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Minimal Log Discrepancy and Orbifold Curves

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Dedicated to Professor Gang Tian for his 65th birthday.

Abstract. We show that the minimal log discrepancy of any isolated Fano cone singularity is at most the dimension of the variety. This is based on its relation with dimensions of moduli spaces of orbifold rational curves. We also propose a conjectural characterization of weighted projective spaces as Fano orbifolds in terms of orbifold rational curves, which would imply that the equality holds only for smooth points.

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1 Introduction and main result

In birational algebraic geometry, particularly in the study of the Minimal Model Program (MMP), the minimal log discrepancy (mld) is an important invariant for Kawamata log terminal (klt) singularities ([3,12]). Indeed, Shokurov showed that two conjectural properties of mld would imply the termination of flips in the MMP. One of these conjectures states that the mld of closed points is lower semicontinuous as the point moves on a fixed normal variety X of dimension n. Since the mld of a smooth point is equal to n, this in particular implies that $\mathrm{mld}(x,X) \le n$ ([3, Conjecture 3.2]). In this short essay, we will prove that this sharp upper bound indeed holds for any isolated Fano cone singularity.

Theorem 1.1. *Let $o \in X$ be an isolated Fano cone singularity of dimension n, then* $\mathrm{mld}(o,X) \le n$.

Remark 1.1. By [15, Theorem 1.1], we find more generally that if the link of an isolated singularity is contactomorphic to the link of an isolated Fano cone singularity, then the same inequality also holds.

Note that when the Fano cone is an ordinary cone over a smooth Fano manifold, the upper bound is equivalent to the well-known fact that the Fano index of a Fano manifold is at most the dimension plus one. This fact can be proved using either the Riemann-Roch theorem or Mori's bend-and-break theory of rational curves. Since the Riemann-Roch approach does not seem to work well in the orbifold setting, our proof uses Mori's theory in the orbifold setting and shows its connection with minimal log discrepancy

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invariants. This connection is motivated by our previous work [14], which in particular expresses the mld invariant in terms of certain symplectic invariants that arise essentially in the study of moduli spaces of pseudo-holomorphic curves. In this paper, we will show that the mld invariant is bounded from above by dimensions of certain moduli spaces of orbifold rational curves with domain a weighted projective line (2.6). We will give two proofs for this crucial inequality, one using purely algebraic geometry and the other from a symplectic perspective. The argument for the above result also leads us to propose a characterization of weighted projective spaces using orbifold rational curves in the same spirit as Mori-Mukai (Conjecture 2.1), which would imply that the equality case occurs only for smooth points. As evidence, we prove an orbifold version of Mori's theorem:

Theorem 1.2. *Let $\mathcal{Y}$ be a complex orbifold whose orbifold tangent bundle $T\mathcal{Y}$ is ample, then $\mathcal{Y}$ is a finite quotient of a weighted projective space.*

See [5] for a related characterization of smooth projective spaces among normal varieties with quotient singularities. The method of proof follows Mori's approach, which uses a family of orbifold rational curves of minimal degree passing through a fixed (orbifold) point to sweep out the whole orbifold. The ampleness is used to deduce that there are no obstructions to the deformation theory of such curves. It is important for this purpose that we use orbifold rational curves with only one non-trivial orbifold point, which guarantees the splitting of any orbifold holomorphic vector bundle into a direct sum of orbifold line bundles (2.7).

2 Proof of main results

We will use the notation from our previous paper [14]. The starting point of our proof is a formula for the mld of an isolated Fano cone singularity (X,o) derived in [14, Proposition 2.12]. To state the formula, we assume that X is given as the orbifold cone $C(\mathcal{Y},\mathcal{L})$ where the orbifold base $\mathcal{Y}=(Y,\Delta)$ is obtained as the quotient $(X\setminus o)/\mathbb{C}^*$ and $\mathcal{L}$ is the associated orbifold line bundle. The klt condition is equivalent to the condition that (Y,Δ) is a klt Fano orbifold and $-K_{\mathcal{Y}}=r\mathcal{L}$ for some $r\in\mathbb{Q}_{>0}$. Throughout this paper, we use $(\mathcal{Y},\mathcal{L})$ to denote a Deligne-Mumford stack (or equivalently an orbifold) equipped with an orbifold line bundle and use (Y,L) to denote the associated coarse moduli space as an algebraic variety equipped with a $\mathbb{Q}$-line bundle.

Let $\mu: X'\to X$ be the extraction of the orbifold base Y such that X' is isomorphic to the total space of the orbifold line bundle $\mathcal{L}^{-1}\to\mathcal{Y}$. Now X', as an affine variety, has only cyclic quotient singularities along the zero section Y_0. Fix any $p\in Y_0$ and choose a neighborhood U of p such that U is locally isomorphic to

$$\mathbb{C}\times\mathbb{C}^{n-1}/\frac{1}{m}(1,b_2,\cdots,b_n) \tag{2.1}$$

The natural projection $\pi: X'\to Y$ is induced by the map $(x_1,\cdots,x_n)\mapsto(x_2,\cdots,x_n)$.

Proposition 2.1. *With the above notation, we have the formula:*

$$\mathrm{mld}(o,X)=\min_{p,g\neq 1}\left\{r,\frac{1}{m}\left(rw_1(g)+\sum_{i=2}^{n}w_i(g)\right)\right\} \tag{2.2}$$

where p ranges over all quotient singularities on $Y_0\subset X'$ and g ranges over all non-identity elements in the stabilizer group $G_p\cong\mathbb{Z}_m$ that satisfy $g^*x_i=e^{2\pi\sqrt{-1}w_i/m}x_i$ with $0\leq w_i<m$.

We give a short sketch of the proof.

