	5.7504e-05	41min 43s
		3000	20	3.2666e-04	2.9458e-04	4h 32min
		6000	15	1.0721e-03	9.3046e-04	47h 24min
Method (4.3)	0.3	1500	40	2.4566e-05	1.8222e-05	12s
		3000	10	7.7799e-03	4.6080e-03	14s
		5000	30	7.9764e-05	5.8091e-05	54s
	0.7	2000	30	6.2330e-05	5.7504e-05	6s
		3000	20	3.2666e-04	2.9458e-04	7s
		6000	15	1.0721e-03	9.3046e-04	18s

Table 13: α-robustness of (4.3) for Example 4.2 with $c = \pm 1/4$.

c		$\alpha = 0.9$	$\alpha = 0.95$	$\alpha = 0.99$	$\alpha = 0.995$
1/4	Error_p	2.1915e-06	1.4302e-06	3.4979e-07	1.7697e-07
	Error_u	1.6673e-05	1.0898e-05	2.6945e-06	1.3763e-06
-1/4	Error_p	2.2249e-06	1.4520e-06	3.5527e-07	1.8008e-07
	Error_u	1.6848e-05	1.1011e-05	2.7222e-06	1.3903e-06

Table 14: Errors and spatial convergence rates of (4.7) for Example 4.2 with $c = 1/4$.

α	N	Error$_p$	Error$_u$	Rate$_p$	Rate$_u$	
0.3	24	1.9228e-02	2.9200e-02	—	—	
	48	4.7933e-03	7.2927e-03	2.0041	2.0014	
	96	1.1975e-03	1.8232e-03	2.0010	1.9999	≈ 2
	192	2.9935e-04	4.5633e-04	2.0001	1.9983	
0.5	24	1.7811e-02	2.9180e-02	—	—	
	48	4.4401e-03	7.2872e-03	2.0041	2.0015	
	96	1.1092e-03	1.8211e-03	2.0010	2.0005	≈ 2
	192	2.7724e-04	4.5512e-04	2.0003	2.0004	
0.7	24	1.6028e-02	2.9162e-02	—	—	
	48	3.9957e-03	7.2826e-03	2.0041	2.0016	
	96	9.9819e-04	1.8197e-03	2.0010	2.0006	≈ 2
	192	2.4947e-04	4.5455e-04	2.0004	2.0011	

Table 15: Comparisons of (4.3) and (4.7) for Example 4.2 with $(N_t, c) = (500, 1/4)$.

	α	N	Error$_p$	Error$_u$	CPU (s)
Method (4.3)	0.4	40	1.5174e-05	1.6568e-05	7s
		60	4.4933e-06	3.0085e-06	8s
		100	5.8798e-07	1.9938e-06	9s
	0.6	40	1.4224e-05	1.5770e-05	7s
		50	5.7209e-06	5.9632e-06	8s
		80	8.7472e-07	3.5406e-06	8s
Method (4.7)	0.4	1000	1.0544e-05	1.4771e-05	35s
		1500	4.6241e-06	5.4875e-06	1min 18s
		4000	6.0327e-07	1.3179e-06	10min 21s
	0.6	800	1.4953e-05	2.2484e-05	25s
		2000	2.2325e-06	1.6333e-06	2min 16s
		3500	7.2200e-07	2.8894e-06	7min 17s

5. Conclusions

We present a fast fourth-order compact BCFD method for the time-fractional reaction-diffusion equations with variably diffusion coefficient and initial weak singularity. For general reaction (positive or negative), and discretization on staggered uniform spatial meshes and graded temporal meshes, an α-robust unconditional stability and optimal-order sharp error analysis are rigorously analyzed. This seems to be the first paper on such analysis of fast high-order finite difference methods. Finally, some numerical experiments are tested to verify the effectiveness, efficiency and robustness of the developed method. Meanwhile, a fast compact BCFD method for Neumann boundary conditions is also developed and tested, numerical results show that the method is also fourth-order accurate in space and $(2-\alpha)$-th order accurate in time. However, up to now stability and error analysis are still lack.

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