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Richardson Extrapolation of the Euler Scheme for Backward Stochastic Differential Equations

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Abstract. In this work, we consider Richardson extrapolation of the Euler scheme for backward stochastic differential equations (BSDEs). First, applying the Adomian decomposition to the nonlinear generator of BSDEs, we introduce a new system of BSDEs. Then we theoretically prove that the solution of the Euler scheme for BSDEs admits an asymptotic expansion, in which the coefficients in the expansions are the solutions of the system. Based on the expansion, we propose Richardson extrapolation algorithms for solving BSDEs. Finally, some numerical tests are carried out to verify our theoretical conclusions and to show the stability, efficiency and high accuracy of the algorithms.

AMS subject classifications: 65C30, 60H10, 60H35

Key words: Backward stochastic differential equations, Euler scheme, Adomian decomposition, Richardson extrapolation, asymptotic error expansion.

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space with $\mathbb{F} = \{\mathcal{F}_t\}_{0\leq t\leq T}$ being the natural filtration generated by a standard d_1-dimensional Brownian motion $W_t = (W_t^1, W_t^2, \cdots, W_t^{d_1})^\top$, $0 \leq t \leq T$. We consider the following BSDE that is defined on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$:

$$Y_t = \varphi(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)\,\mathrm{d}s - \int_t^T Z_s\,\mathrm{d}W_s, \tag{1.1}$$

where T is a deterministic terminal time, $\varphi : \mathbb{R}^d \longrightarrow \mathbb{R}^q$ and $f : [0,T] \times \mathbb{R}^d \times \mathbb{R}^q \times \mathbb{R}^{q\times d_1} \longrightarrow \mathbb{R}^q$ are the terminal condition and the generator of the BSDE (1.1), respectively. Note that the stochastic integral with respect to W_t is of Itô's type, and X_t is a diffussion process. In this paper, we only consider the case where

$$X_t = X_0 + \int_0^t b(s, X_s)\,\mathrm{d}s + \int_0^t \sigma(s, X_s)\,\mathrm{d}W_s, \quad 0 \leq t \leq T, \tag{1.2}$$

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where the functions $b : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ and $\sigma : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^{d\times d_1}$ are called the drift and the diffusion coefficients of the SDE (1.2). A pair of processes (Y_t, Z_t) is called an L^2-adapted solution of (1.1) if it is $\mathcal{F}_t$-adapted, square integrable, and satisfies the BSDE (1.1).

In 1990, the existence and uniqueness of the solution of BSDEs are proved by Pardoux and Peng [27]. Since then, BSDEs becomes an important tool for formulating many problems in various important areas such as mathematical finance, stochastic optimal control, risk measure, game theory, and so on (see, e.g., [11, 25, 28, 31] and references therein).

As it is often difficult to solve BSDEs analytically, even for the linear BSDEs, numerical methods have become popular tools for solving BSDEs. In recent years, great efforts have been made for designing efficient numerical schemes for BSDEs and forward backward stochastic differential equations (FBSDEs). There are two main types of numerical schemes: the first one is based on numerical solution of a parabolic PDE which is related to a FBSDEs [10, 24], while the second type of schemes focus on discretizing FBSDEs directly [3, 5, 9, 17, 23, 32, 37]. From the temporal discretization viewpoint, popular strategies include Euler-type methods [14, 15, 35], θ-schemes [33, 39], Runge-Kutta schemes [8], multistep schemes [7, 13, 38, 40, 41], and strong stability preserving multistep (SSPM) schemes [12], to name a few. For fully coupled FBSDEs, there exists only few numerical studies and satisfactory results [26, 38]. We mention the work in [38], where a class of multistep type schemes are proposed, which turns out to be effective in obtaining highly accurate solutions of FBSDEs, and the work in [34], where the classical deferred correction (DC) method is adopted to design highly accurate numerical methods for fully coupled FBSDEs.

Our objective in this paper is to present a theoretical analysis for the Richardson extrapolation (RE) of the numerical solutions of the Euler scheme for BSDEs. It is well known that the extrapolation method, which was established by Richardson [30], is an efficient procedure for increasing the accuracy of approximations of many problems in numerical analysis. The effectiveness of this method relies heavily on the existence of an asymptotic expansion for the error of the numerical method which is used for the extrapolation procedure. The applications of RE to ordinary differential equations (ODEs) based on one-step schemes, e.g., Runge-Kutta methods are described, for example in [6, 16]. In addition, this method has been well demonstrated in its applications to the finite element and the mixed finite element methods for elliptic partial differential equations [4], Sobolev- and viscoelasticity-type equations [21], partial integro-differential equations [22], Fredholm and Volterra integral equations of the second kind [19], Volterra integro-differential equations [36], and to collocation methods in [20], etc..

The effectiveness and the high accuracy of Richardson extrapolation motivate us to use it to improve the accuracy of the solutions of the Euler scheme for BSDEs. To this end, we first theoretically prove that the solutions of the Euler scheme for BSDEs admit asymptotic expansions where the coefficients in the expansions satisfy a specific BSDEs system. To the best of our knowledge, such theoretical results are new in literature.

Then based on the expansions, we adopt the Aitken-Neville algorithm to present our Richardson extrapolation of Euler methods (RE-Euler, for short) for solving BSDEs, where three kinds of step-number sequences named Romberg, Bulirsch and harmonic sequences are in use for obtaining high-order RE-Euler methods. Finally, numerical tests are given to verify our theoretical results and to demonstrate that our RE-Euler methods 1) enjoy K-order convergence in time discretization for solving BSDEs for $1 \leq K \leq 5$; 2) are stable and more efficient than the multistep schemes proposed in [38].

We in this work mainly focus on analyzing the asymptotic error expansions of the Euler scheme for solving BSDEs. The main contributions of this paper are as follows:

- Applying the Adomian decomposition to the nonlinear generator of BSDEs, we introduce a new system of BSDEs.
- We theoretically prove that the solution of the Euler scheme for BSDEs admits an asymptotic expansion, in which the coefficients in the expansions are the solutions of the new system of BSDEs.
- Based on the expansion, we propose Richardson extrapolation algorithms for solving BSDEs.
- Our numerical tests verify our theoretical conclusions, and show that the RE-Euler methods are easy in use, stable, very efficient and high accurate.

The rest of the paper is organized as follows. In Section 2, we recall the Adomian decomposition and the Richardson extrapolation method in brief. We present the asymptotic error expansions of the solutions of the Euler scheme for BSDEs in Section 3. The construction of the RE-Euler algorithms for BSDEs is presented in Section 4. And in Section 5, numerical tests are carried out to support the theoretical results. Finally, some concluding remarks are given in Section 6.

2. Preliminaries

2.1. Nonlinear Feynman-Kac formula

Let $u \in C^{1,2}([0,T] \times \mathbb{R}^d; \mathbb{R}^q)$ be the solution to the parabolic partial differential equation (PDE)

$$L^0 u(t,x) + f\big(t, x, u(t,x), \nabla_x u(t,x)\sigma(t,x)\big) = 0, \quad (t,x) \in [0,T) \times \mathbb{R}^d \tag{2.1}$$

with the terminal condition $u(T,x) = \varphi(x)$. Here C^{k_1,k_2} refers to the set of functions $g(t,x)$ with continuous partial derivatives up to k_1 with respect to t, and up to k_2 with respect to x. L^0 is a second order differential operator defined by

$$L^0 := \frac{\partial}{\partial t} + \frac{1}{2}\sum_{i,j=1}^{d}\sum_{l=1}^{d_1}(\sigma_{il}\sigma_{jl})(t,x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(t,x)\frac{\partial}{\partial x_i}. \tag{2.2}$$

In 1991, Peng [29] proved that under certain regularity conditions, the solution u of the PDE (2.1) can be expressed as

$$u(t,X_t)=Y_t,\quad \nabla_x u(t,X_t)\sigma(t,X_t)=Z_t,\quad t\in[0,T). \tag{2.3}$$

The representation (2.3) is known as the nonlinear Feynman-Kac formula.

2.2. The diffusion process generator

Definition 2.1. *Let X_t be a diffusion process in $\mathbb{R}^d$ satisfying* (1.2)*. Then the generator D_t^x of X_t on $g:[0,T]\times\mathbb{R}^d$ is defined by*

$$D_t^x g(t,x)=\lim_{s\downarrow t}\frac{\mathbb{E}_t^x[g(s,X_s)]-g(t,x)}{s-t},\quad x\in\mathbb{R}^d, \tag{2.4}$$

if the limit exists, where $\mathbb{E}_t^x[\cdot]$ is the conditional expectation $\mathbb{E}[\cdot|\mathcal{F}_t,X_t=x]$ for $(t,x)\in[0,T]\times\mathbb{R}^d$.

Note that $D_t^x g(t,x)=L^0 g(t,x)$ when $g\in C^{1,2}([0,T]\times\mathbb{R}^d)$. By Definition 2.1, Itô's formula and the tower rule of conditional expectations, we have the following lemma.

Lemma 2.1 ([38])**.** *Let $t\in[0,s]$ be a fixed time. If*

$$g\in C_b^{1,2}\big([0,T]\times\mathbb{R}^d\big),\quad \mathbb{E}_t^x\big[L^0 g(s,X_s)\big]<+\infty,$$

then for $s\in[t,T)$ we have the identity

$$\frac{d\mathbb{E}_t^x[g(s,X_s)]}{ds}=\mathbb{E}_t^x\big[L^0 g(s,X_s)\big].$$

Proof. By Definition 2.1, we have

$$L^0 g(s,X_s)=\lim_{r\downarrow s}\frac{\mathbb{E}_s^{X_s}[g(r,X_r)]-g(s,X_s)}{r-s}. \tag{2.5}$$

Taking the conditional expectation $\mathbb{E}_t^x[\cdot]$ on both sides of (2.5), we have

$$\mathbb{E}_t^x\big[L^0 g(s,X_s)\big]=\mathbb{E}_t^x\left[\lim_{r\downarrow s}\frac{\mathbb{E}_s^{X_s}[g(r,X_r)]-g(s,X_s)}{r-s}\right]. \tag{2.6}$$

Note that we can exchange the order of the limit and the conditional expectation in (2.6) on account of the condition $g\in C_b^{1,2}([0,T]\times\mathbb{R}^d)$. Then we have

$$\begin{aligned}\mathbb{E}_t^x[L^0 g(s,X_s)]&=\mathbb{E}_t^x\left[\lim_{r\downarrow s}\frac{\mathbb{E}_s^{X_s}[g(r,X_r)]-g(s,X_s)}{r-s}\right]\\&=\lim_{r\downarrow s}\frac{\mathbb{E}_t^x[\mathbb{E}_s^{X_s}[g(r,X_r)]]-\mathbb{E}_t^x[g(s,X_s)]}{r-s}\\&=\lim_{r\downarrow s}\frac{\mathbb{E}_t^x[g(r,X_r)]-\mathbb{E}_t^x[g(s,X_s)]}{r-s}\\&=\frac{d\mathbb{E}_t^x[g(s,X_s)]}{ds}.\end{aligned} \tag{2.7}$$

The proof is complete. ☐

As a direct corollary of Lemma 2.1, we have

Corollary 2.1. *If* $g \in C_b^{k,2k}([0,T]\times\mathbb{R}^d)$ *and* $\mathbb{E}_t^x[(L^0)^{(k)}g(s,X_s)] < +\infty$, *then for* $t \in [0,s]$ *we have*

$$\frac{d^k \mathbb{E}_t^x[g(s,X_s)]}{ds^k} = \mathbb{E}_t^x\left[(L^0)^{(k)}g(s,X_s)\right],$$

where $(L^0)^{(k)} = \underbrace{L^0 \circ \cdots \circ L^0}_{k \text{ times}}$.

2.3. Adomian decomposition

Let $G : \mathscr{X} \to \mathscr{Y}$ be a nonlinear operator, where $\mathscr{X}$ and $\mathscr{Y}$ are two Banach spaces, and $\mathbf{u} \in \mathscr{X}$ have the series form $\mathbf{u} = \sum_{j=0}^{\infty} \mathbf{u}_j$. Then $G\mathbf{u}$ can be decomposed into an infinite series of the form

$$G\mathbf{u} = \sum_{j=0}^{\infty} A_j^G, \tag{2.8}$$

where A_j^G are the so-called Adomian polynomials of $\mathbf{u}_0, \mathbf{u}_1, \cdots, \mathbf{u}_j$ and are calculated by

$$A_j^G = \frac{1}{j!}\left[\frac{d^j}{d\lambda^j} G(\sum_{i=0}^{\infty} \lambda^i \mathbf{u}_i)\right]_{\lambda=0}, \quad j = 0,1,2,\ldots. \tag{2.9}$$

Note that the polynomials A_j^G are generated for the nonlinearity, so that A_j^G depends only on $\mathbf{u}_0, \mathbf{u}_1, \cdots, \mathbf{u}_j$ for $j \geq 0$. We call (2.8) the Adomian decomposition of $G\mathbf{u}$. The Adomian decomposition was proposed by Adomian [1,2] initially with the aims to solve frontier problems, linear and nonlinear, in physics, biology and chemical reactions, etc.. To show the use of the Adomian decomposition in solving nonlinear problems, we choose the nonlinear equation as

$$G\mathbf{u} = L\mathbf{u} + F\mathbf{u} = 0, \tag{2.10}$$

where $L\mathbf{u}$ is the linear term, $F\mathbf{u}$ is the nonlinear term, and $\mathbf{u} = (u, v)$.