Proof. The exceptional divisor of $\mu: X' \to X$ is equal to the zero section Y_0 of the orbifold line bundle $\mathcal{L}^{-1}$, which is the underlying variety of the orbifold $\mathcal{Y}_0$. Moreover, we have the formula $K_{X'} = \mu^* K_X + (r-1) Y_0$. So if we set

$$\mathrm{mld}(Y_0; X', (1-r)Y_0) = \min\{A(v; X', (1-r)Y_0); \mathrm{center}(v) \subset Y_0\},$$

where v ranges over all divisorial valuations over X', then there is an equality

$$\mathrm{mld}(o, X) = \min\left\{r, \mathrm{mld}(Y_0; X', (1-r)Y_0)\right\}.$$

Now we can use the formula for mld of quotient singularities (with a boundary divisor) as in [14]. Since cyclic quotient singularities are also locally toric, we can use toric geometry to calculate the quantity as follows. Let $p \in Y_0$ be a point such that a neighborhood of p is modeled on the quotient (2.1). Let $\mathbb{Z}^n$ be the standard lattice in $\mathbb{R}^n$ and denote by Λ' the larger lattice $\mathbb{Z}^n + \mathbb{Z}\frac{1}{m}(1, b_2, \cdots, b_n)$. Let $\Lambda' \cap (0,1]^n = \{v_1, \cdots, v_{d_p}, \mathbf{1}\}$ with $\mathbf{1} = (1, \cdots, 1)$. Then

$$\mathrm{mld}(U \cap Y_0; X', (1-r)Y_0) = \min_{1 \le k \le d_p} \left\{ \sum_{i=2}^{n} x_i(v_k) + r x_1(v_k) \right\},$$

where $x_j(v_k)$ denotes the j-th coordinate of the vector v_k for $1 \le j \le n$. On the other hand, each v_k corresponds to $g_k \in G_p = \Lambda'/\mathbb{Z}^n$ and $x_j(v_k) = \frac{1}{m} w_j(g_k)$. So we obtain the desired formula. $\square$

It is now convenient to introduce $\mathrm{age}(g) = \sum_{i=2}^{n} w_i(g)/m$, which is the age invariant in [7] for the twisted sector corresponding to g on the base orbifold $\mathcal{Y}$. We use $I\mathcal{Y}$ to denote the inertia orbifold of $\mathcal{Y}$ [2, Definition 2.49], on which age is locally constant [2, Lemma 4.6]. Locally, connected components of $I\mathcal{Y}$ are indexed by the conjugacy classes of the isotropy groups. We use $\mathcal{Y}_1 \subset I\mathcal{Y}$ to denote the non-twisted sector, which is diffeomorphic to $\mathcal{Y}$, and the inverse $g \mapsto g^{-1}$ makes sense on $\pi_0(I\mathcal{Y})$. Then we can rewrite (2.2) in the following form:

$$\mathrm{mld}(o, X) = \min_{g \in \pi_0(I\mathcal{Y}), g \ne 1} \left\{ r, \frac{r w_1(g)}{m} + \mathrm{age}(g) \right\} = \min_{g \in \pi_0(I\mathcal{Y}), g \ne 1} \left\{ r, \frac{r w_1(g^{-1})}{m} + \mathrm{age}(g^{-1}) \right\}.$$

As $w_1(g^{-1}) = 0$ if $w_1(g) = 0$ and $w_1(g^{-1}) = m - w_1(g)$ if $w_1(g) \ne 0$, we have

$$\begin{aligned} \mathrm{mld}(o, X) &= \min_{g \in \pi_0(I\mathcal{Y}), g \ne 1} \left\{ r, \frac{r(m - w_1(g))}{m} + \mathrm{age}(g^{-1}) \right\} \\ &= \min_{g \in \pi_0(I\mathcal{Y})} \left\{ \frac{r(m - w_1(g))}{m} + \mathrm{age}(g^{-1}) \right\}. \end{aligned} \tag{2.3}$$

The last equality follows from

$$\frac{r(m - w_1(1))}{m} + \mathrm{age}(1^{-1}) = r.$$

Our main new observation in this note is that the quantity on the right-hand side of the above formula can be related to the dimension of the moduli space of twisted (or orbifold) rational curves. This is motivated on the one hand by the fact that the lSFT invariant calculates the dimension of certain moduli spaces in symplectic field theory (see §3 for an explanation) and on the other hand by similar formulae that appear in the study of orbifold Gromov-Witten invariants [1,7].

Let M be the smooth link of the Fano cone singularity (X,o). We consider the finite set S of isotropy groups (including the trivial group, which is the isotropy group for a generic point) of the S^1 action on M. The set S is equipped with a partial order; we say $G_x > G_y$ if $G_y \subset G_x \subset S^1$ is a subgroup. For $G \in S$, the quotient of the fixed point set M^G/S^1 gives rise to a branch locus $\mathcal{Y}_G$ of the quotient Kähler orbifold $\mathcal{Y}$, giving M a stratification over the partially ordered set S. For non-minimal $G \in S$, we use G^- to denote the unique maximal element that is smaller than G. We formally define $G^- = \emptyset$ when G is the minimal element of S. By [14, Proposition 3.3], the Reeb orbits with period at most the period of a principal orbit (the simple orbit over a generic point) are parameterized by M^G plus a multiplicity $k \in G \backslash G^-$. In particular, the space of unparameterized Reeb orbits with period at most the period of a principal orbit can be identified with $I\mathcal{Y}$.

We use $\mathfrak{g} = (g_1, \cdots, g_k)$ to denote a sequence of k elements in $\pi_0(I\mathcal{Y})$, which may be repetitive. By an orbifold curve $\mathcal{C} = \left(C, Q = \sum_{i=1}^k (1-\ell_i^{-1}) q_i\right)$, we mean a smooth Riemann surface C with an orbifold structure $\mathbb{C}/\mathbb{Z}_{\ell_i}$ with $\ell_i \in \mathbb{Z}_{>0}$ at each $q_i \in \mathrm{Supp}(Q)$. Note that we allow the trivial orbifold structure at q_i, which corresponds to $\ell_i = 1$. We will use the notion of twisted maps in the sense of Abramovich-Vistoli as defined in [4].† In particular, each marked point q_i gives rise to a gerbe $\mathcal{Q}_i$ banded by $\mathbb{Z}_{\ell_i}$ and we have an evaluation map $\mathrm{ev}_{q_i}(f) \in I\mathcal{Y}$. We denote by $\mathrm{Mor}_{\mathfrak{g}}((C,Q),\mathcal{Y})$ the stack of twisted maps $f: \mathcal{C} \to \mathcal{Y}$ that satisfy $\mathrm{ev}_{q_i}(f) \in \mathcal{Y}_{g_i}$ for each $1 \le i \le k$ ([18]). In the setting of differential geometry, this corresponds to the moduli space of representable good orbifold maps defined in [7], or equivalently, representable orbifold maps using the groupoid language in [13, §5]. For each $f \in \mathrm{Mor}_{\mathfrak{g}}((C,Q),\mathcal{Y})$, we denote by $\mathrm{Mor}_{\mathfrak{g}}((C,Q),\mathcal{Y}; f|_Q)$ the subspace of twisted maps $\hat{f}$ in $\mathrm{Mor}_{\mathfrak{g}}((C,Q),\mathcal{Y})$ that satisfy $\mathrm{ev}_{q_i}(\hat{f}) = \mathrm{ev}_{q_i}(f)$ for each $1 \le i \le k$.

