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Minimal Log Discrepancy and Orbifold Curves

Chi Li[1*] and Zhengyi Zhou[2]

[1] *Department of Mathematics, Rutgers University, Piscataway, NJ 08854-8019, USA;*
[2] *Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China.*

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Abstract. We show that the minimal log discrepancy of any isolated Fano cone singularity is at most the dimension of the variety. This is based on its relation with dimensions of moduli spaces of orbifold rational curves. We also propose a conjectural characterization of weighted projective spaces as Fano orbifolds in terms of orbifold rational curves, which would imply that the equality holds only for smooth points.

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1 Introduction and main result

In birational algebraic geometry, particularly in the study of the Minimal Model Program (MMP), the minimal log discrepancy (mld) is an important invariant for Kawamata log terminal (klt) singularities ([3,12]). Indeed, Shokurov showed that two conjectural properties of mld would imply the termination of flips in the MMP. One of these conjectures states that the mld of closed points is lower semicontinuous as the point moves on a fixed normal variety X of dimension n. Since the mld of a smooth point is equal to n, this in particular implies that $\mathrm{mld}(x,X) \leq n$ ([3, Conjecture 3.2]). In this short essay, we will prove that this sharp upper bound indeed holds for any isolated Fano cone singularity.

Theorem 1.1. *Let $o \in X$ be an isolated Fano cone singularity of dimension n, then* $\mathrm{mld}(o,X) \leq n$.

Remark 1.1. By [15, Theorem 1.1], we find more generally that if the link of an isolated singularity is contactomorphic to the link of an isolated Fano cone singularity, then the same inequality also holds.

Note that when the Fano cone is an ordinary cone over a smooth Fano manifold, the upper bound is equivalent to the well-known fact that the Fano index of a Fano manifold is at most the dimension plus one. This fact can be proved using either the Riemann-Roch theorem or Mori's bend-and-break theory of rational curves. Since the Riemann-Roch approach does not seem to work well in the orbifold setting, our proof uses Mori's theory in the orbifold setting and shows its connection with minimal log discrepancy

*Corresponding author. *Email addresses:* `chi.li@rutgers.edu` (Li C), `zhyzhou@amss.ac.cn` (Zhou Z)

invariants. This connection is motivated by our previous work [14], which in particular expresses the mld invariant in terms of certain symplectic invariants that arise essentially in the study of moduli spaces of pseudo-holomorphic curves. In this paper, we will show that the mld invariant is bounded from above by dimensions of certain moduli spaces of orbifold rational curves with domain a weighted projective line (2.6). We will give two proofs for this crucial inequality, one using purely algebraic geometry and the other from a symplectic perspective. The argument for the above result also leads us to propose a characterization of weighted projective spaces using orbifold rational curves in the same spirit as Mori-Mukai (Conjecture 2.1), which would imply that the equality case occurs only for smooth points. As evidence, we prove an orbifold version of Mori's theorem:

Theorem 1.2. *Let $\mathcal{Y}$ be a complex orbifold whose orbifold tangent bundle $T\mathcal{Y}$ is ample, then $\mathcal{Y}$ is a finite quotient of a weighted projective space.*

See [5] for a related characterization of smooth projective spaces among normal varieties with quotient singularities. The method of proof follows Mori's approach, which uses a family of orbifold rational curves of minimal degree passing through a fixed (orbifold) point to sweep out the whole orbifold. The ampleness is used to deduce that there are no obstructions to the deformation theory of such curves. It is important for this purpose that we use orbifold rational curves with only one non-trivial orbifold point, which guarantees the splitting of any orbifold holomorphic vector bundle into a direct sum of orbifold line bundles (2.7).

2 Proof of main results

We will use the notation from our previous paper [14]. The starting point of our proof is a formula for the mld of an isolated Fano cone singularity (X,o) derived in [14, Proposition 2.12]. To state the formula, we assume that X is given as the orbifold cone $C(\mathcal{Y},\mathcal{L})$ where the orbifold base $\mathcal{Y}=(Y,\Delta)$ is obtained as the quotient $(X\setminus o)/\mathbb{C}^*$ and $\mathcal{L}$ is the associated orbifold line bundle. The klt condition is equivalent to the condition that (Y,Δ) is a klt Fano orbifold and $-K_{\mathcal{Y}}=r\mathcal{L}$ for some $r\in\mathbb{Q}_{>0}$. Throughout this paper, we use $(\mathcal{Y},\mathcal{L})$ to denote a Deligne-Mumford stack (or equivalently an orbifold) equipped with an orbifold line bundle and use (Y,L) to denote the associated coarse moduli space as an algebraic variety equipped with a $\mathbb{Q}$-line bundle.

Let $\mu: X'\to X$ be the extraction of the orbifold base Y such that X' is isomorphic to the total space of the orbifold line bundle $\mathcal{L}^{-1}\to\mathcal{Y}$. Now X', as an affine variety, has only cyclic quotient singularities along the zero section Y_0. Fix any $p\in Y_0$ and choose a neighborhood U of p such that U is locally isomorphic to

$$\mathbb{C}\times\mathbb{C}^{n-1}/\frac{1}{m}(1,b_2,\cdots,b_n) \tag{2.1}$$

The natural projection $\pi: X'\to Y$ is induced by the map $(x_1,\cdots,x_n)\mapsto(x_2,\cdots,x_n)$.

Proposition 2.1. *With the above notation, we have the formula:*

$$\mathrm{mld}(o,X)=\min_{p,g\neq 1}\left\{r,\frac{1}{m}\left(rw_1(g)+\sum_{i=2}^{n}w_i(g)\right)\right\} \tag{2.2}$$

where p ranges over all quotient singularities on $Y_0\subset X'$ and g ranges over all non-identity elements in the stabilizer group $G_p\cong\mathbb{Z}_m$ that satisfy $g^*x_i=e^{2\pi\sqrt{-1}w_i/m}x_i$ with $0\le w_i<m$.