Assume the inverse L^{-1} of the linear operator L exist. Taking L^{-1} in both sides of (2.10) gives

$$\mathbf{u} = -L^{-1}F\mathbf{u}. \tag{2.11}$$

Assume

$$\mathbf{u}(t) = \sum_{j=0}^{\infty} \mathbf{u}_j(t) = \left(\sum_{j=0}^{\infty} u_j(t), \sum_{j=0}^{\infty} v_j(t)\right).$$

Then by using the Adomian decomposition to $F\mathbf{u} = F(t, \mathbf{u}(t))$, we have

$$\sum_{j=0}^{\infty} \mathbf{u}_j = -L^{-1}\sum_{j=0}^{\infty} A_j^F, \tag{2.12}$$

where A_j^F are calculated by

$$A_j^F(t) = \frac{1}{j!}\left[\frac{d^j}{d\lambda^j}F\left(t, \sum_{i=0}^{\infty}\lambda^i u_i(t), \sum_{i=0}^{\infty}\lambda^i v_i(t)\right)\right]_{\lambda=0}, \quad j = 0, 1, 2, \ldots.$$

Here we list the first few Adomian polynomials $A_j^F(t)$, $j = 0, 1, 2, 3$ which are

$$\begin{aligned}
A_0^F(t) &= F_{0,0},\\
A_1^F(t) &= u_1(t)F_{1,0} + v_1(t)F_{0,1},\\
A_2^F(t) &= u_2(t)F_{1,0} + v_2(t)F_{0,1} + \big(u_1^2(t)/2!\big)F_{2,0}\\
&\quad + u_1(t)v_1(t)F_{1,1} + \big(v_1^2(t)/2!\big)F_{0,2},\\
A_3^F(t) &= u_3(t)F_{1,0} + v_3(t)F_{0,1} + u_1(t)u_2(t)F_{2,0}\\
&\quad + \big[u_1(t)v_2(t) + u_2(t)v_1(t)\big]F_{1,1} + v_1(t)v_2(t)F_{0,2}\\
&\quad + \big(u_1^3(t)/3!\big)F_{3,0} + \big(u_1^2(t)/2!\big)v_1(t)F_{2,1}\\
&\quad + u_1(t)\big(v_1^2(t)/2!\big)F_{1,2} + \big(v_1^3(t)/3!\big)F_{0,3},
\end{aligned} \tag{2.13}$$

where $F_{\mu,\nu} = (\partial^{\mu+\nu}/\partial u^\mu \partial v^\nu)F(t, u_0(t), v_0(t))$. It is worthy of noting that in (2.13), $A_0^F(t) = F(t, u_0(t), v_0(t))$, and for $j \geq 1$, A_j^F is linear with respect to u_j and v_j.

Given $\mathbf{u}_0$, we solve the $\mathbf{u}_j$ $(j = 1, 2, \ldots)$ by

$$\mathbf{u}_j = -L^{-1}A_{j-1}^F. \tag{2.14}$$

We call the procedure (2.12)-(2.14) the Adomian decomposition method for solving the nonlinear problem (2.10).

2.4. Richardson extrapolation

Consider a problem with exact solution $y(t)$, where $t \in [0, T]$. Let $\tilde{y}(t; \Delta t)$ be a numerical solution of $y(t)$ on a uniform grid $\pi_N := \{t_n \,|\, t_n = n\Delta t,\ \Delta t = T/N, n = 0, 1, \ldots, N\}$, where Δt is the step size, N is a positive integer. Assume that the exact solution $y(t)$ is smooth enough on the domain $[0, T]$ such that $\tilde{y}(t; \Delta t)$ admits the asymptotic expansion on π_N

$$\tilde{y}(t; \Delta t) - y(t) = e_1(t)\Delta t + \cdots + e_{K-1}(t)(\Delta t)^{K-1} + E_K(t)(\Delta t)^K, \tag{2.15}$$

where the $e_j(t)$ are independent of Δt with $e_j(t_0) = 0$, and $E_K(t)$ is bounded.

Now we choose a sequence of positive integers

$$1 = N_0 < N_1 < N_2 < \cdots, \tag{2.16}$$

and define the corresponding uniform grids $\pi_{N,i}$ $(i = 0, 1, \ldots, K-1)$ by

$$\pi_{N,i} = \left\{t_n \mid t_n = n\Delta t_i,\ \Delta t_i = \frac{T}{N \cdot N_i} = \frac{\Delta t}{N_i},\ n = 0, 1, \ldots, N \cdot N_i\right\}. \tag{2.17}$$

Note that $\pi_{N,0} = \pi_N$, and all the $\pi_{N,i}$, $i = 0, 1, \ldots, K-1$ have the common grid points in $\pi_{N,0}$. Then for any $t_n \in \pi_{N,0}$ (sometimes we also say $n \in \pi_{N,0}$ which means n is a nonnegative integer such that $t_n \in \pi_{N,0}$), and $1 \le p \le m \le K-1$, by (2.15), we have

$$\tilde{y}(t_n; \Delta t_i) - y(t_n) = e_1(t_n)\Delta t_i + \cdots + e_p(t_n)(\Delta t_i)^p + \mathcal{O}\left((\Delta t_i)^{p+1}\right). \tag{2.18}$$

By multiplying $\alpha_i \in \mathbb{R}$ on both sides of (2.18) and adding the derived equations up from $i = m-p$ to m, we obtain

$$\begin{aligned}
&\sum_{i=m-p}^{m} \alpha_i \tilde{y}(t_n; \Delta t_i) - \left(\sum_{i=m-p}^{m} \alpha_i\right) y(t_n) \\
=& \sum_{i=m-p}^{m} \sum_{j=1}^{p} \alpha_i e_j(t_n)(\Delta t_i)^j + \mathcal{O}\left((\Delta t_{m-p})^{p+1} \sum_{i=m-p}^{m} \alpha_i\right) \\
=& \sum_{j=1}^{p} \left(\sum_{i=m-p}^{m} \frac{\alpha_i}{N_i^j}\right) e_j(t_n)(\Delta t)^j + \mathcal{O}\left((\Delta t)^{p+1} \sum_{i=m-p}^{m} \alpha_i\right).
\end{aligned} \tag{2.19}$$

Since $N_i \neq N_j$ in (2.16) for $i \neq j$, the system of equations (2.20)

$$\begin{pmatrix} 1 & \cdots & 1 \\ N_{m-p}^{-1} & \cdots & N_m^{-1} \\ \vdots & \ddots & \vdots \\ N_{m-p}^{-p} & \cdots & N_m^{-p} \end{pmatrix} \begin{pmatrix} \alpha_{m-p} \\ \alpha_{m-p+1} \\ \vdots \\ \alpha_m \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{2.20}$$

has a unique solution $\boldsymbol{\alpha} = (\alpha_{m-p}, \alpha_{m-p+1}, \cdots, \alpha_m)^\top$. Then from (2.19), we have

$$\sum_{i=m-p}^{m} \alpha_i \tilde{y}(t_n; \Delta t_i) - y(t_n) = \mathcal{O}\left((\Delta t)^{p+1}\right). \tag{2.21}$$

Let $T_{i,0}^n = \tilde{y}(t_n; \Delta t_i)$, then we define $T_{m,p}^n = \sum_{i=m-p}^{m} \alpha_i T_{i,0}^n$, $1 \le p \le m \le K-1$. All $T_{m,0}^n$, $0 \le m \le K-1$ and $T_{m,p}^n$, $1 \le p \le m \le K-1$ can be arranged in the form of lower triangular matrix

$$\begin{array}{ccccc}
T_{0,0}^n & & & & \\
T_{1,0}^n & T_{1,1}^n & & & \\
T_{2,0}^n & T_{2,1}^n & T_{2,2}^n & & \\
\vdots & \vdots & \vdots & \ddots & \\
T_{K-1,0}^n & T_{K-1,1}^n & T_{K-1,2}^n & \cdots & T_{K-1,K-1}^n
\end{array} \tag{2.22}$$

The above algorithm is called the Richardson extrapolation. And we call $T_{m,p}^n$, $1 \le p \le m \le K-1$ the extrapolation solutions. It is worthy of mentioning that all the values $T_{\cdot,p}^n$ located in the p-th column in (2.22) are the approximations to the exact solution

$y(t_n)$ with error $\mathcal{O}((\Delta t)^{p+1})$. In particular, the entry $T^n_{K-1,K-1}$ is an approximation to $y(t_n)$ with error $\mathcal{O}((\Delta t)^K)$. We can recursively realize the Richardson extrapolation by the following Aitken-Neville algorithm:

$$\begin{aligned} &T^n_{m,0} = \tilde{y}(t_n; \Delta t_m), \\ &T^n_{m,p} = T^n_{m,p-1} + \frac{T^n_{m,p-1} - T^n_{m-1,p-1}}{N_m/N_{m-p} - 1}, \end{aligned} \qquad 1 \le p \le m \le K-1. \tag{2.23}$$

Note that different N_i, $i = 0, 1, \ldots$ in (2.16) leads to different step-number sequences. Here we list three frequently used sequences.

1) Romberg sequence: $N_i = 2^i,\ i = 0, 1, \ldots.$

$$1, 2, 4, 8, 16, 32, 64, \ldots.$$

2) Bulirsch sequence: $N_i = \begin{cases} 1, & i = 0, \\ 2^{(i+1)/2}, & i \text{ is odd}, \\ 1.5 \cdot 2^{i/2}, & i \text{ is even}. \end{cases}$

$$1, 2, 3, 4, 6, 8, 12, \ldots.$$

3) Harmonic sequence: $N_i = i + 1, i = 0, 1, \ldots.$

$$1, 2, 3, 4, 5, 6, 7, \ldots.$$

Note that the above three sequences have the same first two elements 1 and 2. For $i = 2, 3$, Bulirsch and harmonic sequences have the same elements 3 and 4 which are both smaller than the ones of Romberg sequence. And for $i \ge 4$, among the above three step-number sequences, the N_i in the Romberg sequence is the largest one and the N_i in the harmonic sequence is the smallest one. In Section 5, we will compare the efficiency among the three step-number sequenses.

3. Asymptotic expansions of the Euler scheme for BSDEs

We outline this section as follows. In Subsection 3.1, we overview the Euler scheme for BSDEs. Then the asymptotic expansions of the solutions by the Euler scheme are carefully derived in Subsection 3.2, which is the key to investigate the extrapolation approximations. Without loss of generality, we only consider the case of one-dimensional BSDEs (i.e., $d = q = 1$). However, we remark that all results obtained in the sequel also hold for multidimensional BSDEs.

3.1. The Euler scheme for BSDEs

To begin with, we introduce a regular time partition on the time interval $[0,T]$ as

$$\pi_N := \left\{ t_n : t_n = n\Delta t,\ n = 0, 1, \ldots, N,\ \Delta t = \frac{T}{N} \right\}, \tag{3.1}$$

where N is a positive integer. Then we introduce some notations. By $\mathbb{E}_t^x[\cdot]$ we denote the conditional expectation $\mathbb{E}[\cdot|\mathcal{F}_t, X_t = x]$ for $(t,x) \in [0,T] \times \mathbb{R}$, and by $\Delta W_{r,s}$ the increment $W_s - W_r$ of the Brownian motion W_t for $s \geq r$. For simplicity, we represent $W_{t_{n+1}} - W_{t_n}$ by ΔW_{n+1} for $0 \leq n \leq N-1$. Note that the increment ΔW_{n+1} admits the Gaussian distribution with mean zero and variance Δt.

It follows from (1.1) that

$$Y_{t_n} = Y_{t_{n+1}} + \int_{t_n}^{t_{n+1}} f_s \,\mathrm{d}s - \int_{t_n}^{t_{n+1}} Z_s \,\mathrm{d}W_s, \tag{3.2}$$

where $f_s = f(s, X_s, Y_s, Z_s)$. For fixed $x \in \mathbb{R}$, taking the conditional expectation $\mathbb{E}_{t_n}^x[\cdot]$ on (3.2), we obtain

$$Y_{t_n} = \mathbb{E}_{t_n}^x[Y_{t_{n+1}}] + \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^x[f_s] \,\mathrm{d}s. \tag{3.3}$$

By multiplying ΔW_{n+1} on both sides of the Eq. (3.2), taking conditional expectation $\mathbb{E}_{t_n}^x[\cdot]$ and then using the isometry property of Itô integral, we obtain

$$0 = \mathbb{E}_{t_n}^x[Y_{t_{n+1}} \Delta W_{n+1}] + \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^x[f_s \Delta W_{t_n,s}] \,\mathrm{d}s - \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^x[Z_s] \,\mathrm{d}s. \tag{3.4}$$

For the temporal semi-discretizations, we use (Y^n, Z^n) to represent the approximation value of the solution (Y_t, Z_t) of the BSDE (1.1) at the time level $t = t_n$, $n = N, N-1, \ldots, 0$. By simply using the left rule to the integrals in (3.3) and (3.4), we obtain the following Euler scheme, one special case of the generalized θ-scheme proposed in [39].

Scheme 3.1 (Euler Scheme). Given Y^N, for $n = N-1, \ldots, 0$, solve random variables Y^n and Z^n by

$$\begin{aligned} Y^n &= \mathbb{E}_{t_n}^x[Y^{n+1}] + \Delta t f(t_n, x, Y^n, Z^n), \\ \Delta t Z^n &= \mathbb{E}_{t_n}^x[Y^{n+1} \Delta W_{n+1}]. \end{aligned} \tag{3.5}$$

We call $\{(Y^n,\ Z^n)\}_{n=0}^{N-1}$ with the terminal condition Y^N the Euler solution of the BSDE (1.1).