The following estimate is the orbifold version of the standard estimate in the smooth case ([9, 2.11]). Note that on the right side of the estimate, we will use the intersection theory on Deligne-Mumford stacks as explained in [1, §2].

Lemma 2.1. *With the above notation, we have the inequality*

$$\begin{aligned} &\dim_{[f]} \mathrm{Mor}_{\mathfrak{g}}((C,Q),\mathcal{Y}; f|_Q) \\ &\ge -K_{\mathcal{Y}} \cdot f_* \mathcal{C} + (1-g(C)) \dim \mathcal{Y} - \left(\sum_{i=1}^k \mathrm{age}(g_i) + \dim \mathcal{Y}_{g_i} \right). \end{aligned} \tag{2.4}$$

Proof. First, similar to the smooth case, by the deformation theory of stacks ([18]), we get

$$\dim_{[f]} \mathrm{Mor}_{\mathfrak{g}}((C,Q),\mathcal{Y}; f|_Q) \ge \chi(\mathcal{C}, f^* T\mathcal{Y} \otimes \otimes_i \mathcal{I}_{\mathcal{Q}_i}),$$

where $\mathcal{I}_{\mathcal{Q}_i}$ is the ideal sheaf of $\mathcal{Q}_i$. From the exact sequence,

$$0 \to \mathcal{I}_{\mathcal{Q}_i} \to \mathcal{O}_{\mathcal{C}} \to \mathcal{O}_{\mathcal{Q}_i} \to 0$$

†We always identify an orbifold with the corresponding Deligne-Mumford stack.

we obtain the identities:

$$\begin{aligned}\chi(\mathcal{C},f^*T\mathcal{Y}\otimes\otimes_i\mathcal{I}_{\mathcal{Q}_i}) &= \chi(\mathcal{C},f^*T\mathcal{Y})-\sum_i\chi(\mathcal{Q}_i,f^*T\mathcal{Y})\\ &=\chi(\mathcal{C},f^*T\mathcal{Y})-\sum_i\chi(\mathcal{Q}_i,f^*T_{\mathcal{Y}_{g_i}})\\ &=\left(-K_{\mathcal{Y}}\cdot f_*\mathcal{C}+(1-g(C))\dim\mathcal{Y}-\sum_i \mathrm{age}(g_i)\right)-\sum_i\dim\mathcal{Y}_{g_i}.\end{aligned}$$

For the second identity, we used the tangent bundle lemma from [1, Lemma 3.6.1]. The last identity used the Riemann-Roch theorem for twisted curves ([1, Theorem 7.2.1]). □

When $C=\mathbb{P}^1$ and $Q=0\cdot\{0\}+(1-\ell^{-1})\{\infty\}$, the orbifold curve $(\mathbb{P}^1,Q)$ is identified with the weighted projective line $\mathbb{P}^1(1,\ell)$ with $0=[1,0]$ and $\infty=[0,1]$. When $\mathfrak{g}=(1,g)$ we get from (2.4):

$$\begin{aligned}&\dim_{[f]}\mathrm{Mor}_{(1,g)}\left(\mathbb{P}^1(1,\ell),\mathcal{Y};f|_{\{0,\infty\}}\right)\\ &\geq -K_{\mathcal{Y}}\cdot f_*\mathbb{P}(1,\ell)+(\dim\mathcal{Y}-(\mathrm{age}(g)+\dim\mathcal{Y}_g))-\dim\mathcal{Y}\\ &=-K_{\mathcal{Y}}\cdot f_*\mathbb{P}(1,\ell)+\mathrm{age}(g^{-1})-\dim\mathcal{Y}.\end{aligned}$$

The last identity used the fact $\dim\mathcal{Y}-(\mathrm{age}(g)+\dim\mathcal{Y}_g)=\mathrm{age}(g^{-1})$ and is the main evidence that the formula (2.3) is related to the dimension of moduli spaces of twisted rational curves.

Now note that, $f^*\mathcal{L}^{-1}$ is an orbifold line bundle over $\mathbb{P}^1(1,\ell)$ and, according to the definition of twisted curves, the morphism of the stabilizer groups $\mathbb{Z}_\ell\to G_{f(\infty)}\cong\mathbb{Z}_m$ is injective with 1 mapped to g. By setting $\langle g\rangle\cong\mathbb{Z}_\ell$ to be a subgroup of $G_{f(\infty)}$, the twisted curve then satisfies $\mathrm{ev}(f)\in\mathcal{Y}_g$. Assume $f(\infty)$ is locally modeled on

$$\mathbb{C}\times\mathbb{C}^{n-1}/\frac{1}{m}(1,b_2,\cdots,b_n).$$

The generator of $\mathbb{Z}_\ell$ is mapped to $p\in\mathbb{Z}_m$ such that the order of p is ℓ. Note that $f^*\mathcal{L}^{-1}=\mathcal{O}_{\mathcal{C}}(k)$ for $k\in\mathbb{Z}_{<0}$ where $\mathcal{C}=\mathbb{P}(1,\ell)$. We have $\frac{p\ell}{m}=k \mod \ell$. As $w_1(g)=p\in[0,m)$, we have $k\leq\frac{w_1(g)\ell}{m}-\ell$. Since $c_1(\mathcal{O}_{\mathcal{C}}(1))\cdot\mathcal{C}=\frac{1}{\ell}$, we obtain the following inequality:

$$-K_{\mathcal{Y}}\cdot f_*\mathcal{C}=r\mathcal{L}\cdot f_*\mathcal{C}=rf^*\mathcal{L}\cdot\mathcal{C}=r\cdot\frac{-k}{\ell}\geq r\cdot\frac{\ell-\frac{w_1(g)\ell}{m}}{\ell}=r\frac{m-w_1(g)}{m}.$$

