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On Order of a Function of Several Complex Variables Analytic in the Unit Polydisc

Ratan Kumar Dutta [+]

Department of Mathematics, Siliguri Institute of Technology, Post.-Sukna, Siliguri, Dist.-Darjeeling, Pin-734009, West Bengal, India

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Abstract: This paper is concerned with the study of the maximum modulus and the coefficients of the power series expansion of a function of several complex variables analytic in the unit polydisc.

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1. Introduction and Definitions

Let $f(z)=\sum_{n=0}^{\infty} c_n z^n$ be analytic in the unit disc $U=\{z:|z|<1\}$ and $M(r)=M(r,f)$ be the maximum of $|f(z)|$ on $|z|=r$.

In 1968 Sons [8] introduced the following definition of the order ρ and the lower order λ as

$$\begin{matrix}\rho\\ \lambda\end{matrix} = \lim_{r\to 1} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log M(r,f)}{-\log(1-r)}.$$

Maclane [6] and Kapoor [5] proved the following results which are the characterization of order and lower order of a function f analytic in U, in terms of the coefficients c_n.

Theorem 1.1 [6] Let $f(z)=\sum_{n=0}^{\infty} c_n z^n$ be analytic in U, having order $\rho\,(0\le\rho\le\infty)$. Then

$$\frac{\rho}{1+\rho} = \limsup_{n\to\infty} \frac{\log^+\log^+|c_n|}{\log n}.$$

Theorem 1.2 [5] Let $f(z)=\sum_{n=0}^{\infty} c_n z^n$ be analytic in U, having lower order $\lambda\,(0\le\lambda\le\infty)$. Then

$$\frac{\lambda}{1+\lambda} \ge \liminf_{n\to\infty} \frac{\log^+\log^+|c_n|}{\log n}.$$

Notation 1.3 [7] $\log^{[0]} x = x$, $\exp^{[0]} x = x$ and for positive integer m, $\log^{[m]} x = \log(\log^{[m-1]} x)$, $\exp^{[m]} x = \exp(\exp^{[m-1]} x)$.

In a paper [4] Juneja and Kapoor introduced the definition of p-th order and lower p-th order and in 2005 Banerjee [1] generalized Theorem 1.1 and Theorem 1.2 for p-th order and lower p-th order respectively.

Definition 1.4 [4] If $f(z)=\sum_{n=0}^{\infty} c_n z^n$ be analytic in U, its p-th order ρ_p and lower p-th order λ_p are defined as

[+] *E-mail address*: ratan_3128@yahoo.com

$$\begin{matrix}\rho_p \\ \lambda_p\end{matrix} = \lim_{r\to 1} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log^{[p]} M(r)}{-\log(1-r)}, \ p \geq 2.$$

Theorem 1.5 [1] Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be analytic in U and having p-th order ρ_p $(0 \leq \rho_p \leq \infty)$. Then

$$\frac{\rho_p}{1+\rho_p} = \limsup_{n\to\infty} \frac{\log^{+[p]} |c_n|}{\log n}.$$

Theorem 1.6 [1] Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be analytic in U and having lower p-th order λ_p $(0 \leq \lambda_p \leq \infty)$. Then

$$\frac{\lambda_p}{1+\lambda_p} \geq \liminf_{n\to\infty} \frac{\log^{+[p]} |c_n|}{\log n}.$$

In 2008 Banerjee and Dutta [2] introduced the following definition.

Definition 1.7 Let $f(z_1, z_2)$ be a non-constant analytic function of two complex variables z_1 and z_2 holomorphic in the closed unit polydisc

$$P: \{(z_1, z_2) : |z_j| \leq 1; j = 1, 2\}$$

then order of f is denoted by ρ and is defined by

$$\rho = \inf\left\{\mu > 0 : F(r_1, r_2) < \exp\left(\frac{1}{1-r_1} . \frac{1}{1-r_2}\right)^{\mu}; \text{for all } 0 < r_0(\mu) < r_1, r_2 < 1\right\}.$$

Equivalent formula for ρ is

$$\rho = \limsup_{r_1, r_2 \to 1} \frac{\log\log F(r_1, r_2)}{-\log(1-r_1)(1-r_2)}.$$

In a resent paper [3] Banerjee and Dutta introduce the definition of p-th order and lower p-th order of functions of two complex variables analytic in the unit polydisc and generalized the above results for functions of two complex variables analytic in the unit polydisc.

Definition 1.8 Let $f(z_1, z_2) = \sum_{m,n=0}^{\infty} c_{mn} z_1^m z_2^n$ be a function of two complex variables z_1, z_2 holomorphic in the unit polydisc

$$U = \{(z_1, z_2) : |z_j| \leq 1; j = 1, 2\}$$

and

$$F(r_1, r_2) = \max\{|f(z_1, z_2)| : |z_j| \leq r_j; j = 1, 2\},$$

be its maximum modulus. Then the p-th order ρ_p and lower p-th order λ_p are defined as

$$\begin{matrix}\rho_p \\ \lambda_p\end{matrix} = \lim_{r_1, r_2 \to 1} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log^{[p]} F(r_1, r_2)}{-\log(1-r_1)(1-r_2)}, \ p \geq 2.$$

When $p = 2$, Definition 1.8 coincides with Definition 1.7.

Theorem 1.9 Let $f(z_1, z_2)$ be analytic in U and having p-th order ρ_p $(0 \leq \rho_p \leq \infty)$. Then

$$\frac{\rho_p}{1+\rho_p} = \limsup_{m,n\to\infty} \frac{\log^{+[p]} |c_{mn}|}{\log mn}.$$

Theorem 1.10 Let $f(z_1, z_2)$ be analytic in U and having lower p-th order λ_p $(0 \leq \lambda_p \leq \infty)$. Then