Now we define the local truncation errors R_y^n and R_z^n of Scheme 3.1 as

$$\begin{cases} R_y^n = Y_{t_n} - \mathbb{E}_{t_n}^x[Y_{t_{n+1}}] - \Delta t f(t_n, x, Y_{t_n}, Z_{t_n}), \\ R_z^n = \Delta t Z_{t_n} - \mathbb{E}_{t_n}^x[Y_{t_{n+1}} \Delta W_{n+1}]. \end{cases} \tag{3.6}$$

Error estimates were presented in [39] for the above temporal semidiscrete scheme. It was shown that $\mathbb{E}[|R_y^n|] = \mathcal{O}((\Delta t)^2)$ and $\mathbb{E}[|R_z^n|] = \mathcal{O}((\Delta t)^2)$ for sufficiently small time step Δt under certain regularity conditions on f and φ. It was proved in [39] that the Euler Scheme 3.1 possesses convergence rate of 1. In this paper, we pay attention to improve the accuracy of the Euler solutions for BSDEs by the Richardson extrapolation method. To this end, we shall give the asymptotic error expansions of the Euler solutions which are the theoretical basis for our discussions of extrapolation methods.

3.2. Asymptotic expansions of Scheme 3.1

The purpose of this subsection is to deduce the asymptotic expansion of the Euler Scheme 3.1. To this end, we first derive the asymptotic expansions of the truncation errors R_y^n and R_z^n of the Euler scheme in Subsection 3.2.1. Then in Subsection 3.2.3, we define two processes $Y^{n,[K]}$ and $Z^{n,[K]}$ with given processes $e_t^{y,[j]}$ and $e_t^{z,[j]}$, $1 \leq j \leq K$, and introduce two truncation errors $R_y^{n,[K]}$ and $R_z^{n,[K]}$, which have the expansions (3.40) and (3.41), respectively. When the $e_t^{y,[j]}$ and $e_t^{z,[j]}$ are defined by the BSDE system (3.16) defined in Subsection 3.2.2, the $R_y^{n,[K]}$ and $R_z^{n,[K]}$ have the estimates given in Lemma 3.2. Finally by using the Euler scheme and Lemma 3.2, we obtain the asymptotic expansion of the Euler scheme in Theorem 3.1. To this end, we shall need Assumption 3.1 below.

Assumption 3.1. The functions φ and f in (1.1) are bounded and smooth enough with bounded derivatives.

3.2.1. Asymptotic expansions of the truncation errors of the Euler scheme

For the sake of simplicity, we define the functions

$$U(t) = \mathbb{E}_{t_n}^x[Y_t], \quad V(t) = \mathbb{E}_{t_n}^x[Z_t], \quad \bar{U}(t) = \mathbb{E}_{t_n}^x[Y_t \Delta W_{t_n,t}]. \tag{3.7}$$

Note that U, V and $\bar{U}$ depend on t, t_n and x.

Under certain regularity conditions on f and φ in (1.1), the Feynman-Kac formula (2.3) implies that U, V and $\bar{U}$ are all deterministic functions satisfying

$$U^{'}(t) = -\mathbb{E}_{t_n}^x[f_t], \quad \bar{U}(t_n) = 0, \quad \bar{U}^{'}(t_n) = V(t_n), \tag{3.8}$$

and by taking the j-th derivative with respect to t on both sides of the first and the last equation in (3.7), and taking the limit $t \to t_n$, one obtains

$$U^{(j)}(t_n) = \left.\frac{d^j \mathbb{E}_{t_n}^x[Y_t]}{dt^j}\right|_{t=t_n}, \tag{3.9}$$

$$\bar{U}^{(j)}(t_n) = \left.\frac{d^j \mathbb{E}_{t_n}^x[Y_t \Delta W_{t_n,t}]}{dt^j}\right|_{t=t_n} \tag{3.10}$$

with $j = 1, 2, \ldots, K+1$, where K is a positive integer.

Lemma 3.1. *Under Assumption* 3.1*, the local truncation errors* R_y^n *and* R_z^n *have the asymptotic expansions*

$$\begin{aligned} R_y^n &= \sum_{j=2}^{K+1} \alpha_{t_n,j}(\Delta t)^j + \mathcal{O}\left((\Delta t)^{K+2}\right), \\ R_z^n &= \sum_{j=2}^{K+1} \beta_{t_n,j}(\Delta t)^j + \mathcal{O}\left((\Delta t)^{K+2}\right), \end{aligned} \tag{3.11}$$

where $\alpha_{t_n,j} = -U^{(j)}(t_n)/j!$ *and* $\beta_{t_n,j} = -\bar{U}^{(j)}(t_n)/j!$.

Proof. By (3.6) and (3.7), we obtain

$$R_y^n = U(t_n) - U(t_{n+1}) - \Delta t f\big(t_n, x, U(t_n), V(t_n)\big), \tag{3.12}$$

$$R_z^n = \Delta t V(t_n) - \bar{U}(t_{n+1}). \tag{3.13}$$

Then by using Taylor's expansions to U, V and $\bar{U}$ at $t = t_n$, and the relations in (3.8), we deduce

$$\begin{aligned} R_y^n &= -\sum_{j=1}^{K+1} \frac{U^{(j)}(t_n)}{j!}(\Delta t)^j - \Delta t f\big(t_n, x, U(t_n), V(t_n)\big) + \mathcal{O}\left((\Delta t)^{K+2}\right) \\ &= -\sum_{j=2}^{K+1} \frac{U^{(j)}(t_n)}{j!}(\Delta t)^j + \mathcal{O}\left((\Delta t)^{K+2}\right), \end{aligned} \tag{3.14}$$

$$\begin{aligned} R_z^n &= \Delta t V(t_n) - \left(\sum_{j=1}^{K+1} \frac{\bar{U}^{(j)}(t_n)}{j!}(\Delta t)^j\right) + \mathcal{O}\left((\Delta t)^{K+2}\right) \\ &= -\left(\sum_{j=2}^{K+1} \frac{\bar{U}^{(j)}(t_n)}{j!}(\Delta t)^j\right) + \mathcal{O}\left((\Delta t)^{K+2}\right). \end{aligned} \tag{3.15}$$

The proof is complete. ☐

3.2.2. A new system of BSDEs

To obtain our asymptotic expansions of the Euler solution of the BSDE (1.1), we introduce $\mathcal{F}_t$-adapted stochastic processes $(e_t^{y,[j]}, e_t^{z,[j]})$, $j = 1, 2, \ldots, K$, which are the solutions of the BSDEs

$$\begin{aligned} e_t^{y,[j]} = &\int_t^T \left(\lambda_s^{y,[j]} + \lambda_s^y e_s^{y,[j]} + \lambda_s^z e_s^{z,[j]}\right) \mathrm{d}s \\ &- \int_t^T \left(e_s^{z,[j]} + \lambda_s^{z,[j]}\right) \mathrm{d}W_s, \quad j = 1, 2, \ldots, K, \quad \forall t \in [0, T], \end{aligned} \tag{3.16}$$

where

$$
\begin{aligned}
&\lambda_s^y = \frac{\partial f}{\partial y}(s, X_s, Y_s, Z_s), \quad \lambda_s^z = \frac{\partial f}{\partial z}(s, X_s, Y_s, Z_s), \\
&\lambda_s^{y,[j]} = \frac{Y_s^{<j+1>}}{(j+1)!} + B_j(s) - \lambda_s^y e_s^{y,[j]} - \lambda_s^z e_s^{z,[j]} + \sum_{l=2}^{j} \frac{1}{l!}\big(e_s^{y,[j-l+1]}\big)^{<l>}, \\
&\lambda_s^{z,[j]} = -\frac{\bar{Y}_s^{<j+1>}}{(j+1)!} - \sum_{l=2}^{j} \frac{1}{l!}\big(\bar{e}_s^{y,[j-l+1]}\big)^{<l>}, \\
&B_j(s) = \frac{1}{j!}\left[\frac{d^j}{d\lambda^j} f\left(s, X_s, Y_s + \sum_{i=1}^{K} \lambda^i e_s^{y,[i]}, Z_s + \sum_{i=1}^{K} \lambda^i e_s^{z,[i]}\right)\right]_{\lambda=0},
\end{aligned} \tag{3.17}
$$

and

$$
\begin{aligned}
Y_s^{<j+1>} &= (L^0)^{(j+1)} u(s, X_s), \\
\bar{Y}_s^{<j+1>} &= (L^0)^{(j+1)} \tilde{u}(s, \Delta W_{t,s}), \\
\big(e_s^{y,[j-l+1]}\big)^{<l>} &= (L^0)^{(l)} u^{[j-l+1]}(s, X_s), \\
\big(\bar{e}_s^{y,[j-l+1]}\big)^{<l>} &= (L^0)^{(l)} \tilde{u}^{[j-l+1]}(s, \Delta W_{t,s}).
\end{aligned} \tag{3.18}
$$

Here

$$
\begin{aligned}
&\tilde{u}(s, \Delta W_{t,s}) = u(s, X_s)\Delta W_{t,s}, \\
&\tilde{u}^{[j-l+1]}(s, \Delta W_{t,s}) = u^{[j-l+1]}(s, X_s)\Delta W_{t,s},
\end{aligned}
$$

L^0 is defined by (2.2), and $(L^0)^{(k)} = \underbrace{L^0 \circ \cdots \circ L^0}_{k \text{ times}}$, where the $u : [0,T] \times \mathbb{R} \to \mathbb{R}$ is the solution of the PDE

$$
L^0 u(t,x) + f\big(t, u(t,x), \nabla_x u(t,x)\big) = 0, \quad (t,x) \in [0,T) \times \mathbb{R}
$$

with the terminal condition $u(T,x) = \varphi(x)$, and $u^{[j-l+1]} : [0,T] \times \mathbb{R} \to \mathbb{R}$, $2 \le l \le j \le K$ are the solutions of the PDEs

$$
\begin{aligned}
&L^0 u^{[j-l+1]}(t,x) + \lambda_t^{y,[j-l+1]} - \lambda_t^z \lambda_t^{z,[j-l+1]} + \lambda_t^y u^{[j-l+1]}(t,x) \\
&\quad + \lambda_t^z \nabla_x u^{[j-l+1]}(t,x) = 0, \quad (t,x) \in [0,T) \times \mathbb{R}
\end{aligned}
$$

with the terminal conditions $u^{[j-l+1]}(T,x) = 0$.

Remark 3.1. Let $\tilde{e}_s^{z,[j]} = e_s^{z,[j]} + \lambda_s^{z,[j]}$, then (3.16) can be written as

$$
\begin{aligned}
e_t^{y,[j]} = &\int_t^T \big(\lambda_s^{y,[j]} - \lambda_s^z \lambda_s^{z,[j]} + \lambda_s^y e_s^{y,[j]} + \lambda_s^z \tilde{e}_s^{z,[j]}\big)\, \mathrm{d}s \\
&- \int_t^T \tilde{e}_s^{z,[j]}\, \mathrm{d}W_s, \quad j = 1, 2, \ldots, K, \quad \forall t \in [0,T].
\end{aligned} \tag{3.19}
$$

Note that the BSDEs (3.19) are linear with unknown $(e_t^{y,[j]}, \tilde{e}_t^{z,[j]})$ and the unique solvability of (3.19) can be guaranteed by Assumption 3.1 which implies that the BSDEs (3.16) have the unique solutions $(e_t^{y,[j]}, e_t^{z,[j]})$, $j = 1, 2, \ldots, K$.

Taking the conditional expectation $\mathbb{E}_t^x[\cdot]$ on $Y_s^{<j+1>}$, $\bar{Y}_s^{<j+1>}$, $(e_s^{y,[j-l+1]})^{<l>}$ and $(\bar{e}_s^{y,[j-l+1]})^{<l>}$ defined in (3.18) for $t \leq s \leq T$. Then by Corollary 2.1, we have the identities

$$\begin{aligned}
&\mathbb{E}_t^x\big[Y_s^{<j+1>}\big] = \frac{d^{j+1}\mathbb{E}_t^x[Y_s]}{dt^{j+1}},\\
&\mathbb{E}_t^x\big[\bar{Y}_s^{<j+1>}\big] = \frac{d^{j+1}\mathbb{E}_t^x[Y_s \Delta W_{t,s}]}{dt^{j+1}},\\
&\mathbb{E}_t^x\big[(e_s^{y,[j-l+1]})^{<l>}\big] = \frac{d^l \mathbb{E}_t^x[e_s^{y,[j-l+1]}]}{dt^l},\\
&\mathbb{E}_t^x\big[(\bar{e}_s^{y,[j-l+1]})^{<l>}\big] = \frac{d^l \mathbb{E}_t^x[e_s^{y,[j-l+1]}\Delta W_{t,s}]}{dt^l}.
\end{aligned} \tag{3.20}$$

3.2.3. Asymptotic expansions of the Euler scheme

Now we define $Y^{n,[K]}$ and $Z^{n,[K]}$ by

$$Y^{n,[K]} = Y^n - \sum_{j=1}^K e_{t_n}^{y,[j]}(\Delta t)^j, \quad Z^{n,[K]} = Z^n - \sum_{j=1}^K e_{t_n}^{z,[j]}(\Delta t)^j. \tag{3.21}$$