For simplicity of notation, we set $d_{g^{-1}}(f)=-K_{\mathcal{Y}}\cdot f_*\mathcal{C}+\mathrm{age}(g^{-1})$. Then the above discussion yields:

$$\dim_{[f]}\mathrm{Mor}_{(1,g)}\left(\mathbb{P}^1(1,\ell),\mathcal{Y};f|_{\{0,\infty\}}\right)+\dim\mathcal{Y}\geq d_{g^{-1}}(f)\geq\frac{r(m-w_1(g))}{m}+\mathrm{age}\left(g^{-1}\right). \quad (2.5)$$

Because the minimum (2.3) is taken over all strata, we get

$$\mathrm{mld}(o,X)\leq\min\left\{d_{g^{-1}}(f);f\in\mathrm{Mor}_{(1,g)}\left(\mathbb{P}^1(1,\ell);f|_{\{0,\infty\}}\right),\ell\in\mathbb{Z}_{>0}\right\}. \quad (2.6)$$

So to prove the main theorem, we just need to prove the inequality $d_{g^{-1}}(f)\leq n=\dim\mathcal{Y}+1$ for some twisted map $f:\mathbb{P}^1(1,\ell)\to\mathcal{Y}$. If $\mathcal{Y}$ is a smooth Fano manifold without orbifold points, then this was achieved by Mori via a reduction to characteristic $p>0$ and the bend-and-break method [16]. In [8], this has been partly generalized to the orbifold setting, but the authors were mainly interested in singular Fano varieties with special singularities. We will explain that Mori's method indeed gives us the sharp upper bound of mld in our problem.

In the bend-and-break method, there are two steps. First, we need to obtain a non-trivial twisted map $f:\mathbb{P}(1,\ell)\to\mathcal{Y}$. This is achieved by deforming a morphism from a smooth curve of genus $g\geq 1$. Let C be a smooth Riemann surface of genus $g\geq 1$ and $f:C\to Y$ be a non-trivial morphism. First, by [4, Proof of Theorem 7.1.1], there exists a ramified cover $C'\to C$ such that f lifts to a (twisted) morphism $f':C'\to\mathcal{Y}$ where C' does not have a non-trivial orbifold structure.

Proposition 2.2. *Let $f:C\to\mathcal{Y}$ be a morphism from a smooth curve of genus $g(C)\geq 1$. Fix any $c\in C$, if $\dim_{[f]}\mathrm{Mor}(C,\mathcal{Y};f|_c)\geq 1$, then there exists a nontrivial twisted map $\hat{f}:\mathbb{P}^1(1,\ell)\to\mathcal{Y}$.*

Proof. First, we use the argument as in [12, Proof of Corollary 1.7] or [9, Proof of Proposition 3.1]. Let T be the normalization of a 1-dimensional substack of $\mathrm{Mor}(C,\mathcal{Y};f|_c)$ passing through $[f]$ and let $\bar{T}$ be a smooth compactification of T. By the same reasoning as in [12, Proof of Corollary 1.7] or [9, Proof of Proposition 3.1], there exists $t_0\in\bar{T}$ such that ev is not defined at (c,t_0). The indeterminacies of the induced rational map $\mathrm{ev}:C\times\bar{T}\dashrightarrow Y$ can be resolved by blowing up points to get a morphism $\widetilde{\mathrm{ev}}:S\xrightarrow{\epsilon}C\times\bar{T}\overset{\mathrm{ev}}{\dashrightarrow}X$. The fiber F_{t_0} of t_0 under the projection $S\to\bar{T}$ is the union of the strict transform $\hat{C}$ of $C\times\{t_0\}$ and a connected exceptional rational 1-cycle E which is not entirely contracted by $\widetilde{\mathrm{ev}}$. There is an irreducible $\mathbb{P}^1$-component E_1 of E such that E_1 intersects $\overline{F_{t_0}\setminus E_1}$ at a single node p_0.

Now by the proof of the valuative criterion for twisted stable maps as in [4, Proof of Proposition 6.0.1] (also [7]), we can add orbifold structures to the nodes of $F_{t_0}=\hat{C}\cup E$ so that the morphism $\widetilde{\mathrm{ev}}:F_{t_0}\to Y$ lifts to become a twisted map $\widetilde{\mathrm{ev}}:\mathcal{F}_{t_0}\to\mathcal{Y}$. In more detail, at smooth points of F_{t_0} we do not need to add a new orbifold structure. This is proved based on a base-change construction and a purity lemma ([4, Lemma 2.4.1]). On a node $\{xy=0\}$ of F_{t_0}, there exists a chart $\{uv=0\}/\mathbb{Z}_\ell$ where the action of $\mathbb{Z}_\ell$ is described by $(u,v)\mapsto(\zeta u,\zeta^{-1}v)$ with $\zeta=\exp(2\pi\sqrt{-1}/\ell)$. In this chart, the map $\mathcal{F}_{t_0}\to F_{t_0}$ is given by $x=u^\ell$ and $y=v^\ell$. Let $\mathcal{E}_1$ be the orbifold curve $(E_1,(1-\ell^{-1})p_0)$. Then the induced twisted map $\hat{f}:\mathcal{E}_1\cong\mathbb{P}^1(1,\ell)\to\mathcal{Y}$ is the desired map. □

In the second step, we can degenerate the twisted map $\mathbb{P}^1$ to another one if the dimension of the deformation space is large.

Proposition 2.3. *Let $f:\mathcal{C}=\mathbb{P}^1(1,\ell)\to\mathcal{Y}$ be a twisted map such that $f(0)\in\mathcal{Y}_1$ and $f(\infty)\in\mathcal{Y}_g$. Assume that*

$$\dim_{[f]}\mathrm{Mor}_{(1,g)}\left(\mathbb{P}^1(1,\ell),\mathcal{Y};f|_{\{0,\infty\}}\right)\geq 2,$$

then there exists a non-trivial twisted map $\hat{f}:\hat{\mathcal{C}}=\mathbb{P}^1(1,\hat{\ell})\to\mathcal{Y}$ such that $f_*[C]$ is numerically equivalent to $\hat{f}_*[\hat{\mathcal{C}}]+\sum_i(f_i)_*[\mathcal{C}_i]$ where $\{f_i:\mathcal{C}_i\to\mathcal{Y}\}$ is a nonempty collection of twisted maps.

Note that in general $\hat{f}(\infty)$ is different from $f(\infty)$.

Proof. The proof is similar to that of Proposition 2.3. Let T be the normalization of a 1-dimensional substack of $\mathrm{Mor}(\mathcal{C},\mathcal{Y};f|_{\{0,\infty\}})$ passing through $[f]$ but not contained in its $\mathbb{C}^*$-orbit and let $\bar{T}$ be a smooth compactification of T. By arguing as in [9, Proposition 3.2] in the orbifold setting, there exists $t_0\in\bar{T}$ such that ev is not defined at $(0,t_0)$. We resolve the indeterminacy of the evaluation map $\mathrm{ev}:\mathbb{P}^1\times\overline{T}\dashrightarrow X$ to get a fibered surface $\pi:S\to\bar{T}$ and a map $\tilde{\mathrm{ev}}:S\to X$ such that $F_{t_0}=\pi^{-1}(t_0)$ is a cycle of rational curves. There is an irreducible component E_1 of F_{t_0} that intersects $\overline{F_{t_0}\setminus E_1}$ at a single node. As before, we now use the same argument as [4, Proof of Proposition 6.0.1] to extend orbifold structures on $\mathbb{P}^1\times T$ to S (up to base change). This induces a $\mathbb{Z}_{\hat{l}}$-orbifold structure at one point on $E_1\cong\mathbb{P}^1$ and we get a twisted map $\hat{f}:\mathbb{P}^1(1,\hat{\ell})\to\mathcal{Y}$ as desired. □

We remark that the bend-and-break results in the orbifold setting, similar to the above two propositions, have also appeared in [8, Proposition 3.5 and 3.6]. Since the (image of) a twisted map cannot break infinitely many times, the above two bend-and-break constructions produce a non-trivial bound (by also using (2.5)).

Proposition 2.4. *Assume that there exists a non-trivial smooth morphism $C\to\mathcal{Y}$ that satisfies $\dim_{[f]}\mathrm{Mor}(C,\mathcal{Y};f|_c)\geq 1$. Then there exists a twisted map $f:\mathbb{P}^1(1,\ell)\to\mathcal{Y}$ satisfying $f(0)\in\mathcal{Y}_1$, $f(\infty)\in\mathcal{Y}_g$ and*

$$-K_{\mathcal{Y}}\cdot f_*\mathbb{P}(1,\ell)+\mathrm{age}(g^{-1})-\dim\mathcal{Y}\leq\dim_{[f]}\mathrm{Mor}_{(1,g)}\left(\mathbb{P}^1(1,\ell),\mathcal{Y};f|_{\{0,\infty\}}\right)\leq 1.$$

To achieve the dimension condition in the above results, Mori introduced a groundbreaking method of reducing to the field of characteristic $p>0$ and using the Frobenius morphism to increase the dimension of the deformation space. He then applied the bend-and-break method to get a rational curve in positive characteristics with a uniform upper bound on their degrees. Finally, Mori concluded the existence of a rational curve in characteristic zero by an algebraic argument ([12, p. 15], [9, p. 62-63]). The same line of arguments can be applied in the setting of orbifolds ([8]). Note that, in the characteristic p argument, we need to assume that the orders of the stabilizers are relatively prime to p. In other words, we need to work with tamed Deligne-Mumford stacks in the sense of [4]. This is not a restriction, since there are only finitely many strata and the prime p can be arbitrarily large. As a consequence, we get

Proposition 2.5. *Let $\mathcal{Y}$ be a Fano orbifold. There exists a twisted map $f:\mathbb{P}^1(1,\ell)\to\mathcal{Y}$ satisfying $f(0)\in\mathcal{Y}_1$, $f(\infty)\in\mathcal{Y}_g$ and*

$$d_{g^{-1}}(f)=-K_{\mathcal{Y}}\cdot f_*\mathbb{P}(1,\ell)+\mathrm{age}(g^{-1})\leq\dim\mathcal{Y}+1.$$

Combining this with the estimates (2.5)-(2.6), we complete the proof of Theorem 1.1.

Example 2.1. Consider the weighted projective line $\mathbb{P}^1(2,3)=\left(\mathbb{C}^2\setminus 0\right)/\mathbb{C}^*$ where $\mathbb{C}^*$ acts on $\mathbb{C}^2$ by $\lambda\circ(z_1,z_2)=\left(\lambda^2z_1,\lambda^3z_2\right)$. We have

$$I\mathcal{Y}=Y\sqcup\{([1,0]=0,-1)\}\sqcup\{([0,1]=\infty,\epsilon)\}\sqcup\{(\infty,\epsilon^{-1})\}$$

with $\epsilon=e^{2\pi\sqrt{-1}/3}$. We consider the family of maps $f:\mathbb{C}^*\times\mathbb{P}^1(1,2)\to\mathcal{Y}$ given by

$$f(t,[u_1,u_2])=f_t([u_1,u_2])=\left[t^2u_1^2+u_2,t^3u_1^3+u_1u_2\right].$$

Then $\mathrm{ev}_0(f_t)=\left[t^2,t^3\right]=[1,1]$ and $\mathrm{ev}_\infty(f_t)=\{([1,0],-1)\}$ and

$$d_{-1}(f)=-K_{\mathcal{Y}}\cdot f_*C+\mathrm{age}(-1)=5\cdot\frac{1}{2}+\frac{1}{2}=3>2=\dim\mathcal{Y}+1.$$