By the Euler scheme (3.5), we have the two identities

$$\begin{cases}
Y^{n,[K]} = \mathbb{E}_{t_n}^x\big[Y^{n+1,[K]}\big] + \Delta t f^{[K]}\big(t_n, x, Y^{n,[K]}, Z^{n,[K]}\big),\\
\Delta t Z^{n,[K]} = \mathbb{E}_{t_n}^x\big[Y^{n+1,[K]}\Delta W_{n+1}\big]\\
\qquad\qquad + \sum\limits_{j=1}^K \big(\bar{E}^{y,[j]}(t_{n+1}) - \Delta t E^{z,[j]}(t_n)\big)(\Delta t)^j,
\end{cases} \tag{3.22}$$

where

$$\begin{aligned}
\Delta t f^{[K]}(t_n, x, y, z) = \Delta t f\left(t_n, x, y + \sum_{j=1}^K E^{y,[j]}(t_n)(\Delta t)^j, z + \sum_{j=1}^K E^{z,[j]}(t_n)(\Delta t)^j\right)\\
+ \sum_{j=1}^K \big(E^{y,[j]}(t_{n+1}) - E^{y,[j]}(t_n)\big)(\Delta t)^j,
\end{aligned} \tag{3.23}$$

and

$$\begin{aligned}
&E^{y,[k]}(t) = \mathbb{E}_{t_n}^x\big[e_t^{y,[k]}\big],\\
&E^{z,[k]}(t) = \mathbb{E}_{t_n}^x\big[e_t^{z,[k]}\big], \qquad\qquad k = 1, 2, \ldots, K.\\
&\bar{E}^{y,[k]}(t) = \mathbb{E}_{t_n}^x\big[e_t^{y,[k]}\Delta W_{t_n,t}\big],
\end{aligned} \tag{3.24}$$

We define local truncation errors $R_y^{n,[K]}$ and $R_z^{n,[K]}$ as

$$\begin{cases} R_y^{n,[K]} = Y_{t_n} - \mathbb{E}_{t_n}^x[Y_{t_{n+1}}] - \Delta t f^{[K]}(t_n, x, Y_{t_n}, Z_{t_n}), \\ R_z^{n,[K]} = \Delta t Z_{t_n} - \mathbb{E}_{t_n}^x[Y_{t_{n+1}} \Delta W_{n+1}] \\ \qquad - \sum_{j=1}^{K} \left(\mathbb{E}_{t_n}^x \left[e_{t_{n+1}}^{y,[j]} \Delta W_{n+1} \right] - \Delta t e_{t_n}^{z,[j]} \right) (\Delta t)^j, \end{cases} \tag{3.25}$$

where (Y_t, Z_t) is the solution of the BSDE (1.1). Then we have the following Lemma 3.2.

Lemma 3.2. *Let Assumption* 3.1 *hold, and* $(e_t^{y,[j]}, e_t^{z,[j]})$*,* $j = 1, 2, \ldots, K$*, are the solutions of BSDEs* (3.16)*, then we have* $R_y^{n,[K]} = \mathcal{O}((\Delta t)^{K+2})$ *and* $R_z^{n,[K]} = \mathcal{O}((\Delta t)^{K+2})$*.*

Proof. For $t \in [t_n, T]$, by (3.19), we obtain

$$e_{t_n}^{y,[j]} = e_t^{y,[j]} + \int_{t_n}^{t} \left(\lambda_s^{y,[j]} - \lambda_s^z \lambda_s^{z,[j]} + \lambda_s^y e_s^{y,[j]} + \lambda_s^z \tilde{e}_s^{z,[j]} \right) \mathrm{d}s - \int_{t_n}^{t} \tilde{e}_s^{z,[j]} \, \mathrm{d}W_s. \tag{3.26}$$

For fixed $x \in \mathbb{R}$, taking the conditional expectation $\mathbb{E}_{t_n}^x[\cdot]$ on (3.26), we obtain

$$e_{t_n}^{y,[j]} = \mathbb{E}_{t_n}^x \left[e_t^{y,[j]} \right] + \int_{t_n}^{t} \mathbb{E}_{t_n}^x \left[\lambda_s^{y,[j]} - \lambda_s^z \lambda_s^{z,[j]} + \lambda_s^y e_s^{y,[j]} + \lambda_s^z \tilde{e}_s^{z,[j]} \right] \mathrm{d}s. \tag{3.27}$$

By taking the derivative with respect to t on both sides of (3.27), and taking the limit $t \to t_n$, one obtains

$$\begin{aligned} \left. \frac{d\mathbb{E}_{t_n}^x[e_t^{y,[j]}]}{dt} \right|_{t=t_n} &= -\lambda_{t_n}^{y,[j]} + \lambda_{t_n}^z \lambda_{t_n}^{z,[j]} - \lambda_{t_n}^y e_{t_n}^{y,[j]} - \lambda_{t_n}^z \tilde{e}_{t_n}^{z,[j]} \\ &= -\lambda_{t_n}^{y,[j]} - \lambda_{t_n}^y e_{t_n}^{y,[j]} - \lambda_{t_n}^z e_{t_n}^{z,[j]}. \end{aligned} \tag{3.28}$$

Then, by (3.17) and (3.20), we deduce

$$\begin{aligned} \left. \frac{d\mathbb{E}_{t_n}^x[e_t^{y,[j]}]}{dt} \right|_{t=t_n} = & \left. \frac{-1}{(j+1)!} \frac{d^{j+1}\mathbb{E}_{t_n}^x[Y_t]}{dt^{j+1}} \right|_{t=t_n} - B_j(t_n) \\ & - \sum_{l=2}^{j} \frac{1}{l!} \left. \frac{d^l \mathbb{E}_{t_n}^x[e_t^{y,[j-l+1]}]}{dt^l} \right|_{t=t_n}. \end{aligned} \tag{3.29}$$

By multiplying $\Delta W_{t_n,t}$ on both sides of the Eq. (3.26), taking conditional expectation $\mathbb{E}_{t_n}^x[\cdot]$ and then using the isometry property of Itô integral, we have

$$\begin{aligned} -\mathbb{E}_{t_n}^x \left[e_t^{y,[j]} \Delta W_{t_n,t} \right] = & \int_{t_n}^{t} \mathbb{E}_{t_n}^x \left[(\lambda_s^{y,[j]} - \lambda_s^z \lambda_s^{z,[j]} + \lambda_s^y e_s^{y,[j]} + \lambda_s^z \tilde{e}_s^{z,[j]}) \Delta W_{t_n,s} \right] \mathrm{d}s \\ & - \int_{t_n}^{t} \mathbb{E}_{t_n}^x \left[\tilde{e}_s^{z,[j]} \right] \mathrm{d}s. \end{aligned} \tag{3.30}$$

Similarly, by (3.17) and (3.20), taking the derivative with respect to t on both sides of (3.30), and taking the limit $t \to t_n$, we deduce

$$\begin{aligned}\left.\frac{d\mathbb{E}^x_{t_n}[e^{y,[j]}_t \Delta W_{t_n,t}]}{dt}\right|_{t=t_n} &= \frac{-1}{(j+1)!}\left.\frac{d^{j+1}\mathbb{E}^x_{t_n}[Y_t\Delta W_{t_n,t}]}{dt^{j+1}}\right|_{t=t_n} \\ &\quad + e^{z,[j]}_{t_n} - \sum_{l=2}^{j}\frac{1}{l!}\left.\frac{d^l\mathbb{E}^x_{t_n}[e^{y,[j-l+1]}_t\Delta W_{t_n,t}]}{dt^l}\right|_{t=t_n}. \end{aligned} \tag{3.31}$$

By (3.7), (3.24) and (3.25), we obtain

$$\begin{aligned} R^{n,[K]}_y &= U(t_n) - U(t_{n+1}) - \Delta t f\big(t_n, x, Y^{[K]}_{t_n}, Z^{[K]}_{t_n}\big) \\ &\quad - \sum_{j=1}^{K}\big(E^{y,[j]}(t_{n+1}) - E^{y,[j]}(t_n)\big)(\Delta t)^j, \end{aligned} \tag{3.32}$$

$$\begin{aligned} R^{n,[K]}_z &= \Delta t V(t_n) - \bar{U}(t_{n+1}) \\ &\quad - \sum_{j=1}^{K}\big(\bar{E}^{y,[j]}(t_{n+1}) - \Delta t E^{z,[j]}(t_n)\big)(\Delta t)^j, \end{aligned} \tag{3.33}$$

where

$$Y^{[K]}_{t_n} = Y_{t_n} + \sum_{j=1}^{K} E^{y,[j]}(t_n)(\Delta t)^j,$$

$$Z^{[K]}_{t_n} = Z_{t_n} + \sum_{j=1}^{K} E^{z,[j]}(t_n)(\Delta t)^j.$$

By using the Adomian decomposition to $f(t_n, x, Y^{[K]}_{t_n}, Z^{[K]}_{t_n})$ in (3.32), we deduce

$$f\big(t_n, x, Y^{[K]}_{t_n}, Z^{[K]}_{t_n}\big) = \sum_{j=0}^{K} B_j(t_n)(\Delta t)^j + \mathcal{O}\left((\Delta t)^{K+1}\right), \tag{3.34}$$

where $B_j(t_n)$ is calculated by

$$\begin{aligned} B_j(t_n)(\Delta t)^j = \frac{1}{j!}\Bigg[\frac{d^j}{d\lambda^j} f\Bigg(&t_n, X_{t_n}, Y_{t_n} + \sum_{i=1}^{K}\lambda^i E^{y,[i]}(t_n)(\Delta t)^i, \\ & Z_{t_n} + \sum_{i=1}^{K}\lambda^i E^{z,[i]}(t_n)(\Delta t)^i\Bigg)\Bigg]_{\lambda=0}. \end{aligned} \tag{3.35}$$

Using Taylor expansions and (3.8)-(3.10), we deduce that

$$U(t_n) - U(t_{n+1}) = -\sum_{j=1}^{K+1}\frac{U^{(j)}(t_n)}{j!}(\Delta t)^j + \mathcal{O}\left((\Delta t)^{K+2}\right)$$

$$
= -\sum_{j=1}^{K+1} \frac{1}{j!} \left. \frac{d^j \mathbb{E}_{t_n}^x [Y_t]}{dt^j} \right|_{t=t_n} (\Delta t)^j + \mathcal{O}\left((\Delta t)^{K+2}\right), \tag{3.36}
$$

$$
\begin{aligned}
\Delta t V(t_n) - \bar{U}(t_{n+1}) &= \Delta t V(t_n) - \sum_{j=1}^{K+1} \frac{\bar{U}^{(j)}(t_n)}{j!} (\Delta t)^j + \mathcal{O}\left((\Delta t)^{K+2}\right) \\
&= -\sum_{j=2}^{K+1} \frac{1}{j!} \left. \frac{d^j \mathbb{E}_{t_n}^x [Y_t \Delta W_{t_n,t}]}{dt^j} \right|_{t=t_n} (\Delta t)^j + \mathcal{O}\left((\Delta t)^{K+2}\right).
\end{aligned} \tag{3.37}
$$

By the definitions of $E^{y,[j]}$, $\bar{E}^{y,[j]}$ and $E^{z,[j]}$ in (3.24), and using Taylor expansions again, we obtain

$$
\begin{aligned}
& E^{y,[j]}(t_{n+1}) - E^{y,[j]}(t_n) \\
= & \sum_{l=1}^{K-j+1} \frac{1}{l!} \left(E^{y,[j]}\right)^{(l)} (t_n) (\Delta t)^l + \mathcal{O}\left((\Delta t)^{K-j+2}\right) \\
= & \sum_{l=1}^{K-j+1} \frac{1}{l!} \left. \frac{d^l \mathbb{E}_{t_n}^x [e_t^{y,[j]}]}{dt^l} \right|_{t=t_n} (\Delta t)^l + \mathcal{O}\left((\Delta t)^{K-j+2}\right),
\end{aligned} \tag{3.38}
$$

$$
\begin{aligned}
& \bar{E}^{y,[j]}(t_{n+1}) - \Delta t E^{z,[j]}(t_n) \\
= & \sum_{l=1}^{K-j+1} \left(\frac{1}{l!} \left(\bar{E}^{y,[j]}\right)^{(l)} (t_n) (\Delta t)^l \right) - \Delta t E^{z,[j]}(t_n) + \mathcal{O}\left((\Delta t)^{K-j+2}\right) \\
= & \sum_{l=1}^{K-j+1} \left. \frac{d^l \mathbb{E}_{t_n}^x [e_t^{y,[j]} \Delta W_{t_n,t}]}{dt^l} \right|_{t=t_n} (\Delta t)^l - \Delta t e_{t_n}^{z,[j]} + \mathcal{O}\left((\Delta t)^{K-j+2}\right).
\end{aligned} \tag{3.39}
$$

Inserting (3.34), (3.36) and (3.38) into (3.32), then by (3.8), then let $k = j + l - 1$ and using the constraint $1 \le j \le K$, last let $j = k$, we deduce

$$
\begin{aligned}
R_y^{n,[K]} = & -\sum_{j=2}^{K+1} \frac{1}{j!} \left. \frac{d^j \mathbb{E}_{t_n}^x [Y_t]}{dt^j} \right|_{t=t_n} (\Delta t)^j - \Delta t \sum_{j=1}^{K} B_j(t_n) (\Delta t)^j \\
& - \sum_{j=1}^{K} \sum_{l=1}^{K-j+1} \frac{1}{l!} \left. \frac{d^l \mathbb{E}_{t_n}^x [e_t^{y,[j]}]}{dt^l} \right|_{t=t_n} (\Delta t)^{j+l} + \mathcal{O}\left((\Delta t)^{K+2}\right) \\
= & \sum_{j=1}^{K} \Bigg(-\left. \frac{d \mathbb{E}_{t_n}^x [e_t^{y,[j]}]}{dt} \right|_{t=t_n} - \frac{1}{(j+1)!} \left. \frac{d^{j+1} \mathbb{E}_{t_n}^x [Y_t]}{dt^{j+1}} \right|_{t=t_n} - B_j(t_n) \\
& \qquad - \sum_{l=2}^{j} \frac{1}{l!} \left. \frac{d^l \mathbb{E}_{t_n}^x [e_t^{y,[j-l+1]}]}{dt^l} \right|_{t=t_n} \Bigg) (\Delta t)^{j+1} + \mathcal{O}\left((\Delta t)^{K+2}\right).
\end{aligned} \tag{3.40}
$$