Note that f is undefined at $(0,[1,0])$. Indeed, in a chart near $0=[1,0]$, we set $v=u_2/u_1^2$ to get $f(t,v)=\left[t^2+v,t^3+v\right]$. We can now resolve the indeterminacy by a weighted blowup of weight $(1,3)$ to map $\hat{f}:\hat{\mathbb{C}}^2:=Bl_{(1,3)}\mathbb{C}^2\to Y$. Denote the exceptional divisor by $E_1\cong\mathbb{P}^1(1,3)$, the strict transform of $\{t=0\}$ by E_0 and the strict transform of $\{v=0\}$ by H. Near $E_1\cap E_0$ we have a chart $\mathbb{C}^2/\frac{1}{3}(1,-1)\mapsto\hat{\mathbb{C}}^2$ given by $(x,w)\mapsto\left(xw,w^3\right)$ such that $E_0=\{x=0\},E_1=\{w=0\}$ and

$$\hat{f}(x,w)=f(xw,w^3)=\left[x^2+w,x^3+1\right].$$

Note that when $x=0$, we get $\hat{f}(0,w)=[w,1]$, which can be seen as the restriction of the identity self-map of $\mathbb{P}^1(2,3)$. Near $E_1\cap H$ we have a chart given by $(t,y)\mapsto(t,t^3y)$ such that $E_1=\{t=0\}$ and

$$\hat{f}(t,y)=f\left(t,t^3y\right)=[1+ty,1+y].$$

Note that when $t=0$, we get $\hat{f}(0,y)=[1,1+y]$ which can be seen as the restriction of the map $\hat{f}:\mathcal{E}=\mathbb{P}^1(1,3)\to\mathbb{P}^1(2,3)$ given by

$$[y_1,y_2]\to\left[1,1+\frac{y_2}{y_1^3}\right]=[y_1^2,y_1^3+y_2].$$

So we indeed bend-and-break f to a twisted map

$$f':\mathbb{P}^1(1,3)\cup_{[0,1]_1=[0,1]_2}\mathbb{P}^1(2,3)\to\mathbb{P}^1(2,3)$$

and by restriction get the above twisted map $\hat{f}:\mathcal{E}=\mathbb{P}^1(1,3)\to\mathbb{P}^1(2,3)$ that satisfies $\mathrm{ev}_\infty(\hat{f})\in([0,1],\epsilon)$ and $d_{\epsilon^{-1}}(\hat{f})=5\cdot\frac{1}{3}+\frac{1}{3}=2=\dim\mathcal{Y}+1$.

Example 2.2. Consider $\mathbb{P}^1(2,3,5)=(\mathbb{C}^3\backslash 0)/\mathbb{C}^*$ where $\mathbb{C}^*$ acts on $\mathbb{C}^3$ by $\lambda\circ(z_1,z_2,z_3)=(\lambda^2z_1,\lambda^3z_2,\lambda^5z_3)$. There is a family $\{f_t:\mathbb{P}^1(1,3)\to\mathcal{Y}\}_{t\in\mathbb{C}^*}$ of twisted maps with fixed $f_t(0)$ and $f_t(\infty)$ given by:

$$[u_1,u_2]\to\left[t^2u_1^2,\,t^3u_1^3+u_2,\,t^5u_1^5+u_1^2u_2\right].$$

We have

$$d_{\epsilon^{-1}}(f_t)=10\cdot\frac{1}{3}+\frac{1}{3}+\frac{1}{3}=4>3=\dim\mathcal{Y}+1.$$

Note that $f_t([1,0])$ is not defined at $t=0$ and one can bend-and-break f_t to

$$f':\mathbb{P}^1(1,5)\cup_{[0,1]_1=[0,1]_2}\mathbb{P}^1(3,5)\to\mathbb{P}^2(2,3,5)$$

and get a new twisted map $\hat{f}:\mathbb{P}^1(1,5)\to\mathbb{P}^2(2,3,5)$ given by $[y_1,y_2]\mapsto\left[y_1^2,y_1^3,y_1^5+y_2\right]$ with

$$d_{e^{-2\pi\sqrt{-1}/5}}(\hat{f})=10\cdot\frac{1}{5}+\frac{3}{5}+\frac{2}{5}=3=\dim\mathcal{Y}+1.$$

Based on our discussion and examples above, we propose a conjectural characterization for weighted projective spaces as Fano orbifolds:

Conjecture 2.1. *Let $\mathcal{Y}$ be a Fano orbifold of dimension n. Assume that for any $\ell\in\mathbb{Z}_{>0}$, $g\in\pi_0(I\mathcal{Y})$ and any $f\in\mathrm{Mor}_{(1,g)}(\mathbb{P}^1(1,\ell),\mathcal{Y})$ we have*

$$d_{g^{-1}}(f)=-K_{\mathcal{Y}}\cdot f_*\mathbb{P}^1(1,\ell)+\mathrm{age}(g^{-1})\geq n+1.$$

Then $\mathcal{Y}$ is isomorphic to a finite quotient of a weighted projective space.

Note that, this conjecture would imply that the equality in Theorem 1.1 holds only for smooth points. Indeed, the quotient orbifold $\mathcal{Y}=(X\backslash o)/\mathbb{C}^*$ is a quotient of a weighted projective space if the equality in Theorem 1.1 holds. As a consequence (X,o) is an isolated quotient singularity, for which we know that the minimal discrepancy is equal to the dimension only when it is a smooth point.

In the smooth case, the above conjecture is essentially the conjecture of Mori-Mukai as proved by Cho-Miyaoka-Shepherd-Barron ([6], [5,11]). We expect that a careful implementation of the argument from [6,11] in the orbifold setting will prove such a statement. We leave this to a future study and carry out such arguments here for Fano orbifolds with ample orbifold tangent bundles.

Proof of Theorem 1.2. We sketch a proof that is modeled on Mori's proof in the case of smooth projective manifolds, which depends on the study of the tangent map of the rational curves at a fixed point. Let S be an irreducible component of $\mathrm{Mor}_{(1,g)}\left(\mathbb{P}^1(1,\ell),\mathcal{Y};f|_{\{\infty\}}\right)$ that satisfies

$$d_{g^{-1}}(f)=\chi(f^*T\mathcal{Y})-\dim\mathcal{Y}_g=n+1,$$

and $\mathrm{Mor}_{(1,g)}\left(\mathbb{P}^1(1,\ell),\mathcal{Y};f|_{\{\infty\}}\right)/G$ is proper where G is the automorphism group of $\mathbb{P}(1,\ell)$ fixing ∞. Since $\mathbb{P}(1,\ell)$ has only one orbifold point, by [17, Variation 2], we have a splitting

$$f^*T\mathcal{Y}=\mathcal{O}(b_1)\oplus\cdots\oplus\mathcal{O}(b_n). \tag{2.7}$$

Let $\{a_i\in[1,\ell]\}$ be the weights of $\mathbb{Z}_\ell\hookrightarrow G_{f(\infty)}$ on $f^*T\mathcal{Y}|_\infty$ satisfying $a_1\le\cdots\le a_n$. Then we have $a_i\equiv b_i \mod \ell$. By the ampleness of $T\mathcal{Y}$, we have $b_i\ge 1$. By Bochner-Kodaira's vanishing, we have

$$H^1\Big(\mathbb{P}^1(1,\ell),\mathcal{O}(b_i-k)\Big)=0,\quad \text{for } k<1+\ell+b_i. \tag{2.8}$$

By the Riemann-Roch theorem for orbifold line bundles, we have

$$\chi(\mathcal{O}(b_i))=1+\frac{b_i}{\ell}-\frac{b_i \mod \ell}{\ell}\ge 1$$

and the equality holds if and only if $b_i\in[1,\ell-1]$. Assume $\dim\mathcal{Y}_g=d$. Then for $j\ge n-d+1$, $a_j(g)=\ell$ and $b_j\equiv 0 \mod \ell$. We have

$$n+1=\chi(f^*T\mathcal{Y})-\dim\mathcal{Y}_g=\sum_{j=1}^{n-d}\left(1+\frac{b_i-b_i \mod \ell}{\ell}\right)+\sum_{j=n-d+1}^{n}\frac{b_j}{\ell}.$$

Since each term is an integer at least 1, there must exist $i\in\{1,\cdots,n\}$ such that:

$$b_i=a_i+\ell,\quad \text{and}\quad b_j=a_j \text{ for } j\ne i.$$

In other words, we have a splitting

$$f^*T\mathcal{Y}=\mathcal{O}(a_1)\oplus\cdots\oplus\mathcal{O}(a_{i-1})\oplus\mathcal{O}(a_i+\ell)\oplus\mathcal{O}(a_{i+1})\cdots\oplus\mathcal{O}(a_n). \tag{2.9}$$

The differential df corresponds to a section of

$$\begin{aligned}\mathcal{O}(-(1+\ell))\otimes f^*T\mathcal{Y}=&\mathcal{O}(a_1-\ell-1)\oplus\cdots\oplus\mathcal{O}(a_{i-1}-\ell-1)\\&\oplus\mathcal{O}(a_i-1)\oplus\mathcal{O}(a_{i+1}-\ell-1)\cdots\oplus\mathcal{O}(a_n-\ell-1),\end{aligned} \tag{2.10}$$

which is equivalently a section of $H^0(\mathbb{P}^1(1,\ell),a_i-1)\cong\mathbb{C}$ since $\mathcal{O}(a_j-\ell-1)$ with $a_j-\ell-1<0$ has no holomorphic sections. Locally $f:\mathbb{C}/\frac{1}{\ell}(1)\to\mathbb{C}^n/G_{f(\infty)}$ factors through $\mathbb{C}^n/\mathbb{Z}_\ell$ and has a lifting to $\mathbb{C}\to\mathbb{C}^n$ whose leading order terms are given as:

$$f(z)\sim\left(t_1z^{a_1}+t_1'z^{a_1+\ell},\cdots,t_iz^{a_i}+t_i'z^{a_i+\ell},\cdots,t_nz^{a_n}+t_n'z^{a_n+\ell}\right).$$

Then $(t_1,\cdots,t_n)\neq 0$, for otherwise, $\mathrm{d}f$ will have a zero of order $\geq\ell$ in a component of the splitting (2.10), contradicting the splitting. Then we can define a tangent map:

$$\tilde{\phi}(f)=(t_1,\cdots,t_n)\in\mathbb{C}^n/\frac{1}{\ell}(a_1,\cdots,a_n)$$

and the corresponding projective tangent map $\phi:S\to\mathbb{P}_{\mathbf{w}}$ where $\mathbf{w}=(a_1,\cdots,a_n)$. Since the infinitesimal variation of f with a fixed $\tilde{\phi}(f)=(t)$ corresponds to variation of t_i', the fiber of $\tilde{\phi}$ over $[c]\neq 0$ has an orbifold tangent space and an obstruction space given by $H^j\left(\mathbb{P}^1(1,\ell),\mathcal{O}\oplus\mathcal{O}(-\ell)^{n-1}\right)$ which is equal to $\mathbb{C}$ for $j=0$ and vanishes for $j=1$.

The automorphism group G of $\mathbb{P}(1,\ell)$ fixing ∞ is given by

$$\left\{[z,w]\mapsto[az,bz^\ell+cw]\,|\,a,c\neq 0\right\},$$

where $\infty=[0,1]$. The group G acts on $S\times\mathbb{P}^1(1,\ell)$ and we take the quotient $\mathcal{M}=S/G$ as an orbifold. The tangent space of S at f is given by

$$H^0\left(\mathbb{P}^1(1,\ell),\mathcal{O}^{n-1}\oplus\mathcal{O}(\ell)\right)\cong\mathbb{C}^{n+1}$$

and the tangent space of the G-action is $H^0(\mathbb{P}(1,\ell),\mathcal{O}(\ell))$, hence the tangent space of $\mathcal{M}$ at $[f]$ is $H^0\left(\mathbb{P}^1(1,\ell),\mathcal{O}^{n-1}\right)\cong\mathbb{C}^{n-1}$. It is straightforward to check that the induced smooth orbifold morphism $\bar{\phi}:\mathcal{M}\to\mathbb{P}_{\mathbf{w}}$ is finite of maximal rank and injects the stabilizer of $[f]$ into the stabilizer of $\bar{\phi}([f])$, i.e., $\bar{\phi}$ is representable. In particular, $\bar{\phi}$ is an orbifold covering in the sense of [2, Definition 2.16], therefore we conclude that $\mathcal{M}$ is a weighted projective space of weight $\bar{\mathbf{w}}=\mathbf{w}/k$ where k is a divisor of $\gcd(\bar{\mathbf{w}}):=\gcd(a_1,\cdots,a_n)$ such that $\gcd(\bar{\mathbf{w}})=\gcd(\mathbf{w})/k$ is equal to the order of the stabilizer at the generic point of $\mathcal{M}$.