Similarly, inserting (3.37) and (3.39) into (3.33), then let $k = j + l - 1$ and using the constraint $1 \le j \le K$, last let $j = k$, we obtain

$$\begin{aligned}
R_z^{n,[K]} =& -\sum_{j=2}^{K+1} \frac{1}{j!} \frac{d^j \mathbb{E}_{t_n}^x [Y_t \Delta W_{t_n,t}]}{dt^j}\bigg|_{t=t_n} (\Delta t)^j \\
&- \sum_{j=1}^{K} \left(\sum_{l=1}^{K-j+1} \frac{d^l \mathbb{E}_{t_n}^x [e_t^{y,[j]} \Delta W_{t_n,t}]}{dt^l}\bigg|_{t=t_n} (\Delta t)^l - \Delta t e_{t_n}^{z,[j]} \right) (\Delta t)^j + \mathcal{O}\left((\Delta t)^{K+2}\right) \\
=& \sum_{j=1}^{K} \left(-\frac{d\mathbb{E}_{t_n}^x [e_t^{y,[j]} \Delta W_{t_n,t}]}{dt}\bigg|_{t=t_n} + e_{t_n}^{z,[j]} - \frac{1}{(j+1)!} \frac{d^{j+1} \mathbb{E}_{t_n}^x [Y_t \Delta W_{t_n,t}]}{dt^{j+1}}\bigg|_{t=t_n} \right. \\
&\left. - \sum_{l=2}^{j} \frac{1}{l!} \frac{d^l \mathbb{E}_{t_n}^x [e_t^{y,[j-l+1]} \Delta W_{t_n,t}]}{dt^l}\bigg|_{t=t_n} \right) (\Delta t)^{j+1} + \mathcal{O}\left((\Delta t)^{K+2}\right). \quad (3.41)
\end{aligned}$$

We observe that the coefficients of $(\Delta t)^j$, $j = 2, 3, \ldots, K+1$ in (3.40) and (3.41) vanish by (3.29) and (3.31), respectively. The proof is complete. $\square$

Now we state our asymptotic expansions results for the Euler scheme 3.1 in the following theorem.

Theorem 3.1 (Asymptotic Expansions of the Euler Scheme)**.** *Under Assumption* 3.1*, and if* $\mathbb{E}[|\eta_{t_N}^{y,[K+1]}|^2] = \mathcal{O}((\Delta t)^{2K+2})$*, then the numerical solutions* Y^n *and* Z^n *by the Euler scheme* 3.1 *have the expansions*

$$Y^n = Y_{t_n} + \sum_{j=1}^{K} e_{t_n}^{y,[j]} (\Delta t)^j + \eta_{t_n}^{y,[K+1]}, \quad (3.42)$$

$$Z^n = Z_{t_n} + \sum_{j=1}^{K} e_{t_n}^{z,[j]} (\Delta t)^j + \eta_{t_n}^{z,[K+1]} \quad (3.43)$$

with the estimate

$$\mathbb{E}\left[|\eta_{t_n}^{y,[K+1]}|^2\right] + \Delta t \sum_{i=n}^{N-1} \mathbb{E}\left[|\eta_{t_n}^{z,[K+1]}|^2\right] \le C(\Delta t)^{2K+2}, \quad (3.44)$$

where $(e_t^{y,[j]}, e_t^{z,[j]})$ *are the solutions of the BSDEs* (3.16)*, and* C *is a positive constant depending only on* T, f*, and* φ*.*

Proof. Let $\eta_{t_n}^{y,[K+1]} = Y^{n,[K]} - Y_{t_n}$ and $\eta_{t_n}^{z,[K+1]} = Z^{n,[K]} - Z_{t_n}$, where $Y^{n,[K]}$ and $Z^{n,[K]}$ are defined by (3.21). Then we have

$$Y^n = Y_{t_n} + \sum_{j=1}^{K} e_{t_n}^{y,[j]} (\Delta t)^j + \eta_{t_n}^{y,[K+1]}, \quad (3.45)$$

$$Z^n = Z_{t_n} + \sum_{j=1}^{K} e_{t_n}^{z,[j]} (\Delta t)^j + \eta_{t_n}^{z,[K+1]}. \quad (3.46)$$

In the rest of the proof, we will go to estimate $(\eta_{t_n}^{y,[K+1]}, \eta_{t_n}^{z,[K+1]})$.

1) **The estimate of $\eta_{t_n}^{y,[K+1]}$.** For $0 \le n \le N-1$, by (3.22) and (3.25), we obtain

$$\begin{aligned}\eta_{t_n}^{y,[K+1]} &= \mathbb{E}_{t_n}^x\big[Y^{n+1,[K]} - Y_{t_{n+1}}\big] + \Delta t\big(f^{n,[K]} - f_{t_n}^{[K]}\big) - R_y^{n,[K]} \\ &= \mathbb{E}_{t_n}^x\big[\eta_{t_{n+1}}^{y,[K+1]}\big] + \Delta t \mathcal{E}_f^{n,[K]} - R_y^{n,[K]}, \end{aligned} \tag{3.47}$$

where

$$\begin{aligned} &\mathcal{E}_f^{n,[K]} = f^{n,[K]} - f_{t_n}^{[K]}, \\ &f_{t_n}^{[K]} = f^{[K]}(t_n, X_{t_n}, Y_{t_n}, Z_{t_n}), \\ &f^{n,[K]} = f^{[K]}\big(t_n, X_{t_n}, Y^{n,[K]}, Z^{n,[K]}\big), \end{aligned}$$

and $f^{[K]}$ is defined by (3.23). By Assumption 3.1 and (3.23), $f^{[K]}$ is uniformly Lipschitz continuous in (y,z), and assume the associated Lipschitz constant is L. Then we have

$$\begin{aligned}\big|\eta_{t_n}^{y,[K+1]}\big| &\le \big|\mathbb{E}_{t_n}^x\big[\eta_{t_{n+1}}^{y,[K+1]}\big]\big| + \Delta t\big|\mathcal{E}_f^{n,[K]}\big| + \big|R_y^{n,[K]}\big| \\ &\le \big|\mathbb{E}_{t_n}^x\big[\eta_{t_{n+1}}^{y,[K+1]}\big]\big| + \Delta t L\left(\big|\eta_{t_n}^{y,[K+1]}\big| + \big|\eta_{t_n}^{z,[K+1]}\big|\right) + \big|R_y^{n,[K]}\big|. \end{aligned} \tag{3.48}$$

Using the inequalities

$$(a+b)^2 \le (1+\gamma\Delta t)a^2 + \left(1 + \frac{1}{\gamma\Delta t}\right)b^2, \quad \left(\sum_{n=1}^m a_n\right)^2 \le m\sum_{n=1}^m a_n^2,$$

for any positive real number γ, we deduce

$$\begin{aligned}\big|\eta_{t_n}^{y,[K+1]}\big|^2 \le\ & (1+\gamma\Delta t)\big|\mathbb{E}_{t_n}^x\big[\eta_{t_{n+1}}^{y,[K+1]}\big]\big|^2 \\ &+ \left(1+\frac{1}{\gamma\Delta t}\right)\left(\Delta t L\left(\big|\eta_{t_n}^{y,[K+1]}\big| + \big|\eta_{t_n}^{z,[K+1]}\big|\right) + \big|R_y^{n,[K]}\big|\right)^2 \\ \le\ & (1+\gamma\Delta t)\big|\mathbb{E}_{t_n}^x\big[\eta_{t_{n+1}}^{y,[K+1]}\big]\big|^2 \\ &+ 3\left(1+\frac{1}{\gamma\Delta t}\right)\left((\Delta t L)^2\left(\big|\eta_{t_n}^{y,[K+1]}\big|^2 + \big|\eta_{t_n}^{z,[K+1]}\big|^2\right) + \big|R_y^{n,[K]}\big|^2\right) \\ =\ & (1+\gamma\Delta t)\big|\mathbb{E}_{t_n}^x\big[\eta_{t_{n+1}}^{y,[K+1]}\big]\big|^2 \\ &+ \left\{3(\Delta t L)^2\left(\big|\eta_{t_n}^{y,[K+1]}\big|^2 + \big|\eta_{t_n}^{z,[K+1]}\big|^2\right) + 3\big|R_y^{n,[K]}\big|^2\right\} \\ &+ \frac{1}{\gamma}\left\{3\Delta t L^2\left(\big|\eta_{t_n}^{y,[K+1]}\big|^2 + \big|\eta_{t_n}^{z,[K+1]}\big|^2\right)\right\} + \frac{3}{\gamma\Delta t}\big|R_y^{n,[K]}\big|^2. \end{aligned} \tag{3.49}$$

2) **The estimate of $\eta_{t_n}^{z,[K+1]}$.** By (3.22) and (3.25), we obtain

$$\begin{aligned}\Delta t\eta_{t_n}^{z,[K+1]} &= \mathbb{E}_{t_n}^x\big[(Y^{n+1,[K]} - Y_{t_{n+1}})\Delta W_{n+1}\big] - R_z^{n,[K]} \\ &= \mathbb{E}_{t_n}^x\big[\eta_{t_{n+1}}^{y,[K+1]}\Delta W_{n+1}\big] - R_z^{n,[K]}, \end{aligned} \tag{3.50}$$

and we have

$$\left|\eta_{t_n}^{z,[K+1]}\right| \leq \frac{1}{\Delta t}\left|\mathbb{E}_{t_n}^{x}\left[\eta_{t_{n+1}}^{y,[K+1]}\Delta W_{n+1}\right]\right| + \frac{1}{\Delta t}\left|R_z^{n,[K]}\right|. \tag{3.51}$$

Then by using Hölder inequality and the inequality $(a+b)^2 \leq (1+\epsilon)a^2+(1+1/\epsilon)b^2$ for any positive real number ϵ, we obtain the inequality

$$\begin{aligned}\left|\eta_{t_n}^{z,[K+1]}\right|^2 \leq{}& (1+\epsilon)\frac{1}{(\Delta t)^2}\left|\mathbb{E}_{t_n}^{x}\left[\eta_{t_{n+1}}^{y,[K+1]}\Delta W_{n+1}\right]\right|^2 \\ &+ \left(1+\frac{1}{\epsilon}\right)\frac{1}{(\Delta t)^2}\left|R_z^{n,[K]}\right|^2. \end{aligned} \tag{3.52}$$

Then, by using

$$\begin{aligned}&\left|\mathbb{E}_{t_n}^{x}\left[\eta_{t_{n+1}}^{y,[K+1]}\Delta W_{n+1}\right]\right|^2 \\ ={}& \left|\mathbb{E}_{t_n}^{x}\left[\left(\eta_{t_{n+1}}^{y,[K+1]} - \mathbb{E}_{t_n}^{x}\left[\eta_{t_{n+1}}^{y,[K+1]}\right]\right)\Delta W_{n+1}\right]\right|^2 \\ \leq{}& \mathbb{E}_{t_n}^{x}\left[\left|\Delta W_{n+1}\right|^2\right]\mathbb{E}_{t_n}^{x}\left[\left(\eta_{t_{n+1}}^{y,[K+1]} - \mathbb{E}_{t_n}^{x}\left[\eta_{t_{n+1}}^{y,[K+1]}\right]\right)^2\right] \\ ={}& \Delta t\left(\mathbb{E}_{t_n}^{x}\left[\left|\eta_{t_{n+1}}^{y,[K+1]}\right|^2\right] - \left|\mathbb{E}_{t_n}^{x}\left[\eta_{t_{n+1}}^{y,[K+1]}\right]\right|^2\right),\end{aligned}$$

we deduce

$$\frac{\Delta t}{1+\epsilon}\left|\eta_{t_n}^{z,[K+1]}\right|^2 \leq \mathbb{E}_{t_n}^{x}\left[\left|\eta_{t_{n+1}}^{y,[K+1]}\right|^2\right] - \left|\mathbb{E}_{t_n}^{x}\left[\eta_{t_{n+1}}^{y,[K+1]}\right]\right|^2 + \frac{1}{\epsilon\Delta t}\left|R_z^{n,[K]}\right|^2. \tag{3.53}$$

3) **The estimate of** (3.44)**.** Add the inequality (3.53) to the inequality (3.49), we get

$$\begin{aligned}&\left|\eta_{t_n}^{y,[K+1]}\right|^2 + \frac{\Delta t}{1+\epsilon}\left|\eta_{t_n}^{z,[K+1]}\right|^2 \\ \leq{}& \mathbb{E}_{t_n}^{x}\left[\left|\eta_{t_{n+1}}^{y,[K+1]}\right|^2\right] + \gamma\Delta t\left|\mathbb{E}_{t_n}^{x}\left[\eta_{t_{n+1}}^{y,[K+1]}\right]\right|^2 \\ &+ \left\{3(\Delta t L)^2\left(\left|\eta_{t_n}^{y,[K+1]}\right|^2 + \left|\eta_{t_n}^{z,[K+1]}\right|^2\right) + 3\left|R_y^{n,[K]}\right|^2\right\} \\ &+ \frac{1}{\gamma}\left\{3\Delta t L^2\left(\left|\eta_{t_n}^{y,[K+1]}\right|^2 + \left|\eta_{t_n}^{z,[K+1]}\right|^2\right)\right\} + \frac{3}{\gamma\Delta t}\left|R_y^{n,[K]}\right|^2 + \frac{1}{\epsilon\Delta t}\left|R_z^{n,[K]}\right|^2 \\ \leq{}& (1+\gamma\Delta t)\mathbb{E}_{t_n}^{x}\left[\left|\eta_{t_{n+1}}^{y,[K+1]}\right|^2\right] + \left(3\Delta t + \frac{3}{\gamma}\right)L^2\Delta t\left(\left|\eta_{t_n}^{y,[K+1]}\right|^2 + \left|\eta_{t_n}^{z,[K+1]}\right|^2\right) \\ &+ \left(3+\frac{3}{\gamma\Delta t}\right)\left|R_y^{n,[K]}\right|^2 + \frac{1}{\epsilon\Delta t}\left|R_z^{n,[K]}\right|^2, \end{aligned} \tag{3.54}$$

which can be further simplified to

$$\begin{aligned}&(1-C_1\Delta t)\mathbb{E}\left[\left|\eta_{t_n}^{y,[K+1]}\right|^2\right] + C_3\Delta t\mathbb{E}\left[\left|\eta_{t_n}^{z,[K+1]}\right|^2\right] \\ \leq{}& (1+C_2\Delta t)\mathbb{E}\left[\left|\eta_{t_{n+1}}^{y,[K+1]}\right|^2\right] + \frac{C_4}{\Delta t}\mathbb{E}\left[\left|R_y^{n,[K]}\right|^2\right] + \frac{1}{\epsilon\Delta t}\mathbb{E}\left[\left|R_z^{n,[K]}\right|^2\right], \end{aligned} \tag{3.55}$$

where

$$C_1=\left(3\Delta t+\frac{3}{\gamma}\right)L^2,\qquad C_2=\gamma,$$
$$C_3=\frac{1}{1+\epsilon}-\left(3\Delta t+\frac{3}{\gamma}\right)L^2,\quad C_4=\frac{3+3\gamma\Delta t}{\gamma}.$$

Now we choose $\epsilon=1$, γ large enough, and Δt_0 sufficiently small, such that if $0<\Delta t\le\Delta t_0$ then $C_1\le C$, $C_2\le C$, $C_4\le C$, $1-C\Delta t>0$, and $C_3>C^*>0$, where C and C^* are two positive constants depending on L. Then for $0<\Delta t\le\Delta t_0$, we obtain from (3.55)

$$\begin{aligned}&(1-C\Delta t)\mathbb{E}\left[\left|\eta_{t_n}^{y,[K+1]}\right|^2\right]+C_3\Delta t\mathbb{E}\left[\left|\eta_{t_n}^{z,[K+1]}\right|^2\right]\\ \le&(1+C\Delta t)\mathbb{E}\left[\left|\eta_{t_{n+1}}^{y,[K+1]}\right|^2\right]+\frac{C}{\Delta t}\mathbb{E}\left[\left|R_y^{n,[K]}\right|^2\right]+\frac{1}{\Delta t}\mathbb{E}\left[\left|R_z^{n,[K]}\right|^2\right].\end{aligned}\tag{3.56}$$

Dividing both sizes of the inequality (3.56) by $1-C\Delta t$, we deduce

$$\begin{aligned}&\mathbb{E}\left[\left|\eta_{t_n}^{y,[K+1]}\right|^2\right]+C_3\Delta t\mathbb{E}\left[\left|\eta_{t_n}^{z,[K+1]}\right|^2\right]\\ \le&\frac{1+C\Delta t}{1-C\Delta t}\mathbb{E}\left[\left|\eta_{t_{n+1}}^{y,[K+1]}\right|^2\right]+\frac{C}{\Delta t(1-C\Delta t)}\mathbb{E}\left[\left|R_y^{n,[K]}\right|^2\right]\\ &+\frac{1}{\Delta t(1-C\Delta t)}\mathbb{E}\left[\left|R_z^{n,[K]}\right|^2\right].\end{aligned}\tag{3.57}$$

From the inequality (3.57), by recursively inserting $\eta_{t_i}^{y,[K+1]}, i=n+1,\ldots,N-1$, we deduce

$$\begin{aligned}&\mathbb{E}\left[\left|\eta_{t_n}^{y,[K+1]}\right|^2\right]+C^*\Delta t\sum_{i=n}^{N-1}\left(\frac{1+C\Delta t}{1-C\Delta t}\right)^{i-n}\mathbb{E}\left[\left|\eta_{t_i}^{z,[K+1]}\right|^2\right]\\ \le&\left(\frac{1+C\Delta t}{1-C\Delta t}\right)^{N-n}\mathbb{E}\left[\left|\eta_{t_N}^{y,[K+1]}\right|^2\right]\\ &+\sum_{i=n}^{N-1}\left(\frac{1+C\Delta t}{1-C\Delta t}\right)^{i-n}\frac{C}{\Delta t(1-C\Delta t)}\mathbb{E}\left[\left|R_y^{i,[K]}\right|^2\right]\\ &+\sum_{i=n}^{N-1}\left(\frac{1+C\Delta t}{1-C\Delta t}\right)^{i-n}\frac{1}{\Delta t(1-C\Delta t)}\mathbb{E}\left[\left|R_z^{i,[K]}\right|^2\right].\end{aligned}\tag{3.58}$$

By Lemma 3.2 and (3.58), for sufficiently small time step Δt, we obtain

$$\mathbb{E}\left[\left|\eta_{t_n}^{y,[K+1]}\right|^2\right]+\Delta t\sum_{i=n}^{N-1}\mathbb{E}\left[\left|\eta_{t_n}^{z,[K+1]}\right|^2\right]\le C(\Delta t)^{2K+2}.\tag{3.59}$$

By (3.45), (3.46) and (3.59), we end the proof. □

4. Extrapolation algorithms of the Euler scheme for BSDEs

In this section, based on the asymptotic expansions (3.42) and (3.43) in Theorem 3.1, we will apply the Richardson extrapolation to the Euler solutions of the Euler scheme 3.1 to obtain much accurate approximations to the solution of BSDE (1.1). To this end, we construct our extrapolation algorithms for BSDEs.

For any $t_n \in \pi_N$, let $(\mathcal{Y}_{i,0}^n, \mathcal{Z}_{i,0}^n)$ be the numerical approximations of the exact solution (Y_{t_n}, Z_{t_n}) of the BSDE (1.1) by Scheme 3.1 with time step sizes Δt_i, $i = 0, 1, \ldots, K-1$. Then we define the extrapolation solutions by $\mathcal{Y}_{m,p}^n = \sum_{i=m-p}^m \alpha_i \mathcal{Y}_{i,0}^n$ and $\mathcal{Z}_{m,p}^n = \sum_{i=m-p}^m \alpha_i \mathcal{Z}_{i,0}^n$, $1 \le p \le m \le K-1$. Here π_N, Δt_i and α_i are defined in Subsection 2.4.

All the extrapolation solutions $\mathcal{Y}_{m,p}^n$ and $\mathcal{Z}_{m,p}^n$, $1 \le p \le m \le K-1$ can be obtained by the Aitken-Neville algorithm in Subsection 2.4

$$\begin{aligned} \mathcal{Y}_{m,p}^n &= \mathcal{Y}_{m,p-1}^n + \frac{\mathcal{Y}_{m,p-1}^n - \mathcal{Y}_{m-1,p-1}^n}{N_m/N_{m-p} - 1}, \\ \mathcal{Z}_{m,p}^n &= \mathcal{Z}_{m,p-1}^n + \frac{\mathcal{Z}_{m,p-1}^n - \mathcal{Z}_{m-1,p-1}^n}{N_m/N_{m-p} - 1}, \end{aligned} \qquad 1 \le p \le m \le K-1. \tag{4.1}$$

Now we summarize our Richardson extrapolation algorithms for solving the BSDEs (1.1) in the following algorithm.

Algorithm 4.1 Richardson Extrapolation of the Euler Solutions for BSDEs

Input: $n_0 \in \pi_{N,0}$, K, $\{N_m\}_{m=0}^{K-1}$, X^{n_0}, $Y^{N_{K-1}}$.
for $m = 0, 1, \ldots, K-1$ **do**
  Set $N = N_0 * N_m$;
  Solve $\{(Y^n, Z^n)\}_{n=n_0}^{N-1}$ by Scheme 3.1 on $\pi_{N,m}$;
  Set $\mathcal{Y}_{m,0}^{n_0} = Y^{n_0}$, $\mathcal{Z}_{m,0}^{n_0} = Z^{n_0}$.
end for
for $m = 1, 2, \ldots, K-1$ **do**
  for $p = 1, 2, \ldots, m$ **do**
    $\mathcal{Y}_{m,p}^{n_0} = \mathcal{Y}_{m,p-1}^{n_0} + \frac{\mathcal{Y}_{m,p-1}^{n_0} - \mathcal{Y}_{m-1,p-1}^{n_0}}{\frac{N_m}{N_{m-p}} - 1}$;
    $\mathcal{Z}_{m,p}^{n_0} = \mathcal{Z}_{m,p-1}^{n_0} + \frac{\mathcal{Z}_{m,p-1}^{n_0} - \mathcal{Z}_{m-1,p-1}^{n_0}}{\frac{N_m}{N_{m-p}} - 1}$.
  end for
end for
return $\mathcal{Y}_{K-1,K-1}^{n_0}$, $\mathcal{Z}_{K-1,K-1}^{n_0}$.

For Algorithm 4.1, we have the following conclusion.

Theorem 4.1. *Under Assumption* 3.1, *and if* $\mathbb{E}[|Y^N - Y_{t_N}|^2] = \mathcal{O}((\Delta t)^{2K+2})$, $\mathbb{E}[|Z^N - Z_{t_N}|^2] = \mathcal{O}((\Delta t)^{2K+2})$, *the numerical solutions* $\mathcal{Y}_{K-1,K-1}^{n_0}$ *and* $\mathcal{Z}_{K-1,K-1}^{n_0}$ *of Algorithm* 4.1

have the estimates

$$
\begin{aligned}
&\mathbb{E}\left[\left|\mathcal{Y}^{n_0}_{K-1,K-1}-Y_0\right|^2\right]\le C(\Delta t)^{2K+2},\\
&\mathbb{E}\left[\left|\mathcal{Z}^{n_0}_{K-1,K-1}-Z_0\right|^2\right]\le C(\Delta t)^{2K+2},
\end{aligned}
\tag{4.2}
$$

where (Y_0, Z_0) *refers to the exact solution of BSDE* (1.1) *at* $t = 0$.

Based on the asymptotic expansion (3.42) and (3.43) in Theorem 3.1, the estimates (4.2) can be obtained by the convergence result of the Aitken-Neville algorithm [18].

Remark 4.1. The $N_m, m = 0, 1, \dots, K-1$ are the first K elements of any step-number sequence for Richardson extrapolation, and different $\{N_m\}_{m=0}^{K-1}$ leads different extrapolation algorithm. Compared with other high accurate multistep methods [38, 41], the RE-Euler methods is self-starting ones. So the RE-Euler methods can be used to give the initializations of numerical solutions of other multistep schemes. The Algorithm 4.1 returns the Euler solution when $K = 1$. $(\mathcal{Y}^0_{m,p}, \mathcal{Z}^0_{m,p})$ is an approximation to the exact solution (Y_{t_0}, Z_{t_0}) of the BSDE (1.1) with error $\mathcal{O}((\Delta t)^{p+1})$. Theoretically, our Algorithm 4.1 can achieve any high-order convergence provided the exact solution of the BSDE (1.1) is smooth enough.

5. Numerical tests

In this section, we will provide several numerical tests to verify our theoretical results and to show effectiveness, efficiency and high-order convergence rate of the proposed RE-Euler methods.

The conditional mathematical expectations $\mathbb{E}^x_{t_n}[\cdot]$ in Scheme 3.1 are evaluated by the Gauss-Hermite quadrature rule with r Gaussian nodes, where the values of the integrands at non-grid points are approximated by local Lagrange interpolations with degree l. For more details about the Gauss-Hermite quadrature rule for $\mathbb{E}^x_{t_n}[\cdot]$, readers may refer to [38, Section 3.4]. To show the accuracy and the efficiency of the extrapolation methods, we will report the errors $|Y_0 - \mathcal{Y}^0_{K-1,K-1}|$ and $|Z_0 - \mathcal{Z}^0_{K-1,K-1}|$ between the numerical solution $(\mathcal{Y}^0_{K-1,K-1}, \mathcal{Z}^0_{K-1,K-1})$ of the RE-Euler methods at $n = 0$ and the exact solution (Y_0, Z_0) at $t = 0$ and the related running times (R.T.). For all tests, if not specified, we take $X_0 = 0.0$ and $T = 1.0$. The space grid points are $x_i = ih, i = 0, \pm1, \pm2, \dots$, with h the spatial step size. The time convergence rates (C.R.) are obtained by linear square fitting to the logarithmic errors and the negative logarithm of the time step size $\Delta t = T/N$. All the numerical tests are implemented in Python 3.9.16 on a laptop with Intel Core i5-12500H 12-Core Processor (2.5GHz), and 16 GB DDR5 RAM (4800MHz).

5.1. Time convergence tests

First, we choose an 1-dimensional BSDE in the form

$$
-\,\mathrm{d}Y_t = f_t\,\mathrm{d}t - Z_t\,\mathrm{d}W_t \tag{5.1}
$$

with the terminal condition

$$Y_T = \frac{\exp\left(T/4 + X_T\right)}{1 + \exp\left(T/4 + X_T\right)},$$

where $f_t = Y_t Z_t - 3Z_t/4$. The analytic solutions Y_t and Z_t of (5.1) are

$$Y_t = \frac{\exp\left(t/4 + X_t\right)}{1 + \exp\left(t/4 + X_t\right)}, \quad Z_t = \frac{\exp\left(t/4 + X_t\right)}{(1 + \exp\left(t/4 + X_t\right))^2}.$$

Remark 5.1. The aim of our choosing numerical example (5.1) is to show the stability, efficiency and high convergency of the algorithms for solving nonlinear BSDEs. In this numerical example, the function f is nonlinear with respect to y and z, but the solution (Y_t, Z_t) is smooth and bounded. Thus the chosen function f can be seen as Lipschitz continuous function.

To demonstrate the accuracy of our extrapolation methods, we test the RE-Euler methods with order up to 5 for three different kinds of step-number sequences named Romberg, Bulirsch and harmonic sequences defined in Subsection 2.4. For the Euler scheme 3.1, we take the spatial step size $h = \Delta t$. And we take $r = 5$ and $l = 9$ such that the errors resulted from the spatial interpolations and approximations of conditional expectations do not limit the accuracy of the extrapolation methods. The relevant results are listed in Tables 1-4.

To show the accuracy of the Richardson extrapolation in (2.22), in Table 1 we list the absolute errors $|Y_0 - \mathcal{Y}^0_{m,p}|$ and $|Z_0 - \mathcal{Z}^0_{m,p}|$, $0 \le p \le m \le 4$ of the RE-Euler method with harmonic sequence for the BSDE (5.1) with time step size $\Delta t = 1/N = 1/8$.

From Table 1, we can find that the errors get smaller and smaller as m and p increase. The reasons are the time step sizes get smaller and smaller as m increases and the time convergence rates get higher and higher as p increases.

Table 1: The absolute errors of the RE-Euler method using harmonic sequence.

m \ p	0	1	2	3	4
0	5.456E-03 3.994E-03				
1	2.793E-03 1.960E-03	1.301E-04 7.330E-05			
2	1.877E-03 1.299E-03	4.403E-05 2.304E-05	9.641E-07 1.338E-06		
3	1.413E-03 9.713E-04	2.213E-05 1.159E-05	2.259E-07 3.504E-07	2.014E-08 2.115E-08	
4	1.133E-03 7.757E-04	1.331E-05 6.900E-06	8.776E-08 1.429E-07	4.334E-09 4.542E-09	3.832E-10 3.897E-10

Table 2: Errors, running times and convergence rates of the RE-Euler method using Romberg sequence.

		$N=10$	$N=12$	$N=14$	$N=16$	$N=18$	C.R.
$K=1$	$\lvert Y_0-\mathcal{Y}^0_{0,0}\rvert$	6.565E-04	5.467E-04	4.684E-04	4.096E-04	3.640E-04	1.003
	$\lvert Z_0-\mathcal{Z}^0_{0,0}\rvert$	3.795E-03	3.171E-03	2.723E-03	2.386E-03	2.123E-03	0.988
$K=2$	$\lvert Y_0-\mathcal{Y}^0_{1,1}\rvert$	1.499E-06	1.112E-06	8.552E-07	6.768E-07	5.485E-07	1.712
	$\lvert Z_0-\mathcal{Z}^0_{1,1}\rvert$	2.968E-05	2.060E-05	1.512E-05	1.158E-05	9.144E-06	2.003
$K=3$	$\lvert Y_0-\mathcal{Y}^0_{2,2}\rvert$	1.047E-07	6.128E-08	3.890E-08	2.621E-08	1.849E-08	2.949
	$\lvert Z_0-\mathcal{Z}^0_{2,2}\rvert$	1.874E-08	1.137E-08	7.383E-09	5.049E-09	3.599E-09	2.808
$K=4$	$\lvert Y_0-\mathcal{Y}^0_{3,3}\rvert$	5.070E-10	2.424E-10	1.299E-10	7.575E-11	4.707E-11	4.044
	$\lvert Z_0-\mathcal{Z}^0_{3,3}\rvert$	3.551E-10	1.579E-10	7.983E-11	4.434E-11	2.645E-11	4.419
$K=5$	$\lvert Y_0-\mathcal{Y}^0_{3,3}\rvert$	9.718E-13	4.177E-13	1.902E-13	8.749E-14	6.928E-14	4.701
	$\lvert Z_0-\mathcal{Z}^0_{4,4}\rvert$	5.855E-12	2.446E-12	1.176E-12	6.211E-13	3.280E-13	4.868

Table 3: Errors, running times and convergence rates of the RE-Euler method using Bulirsch sequence.

		$N=10$	$N=12$	$N=14$	$N=16$	$N=18$	C.R.
$K=1$	$\lvert Y_0-\mathcal{Y}^0_{0,0}\rvert$	6.565E-04	5.467E-04	4.684E-04	4.096E-04	3.640E-04	1.003
	$\lvert Z_0-\mathcal{Z}^0_{0,0}\rvert$	3.795E-03	3.171E-03	2.723E-03	2.386E-03	2.123E-03	0.988
$K=2$	$\lvert Y_0-\mathcal{Y}^0_{1,1}\rvert$	1.499E-06	1.112E-06	8.552E-07	6.768E-07	5.485E-07	1.712
	$\lvert Z_0-\mathcal{Z}^0_{1,1}\rvert$	2.968E-05	2.060E-05	1.512E-05	1.158E-05	9.144E-06	2.003
$K=3$	$\lvert Y_0-\mathcal{Y}^0_{2,2}\rvert$	1.391E-07	8.149E-08	5.175E-08	3.488E-08	2.461E-08	2.947
	$\lvert Z_0-\mathcal{Z}^0_{2,2}\rvert$	2.466E-08	1.502E-08	9.769E-09	6.691E-09	4.775E-09	2.794
$K=4$	$\lvert Y_0-\mathcal{Y}^0_{3,3}\rvert$	1.360E-09	6.498E-10	3.482E-10	2.029E-10	1.261E-10	4.047
	$\lvert Z_0-\mathcal{Z}^0_{3,3}\rvert$	9.973E-10	4.423E-10	2.230E-10	1.236E-11	7.356E-11	4.436
$K=5$	$\lvert Y_0-\mathcal{Y}^0_{3,3}\rvert$	6.449E-12	2.815E-12	1.409E-12	7.375E-13	3.769E-13	4.779
	$\lvert Z_0-\mathcal{Z}^0_{4,4}\rvert$	4.001E-11	1.687E-11	8.050E-12	4.269E-12	2.423E-12	4.771

Table 4: Errors, running times and convergence rates of the RE-Euler method using harmonic sequence.

		$N=10$	$N=12$	$N=14$	$N=16$	$N=18$	C.R.
$K=1$	$\lvert Y_0-\mathcal{Y}^0_{0,0}\rvert$	6.565E-04	5.467E-04	4.684E-04	4.096E-04	3.640E-04	1.003
	$\lvert Z_0-\mathcal{Z}^0_{0,0}\rvert$	3.795E-03	3.171E-03	2.723E-03	2.386E-03	2.123E-03	0.988
$K=2$	$\lvert Y_0-\mathcal{Y}^0_{1,1}\rvert$	1.499E-06	1.112E-06	8.552E-07	6.768E-07	5.485E-07	1.712
	$\lvert Z_0-\mathcal{Z}^0_{1,1}\rvert$	2.968E-05	2.060E-05	1.512E-05	1.158E-05	9.144E-06	2.003
$K=3$	$\lvert Y_0-\mathcal{Y}^0_{2,2}\rvert$	1.391E-07	8.149E-08	5.175E-08	3.488E-08	2.461E-08	2.947
	$\lvert Z_0-\mathcal{Z}^0_{2,2}\rvert$	2.466E-08	1.502E-08	9.769E-09	6.691E-09	4.775E-09	2.794
$K=4$	$\lvert Y_0-\mathcal{Y}^0_{3,3}\rvert$	1.360E-09	6.498E-10	3.482E-10	2.029E-10	1.261E-10	4.047
	$\lvert Z_0-\mathcal{Z}^0_{3,3}\rvert$	9.973E-10	4.423E-10	2.230E-10	1.236E-11	7.356E-11	4.436
$K=5$	$\lvert Y_0-\mathcal{Y}^0_{3,3}\rvert$	7.633E-12	3.370E-12	1.698E-12	9.470E-13	4.300E-13	4.773
	$\lvert Z_0-\mathcal{Z}^0_{4,4}\rvert$	4.780E-11	2.022E-11	9.658E-12	5.151E-12	2.909E-12	4.759

To present the convergence rates with respect to Δt of our RE-Euler methods, we calculate the numerical solutions of the BSDE (5.1) with various time step sizes by the RE-Euler methods with the three step-number sequences and list the absolute errors and the convergence rates in Tables 2-4.

From Tables 2-4, we can draw the following conclusions:

1. Our RE-Euler methods are stable and enjoy the K-order time convergence rates for $1 \leq K \leq 5$ no matter which one of the three step-number sequences is chosen. Such a result is consistent with our theoretical results.

2. For the same time step size $\Delta t = T/N$, when $K = 1, 2$, the RE-Euler methods with three different step-number sequences give the same errors, when $K = 3, 4$, the RE-Euler methods with Bulirsch and harmonic sequences give larger errors than the ones of Romberg sequence and when $K = 5$, the errors from small to large are given by the RE-Euler methods with Romberg, and Bulirsch and harmonic sequences in turn. Such results are consistent with the discussions of the extrapolation algorithm described in Subsection 2.4.

Next we illustrate the accuracy of our RE-Euler methods for a two-dimensional example

$$\begin{cases} \mathrm{d}X_t = \sigma \mathbb{I}_{2\times 2}\,\mathrm{d}W_t, \\ -\,\mathrm{d}Y_t = \left(\left(1 + \dfrac{5}{2}\sigma^2\right) e^{-2t} \dfrac{Y_t}{Y_t^2 + (\boldsymbol{\sigma} Z_t)^2}\right) \mathrm{d}t - Z_t\,\mathrm{d}W_t \end{cases} \tag{5.2}$$

with $X_0 = x_0$ and $Y_T = e^{-T} \sin(X_T^1 + 2X_T^2)$.

Here $Z_t = (Z_t^1, Z_t^2)^\top$, $\mathbb{I}_{2\times 2}$ is the 2 by 2 real identity matrix and $\boldsymbol{\sigma} = (3/\sigma, -1/\sigma)$. The analytical solution is given by

$$\begin{cases} Y_t = e^{-t} \sin(X_t^1 + 2X_t^2), \\ Z_t = \left(e^{-t} \cos(X_t^1 + 2X_t^2), 2e^{-t} \cos(X_t^1 + 2X_t^2)\right). \end{cases} \tag{5.3}$$

Note that $W_t = (W_t^1, W_t^2)^\top$ is a two-dimensional standard Brownian motion. For this example, we focus on tesing the convergence rate in time. We set $x = (0.1, 0.1)^\top$, $\sigma = 0.2$ and $T = 1.0$.

We report the errors $|Y^0 - \mathcal{Y}^0_{K-1,K-1}|$, $|Z_0^1 - \mathcal{Z}^{0,1}_{K-1,K-1}|$ and $|Z_0^2 - \mathcal{Z}^{0,2}_{K-1,K-1}|$ between the numerical solution $(\mathcal{Y}^0_{K-1,K-1}, \mathcal{Z}^{0,1}_{K-1,K-1}, \mathcal{Z}^{0,2}_{K-1,K-1})$ computed by RE-Euler methods at $n = 0$ and the exact solution (Y_0, Z_0^1, Z_0^2) at $t = 0$ in Table 5. Again, it can be clearly seen that the convergence rates of our RE-Euler methods roughly coincide with our theoretical results.

5.1.1. Efficiency tests

In this subsection, we are concerned about the efficiency of our RE-Euler methods. The k-order extrapolation method is denoted by RE-Euler(k). We will use example (5.1) for the efficiency tests of the RE-Euler methods.

Table 5: Errors and convergence rates of the RE-Euler method using harmonic sequence for example (5.2).

K		$N=18$	$N=20$	$N=22$	$N=24$	$N=26$	C.R.
1	$\lvert Y_0-\mathcal{Y}_{0,0}^0\rvert$	1.940E-02	1.735E-02	1.568E-02	1.431E-02	1.316E-02	1.055
	$\lvert Z_0^1-\mathcal{Z}_{0,0}^{0,1}\rvert$	6.328E-04	5.719E-04	5.216E-04	4.794E-04	4.434E-04	0.967
	$\lvert Z_0^2-\mathcal{Z}_{0,0}^{0,2}\rvert$	1.266E-03	1.144E-03	1.043E-03	9.588E-04	8.868E-04	0.967
	R.T.(s)	0.75	1.13	1.56	2.15	3.00	
2	$\lvert Y_0-\mathcal{Y}_{1,1}^0\rvert$	6.300E-04	5.063E-04	4.158E-04	3.476E-04	2.948E-04	2.065
	$\lvert Z_0^1-\mathcal{Z}_{1,1}^{0,1}\rvert$	1.174E-05	9.031E-06	7.168E-06	5.824E-06	4.836E-06	2.413
	$\lvert Z_0^2-\mathcal{Z}_{1,1}^{0,2}\rvert$	2.347E-05	1.806E-05	1.434E-05	1.166E-05	9.670E-06	2.411
	R.T.(s)	11.52	17.97	24.90	34.57	46.98	
3	$\lvert Y_0-\mathcal{Y}_{2,2}^0\rvert$	1.065E-05	7.703E-06	5.750E-06	4.406E-06	3.450E-06	3.065
	$\lvert Z_0^1-\mathcal{Z}_{2,2}^{0,1}\rvert$	1.138E-06	7.621E-07	5.407E-07	3.864E-07	3.018E-07	3.637
	$\lvert Z_0^2-\mathcal{Z}_{2,2}^{0,2}\rvert$	2.255E-06	1.529E-06	1.082E-06	7.882E-07	5.976E-07	3.618
	R.T.(s)	62.62	94.78	136.09	190.49	260.64	
		$N=8$	$N=10$	$N=12$	$N=14$	$N=16$	
4	$\lvert Y_0-\mathcal{Y}_{3,3}^0\rvert$	4.103E-06	1.386E-06	5.755E-07	3.164E-07	1.799E-07	4.507
	$\lvert Z_0^1-\mathcal{Z}_{3,3}^{0,1}\rvert$	1.246E-05	1.746E-06	6.967E-07	3.280E-07	2.129E-07	5.801
	$\lvert Z_0^2-\mathcal{Z}_{3,3}^{0,2}\rvert$	1.294E-05	4.446E-06	1.304E-06	6.449E-07	3.507E-07	5.323
	R.T.(s)	9.33	22.96	45.76	83.11	140.26	

We first compare the RE-Euler methods with the Euler scheme. Then we compare the efficiency of our RE-Euler methods among the three step-number sequences. Finally, we compare our RE-Euler methods, where the harmonic step-number sequence is used, with the multistep schemes proposed in [38], and use DM(k) to denote the k-step k-th order one, where the 'DM' refers to 'Differential Multistep'. All the numerical results are listed in Tables 7-13. In the tables, Y_k^0 and Z_k^0 is the numerical solutions at $n=0$ by the DM(k) scheme.

To compare our RE-Euler methods with the Euler scheme, we calculate the numerical solutions of the BSDE (5.1) with various time step sizes by the Euler scheme and the RE-Euler methods with $K=2$ and list the absolute errors and the runing times in Table 6.

Table 6: Errors and running times of the Euler scheme and the 2-order RE-Euler method.

Euler($K=1$)		$N=32$	$N=64$	$N=128$	$N=256$	$N=512$
	$\lvert Y_0-\mathcal{Y}_{0,0}^0\rvert$	8.033E-05	4.115E-05	2.082E-05	1.047E-05	5.253E-06
	$\lvert Z_0-\mathcal{Z}_{0,0}^0\rvert$	1.459E-03	7.286E-04	3.640E-04	1.819E-04	9.095E-05
	R.T.(s)	0.0810	0.3003	1.6665	9.6058	55.6975
$K=2$		$N=8$	$N=12$	$N=16$	$N=20$	$N=24$
	$\lvert Y_0-\mathcal{Y}_{1,1}^0\rvert$	3.106E-05	1.394E-05	7.873E-06	5.050E-06	3.512E-06
	$\lvert Z_0-\mathcal{Z}_{1,1}^0\rvert$	2.393E-05	1.233E-05	7.423E-06	4.941E-06	3.520E-06
	R.T.(s)	0.0239	0.0563	0.0710	0.1135	0.1645

Table 7: Errors and running times of 3-order RE-Euler methods with different step-number sequences.

Romberg		$N = 20$	$N = 22$	$N = 24$	$N = 26$	$N = 28$
	$\|Y_0 - \mathcal{Y}^0_{2,2}\|$	6.685E-09	4.900E-09	3.695E-09	2.854E-09	2.248E-09
	$\|Z_0 - \mathcal{Z}^0_{2,2}\|$	1.290E-07	9.719E-08	7.505E-08	5.915E-08	4.745E-08
	R.T.(s)	0.541	0.696	0.864	1.013	1.236
Bulirsch(harmonic)		$N = 24$	$N = 26$	$N = 28$	$N = 30$	$N = 32$
	$\|Y_0 - \mathcal{Y}^0_{2,2}\|$	4.983E-09	3.846E-09	3.028E-09	2.426E-09	1.973E-09
	$\|Z_0 - \mathcal{Z}^0_{2,2}\|$	9.993E-08	7.877E-08	6.319E-08	5.146E-08	4.246E-08
	R.T.(s)	0.504	0.608	0.714	0.861	1.028

Table 8: Errors and running times of 4-order RE-Euler methods with different step-number sequences.

Romberg		$N = 20$	$N = 22$	$N = 24$	$N = 26$	$N = 28$
	$\|Y_0 - \mathcal{Y}^0_{3,3}\|$	1.299E-10	8.913E-11	6.318E-11	4.600E-11	3.430E-11
	$\|Z_0 - \mathcal{Z}^0_{3,3}\|$	3.071E-10	2.107E-10	1.493E-10	1.087E-10	8.105E-11
	R.T.(s)	3.004	3.803	4.826	5.899	7.045
Bulirsch(harmonic)		$N = 28$	$N = 30$	$N = 32$	$N = 34$	$N = 36$
	$\|Y_0 - \mathcal{Y}^0_{3,3}\|$	9.110E-11	6.933E-11	5.366E-11	4.221E-11	3.366E-11
	$\|Z_0 - \mathcal{Z}^0_{3,3}\|$	2.153E-10	1.638E-10	1.268E-10	9.972E-11	7.946E-11
	R.T.(s)	1.730	2.040	2.374	2.783	3.184

Table 9: Errors and running times of 5-order RE-Euler methods with Romberg and Bulirsch sequences.

Romberg		$N = 8$	$N = 10$	$N = 12$	$N = 14$	$N = 16$
	$\|Y_0 - \mathcal{Y}^0_{4,4}\|$	2.351E-11	7.463E-12	2.933E-12	1.330E-12	6.587E-13
	$\|Z_0 - \mathcal{Z}^0_{4,4}\|$	3.714E-11	1.361E-11	5.728E-12	2.805E-12	1.510E-12
	R.T.(s)	1.726	3.011	4.732	7.035	9.902
Bulirsch		$N = 12$	$N = 16$	$N = 20$	$N = 24$	$N = 28$
	$\|Y_0 - \mathcal{Y}^0_{4,4}\|$	2.132E-11	4.885E-12	1.518E-12	6.378E-13	3.488E-13
	$\|Z_0 - \mathcal{Z}^0_{4,4}\|$	3.921E-11	1.064E-11	3.676E-12	1.461E-12	6.753E-13
	R.T.(s)	0.577	1.159	1.971	3.068	4.482

Table 10: Errors and running times of 5-order RE-Euler methods with Bulirsch and harmonic sequences.

Bulirsch		$N = 16$	$N = 18$	$N = 20$	$N = 22$	$N = 24$
	$\|Y_0 - \mathcal{Y}^0_{4,4}\|$	4.885E-12	2.643E-12	1.518E-12	9.844E-13	6.378E-13
	$\|Z_0 - \mathcal{Z}^0_{4,4}\|$	1.064E-11	5.619E-12	3.676E-12	2.184E-12	1.461E-12
	R.T.(s)	1.140	1.503	1.947	2.463	3.077
harmonic		$N = 18$	$N = 20$	$N = 22$	$N = 24$	$N = 26$
	$\|Y_0 - \mathcal{Y}^0_{4,4}\|$	3.205E-12	1.893E-12	1.161E-12	7.268E-13	6.057E-13
	$\|Z_0 - \mathcal{Z}^0_{4,4}\|$	7.067E-12	4.332E-12	2.398E-12	1.539E-12	1.255E-12
	R.T.(s)	0.942	1.168	1.497	1.815	2.174

Table 11: Errors and running times the DM(3) scheme and the 3-order RE-Euler method.

DM(3)		$N=64$	$N=68$	$N=72$	$N=76$	$N=80$
	$\lvert Y_0 - Y_3^0\rvert$	2.518E-08	2.122E-08	1.805E-08	1.548E-08	1.338E-08
	$\lvert Z_0 - Z_3^0\rvert$	2.103E-07	1.770E-07	1.503E-07	1.288E-07	1.112E-07
	R.T.(s)	0.845	0.939	1.050	1.179	1.301
RE-Euler(3)		$N=20$	$N=22$	$N=24$	$N=26$	$N=28$
	$\lvert Y_0 - \mathcal{Y}_{2,2}^0\rvert$	9.028E-09	6.613E-09	4.983E-09	3.846E-09	3.028E-09
	$\lvert Z_0 - \mathcal{Z}_{2,2}^0\rvert$	1.717E-07	1.294E-07	9.993E-08	7.877E-08	6.319E-08
	R.T.(s)	0.351	0.435	0.535	0.652	0.774

Table 12: Errors and running times the DM(4) scheme and the 4-order RE-Euler method.

DM(4)		$N=64$	$N=68$	$N=72$	$N=76$	$N=80$
	$\lvert Y_0 - Y_4^0\rvert$	9.056E-10	7.253E-10	5.885E-10	4.713E-10	3.949E-10
	$\lvert Z_0 - Z_4^0\rvert$	1.037E-08	8.322E-09	6.812E-09	5.360E-09	4.746E-09
	R.T.(s)	2.092	2.376	2.688	3.005	3.364
RE-Euler(4)		$N=16$	$N=18$	$N=20$	$N=22$	$N=24$
	$\lvert Y_0 - \mathcal{Y}_{3,3}^0\rvert$	8.286E-10	5.216E-10	3.445E-10	2.365E-10	1.677E-10
	$\lvert Z_0 - \mathcal{Z}_{3,3}^0\rvert$	1.962E-09	1.235E-09	8.145E-10	5.591E-10	3.962E-10
	R.T.(s)	0.480	0.655	0.827	1.029	1.275

Table 13: Errors and running times the DM(5) scheme and the 5-order RE-Euler method.

DM(5)		$N=64$	$N=68$	$N=72$	$N=76$	$N=80$
	$\lvert Y_0 - Y_5^0\rvert$	4.447E-11	3.382E-11	2.607E-11	2.037E-11	1.611E-11
	$\lvert Z_0 - Z_5^0\rvert$	7.502E-10	5.722E-10	4.425E-10	3.466E-10	2.746E-10
	R.T.(s)	3.889	4.149	4.613	5.205	5.797
RE-Euler(5)		$N=12$	$N=14$	$N=16$	$N=18$	$N=20$
	$\lvert Y_0 - \mathcal{Y}_{4,4}^0\rvert$	2.565E-11	1.167E-11	5.951E-12	3.167E-12	1.807E-12
	$\lvert Z_0 - \mathcal{Z}_{4,4}^0\rvert$	4.740E-11	2.309E-11	1.271E-11	6.735E-12	4.527E-12
	R.T.(s)	0.490	0.693	0.939	1.271	1.641

The errors and running times in Table 6 show that to achieve the same or smaller errors, the RE-Euler method with $K = 2$ which enjoys theoretical time convergence rate 2 costs less time than the Euler scheme, which means that our 2-order RE-Euler method is more efficient than the Euler scheme.

To show the efficiency of our RE-Euler methods among the three step-number sequences, we list the errors and the runing times for the K-order ($3 \le K \le 5$) RE-Euler methods with the three step-number sequences in Tables 7-10.

From Tables 7 and 8, we can see that for $K = 3$ and $K = 4$, our RE-Euler methods with Bulirsch and harmonic sequences cost less time than the RE-Euler method with Romberg sequence for achieving the same or smaller errors. From Tables 9 and 10, we can conclude that to achieve the same or smaller errors, RE-Euler method with Bulirsch sequence costs less time than the RE-Euler method with Romberg sequence and RE-Euler method with harmonic sequence costs less time than the RE-Euler method with Bulirsch sequence. Thus the harmonic sequence is the most economical one of the three sequences for our RE-Euler methods. So we will adopt the harmonic sequence for our RE-Euler methods to compare with the DM(k) scheme in the follows.

To compare the efficiency of our RE-Euler(k) method with the harmonic step-number sequence with the DM(k) scheme, we numerically solve the BSDE (5.1) with various time step sizes by the DM(k) scheme and the RE-Euler(k) method, and list the absolute errors and the running times in Tables 11-13.

The errors and the running times in Tables 11-13 show that to achieve the same or smaller errors our RE-Euler methods cost less time than the DM(k) schemes for the same rates of convergence from 3 to 5. So our RE-Euler methods with the harmonic sequence are more efficient than DM(k) schemes.

All the above numerical tests show that:

1. Our RE-Euler methods enjoy K-order convergence in time discretization for solving BSDEs for $1 \leq K \leq 5$.

2. The harmonic sequence is the most economical one of the three sequences for our RE-Euler methods with order from 3 to 5.

3. Our RE-Euler methods are stable and more efficient than the Euler scheme and the DM schemes.

6. Conclusions

In this work, by the theory of backward stochastic differential equations and the Adomian decomposition, we theoretically proved that the solutions of the Euler scheme for solving BSDEs admit asymptotic expansions, where the coefficients in the expansions are determined by a specific BSDE system. Then based on the expansions, we proposed Richardson extrapolation-type algorithms for solving BSDEs. Some numerical tests were carried out. The numerical results of the tests verified our theoretical conclusions, and showed that the algorithms are stable, very efficient and high accurate.

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