Note that, locally near $f(\infty)$, $\mathcal{Y}\cong\mathbb{C}^n/G_{f(\infty)}$ is a quotient of $\mathcal{Y}':=\mathbb{C}^n/\frac{1}{\ell}(a_1,\cdots,a_n)$. Let $\mu':\hat{\mathcal{Y}}'\to\mathcal{Y}'$ denote the weighted blowup of $\mathcal{Y}'$ at $[0]$ with the weight $\mathbf{w}$ and $\mu:\hat{\mathcal{Y}}\to\mathcal{Y}$ denote the weighted blow-up at $f(\infty)$ with weight $\mathbf{w}$. Denote by E' and E the exceptional divisors of μ' and μ respectively. Then $E'\cong\mathbb{P}_{\mathbf{w}}$ with $\mathcal{O}(E')|_{E'}=\mathcal{O}_{\mathbb{P}_{\mathbf{w}}}(-\ell)$. The exceptional divisor E is thus a finite quotient of the weighted projective space $\mathbb{P}_{\mathbf{w}}$ and the orbifold line bundle $\mathcal{O}(E)|_E$ is the quotient of $\mathcal{O}_{\mathbb{P}_{\mathbf{w}}}(-\ell)$.

Consider the evaluation map $\Phi:S\times\mathbb{P}^1(1,\ell)\to\mathcal{Y}$ such that $\Phi|_{\{s\}\times\mathbb{P}(1,\ell)}$ is the morphism represented by s. We have an induced orbifold fibration

$$\mathcal{Z}=\frac{(S\times\mathbb{P}^1(1,\ell))}{G}\to\mathcal{M}\cong\mathbb{P}_{\bar{\mathbf{w}}},$$

which is an orbifold $\mathbb{P}(1,\ell)$ fibration with a section $\sigma:\mathcal{M}\to\mathcal{Z}$ by $\sigma([f])=[(f,\infty)]$. The fiber of $\Phi|_{S\times(\mathbb{P}(1,\ell)\setminus\{\infty\})}$, at (f,q), has an orbifold tangent space $H^0(\mathbb{P}(1,\ell),\mathcal{O}(-\ell)^{n-1}\oplus\mathcal{O})\oplus\mathbb{C}$ and the obstruction vanishes, where the $\mathbb{C}$ summand is from the tangent space of $q\in\mathbb{P}(1,\ell)\setminus\{0\}$ and the first component is the subspace of the tangent space of

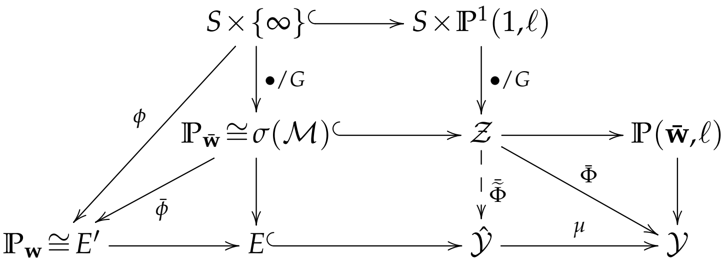
$$S\simeq H^0(\mathbb{P}(1,\ell),\mathcal{O}^{n-1}\oplus\mathcal{O}(\ell))$$

at f subject to the condition that it vanishes at the smooth point $q\in\mathbb{P}(1,\ell)$. As the tangent space of the fiber is the tangent space of the G action, the induced morphism $\bar{\Phi}:\mathcal{Z}\to\mathcal{Y}$ is of maximal rank when restricted to $\mathcal{Z}\backslash\sigma(\mathcal{M})$. We claim the orbifold morphism $\bar{\Phi}|_{\mathcal{Z}\backslash\sigma(\mathcal{M})}$ is also representable. If $g\in G$ fixes $(f,0)$, then on $\mathbb{C}/\frac{1}{\ell}(1)$ neighborhood around $\infty\in\mathbb{P}(1,\ell)$, g is written as $z\mapsto\lambda z$ such that $\lambda\sim\lambda e^{2\pi\sqrt{-1}/\ell}$ if $f\circ g=f$ near 0 in $\mathbb{C}/\frac{1}{\ell}(1)$. Hence the stabilizer $G_{[(f,0)]}$ is a cyclic group. If $G_{[(f,0)]}$ does not inject into the stabilizer of $f(0)$ in $\mathcal{Y}$ with a kernel $\mathbb{Z}_s$, then f is a branched cover over a twisted map $f':\mathbb{P}(1,\ell')\to\mathcal{Y}$ of order s, where ℓ divides ℓ' and ℓ'/ℓ divides s. As a consequence, we have

$$(f')^*T\mathcal{Y}=\mathcal{O}\left(\frac{a_1\ell'}{s\ell}\right)\oplus\cdots\oplus\mathcal{O}\left(\frac{a_{i-1}\ell'}{s\ell}\right)\oplus\mathcal{O}\left(\frac{a_i\ell'}{s\ell}+\frac{\ell'}{s}\right)\oplus\mathcal{O}\left(\frac{a_{i+1}\ell'}{s\ell}\right)\oplus\cdots\oplus\mathcal{O}\left(\frac{a_n\ell'}{s\ell}\right),$$

over $\mathbb{P}(1,\ell')$. All the weights above are no larger than $2\ell'/s\le\ell'$. Since $\chi((f')^*T\mathcal{Y})-\dim\mathcal{Y}_{g'}\ge n+1$ where $f'(\infty)\in\mathcal{Y}_{g'}$, we can argue as in (2.9) that at least one weight is larger than ℓ', a contradiction.

The evaluation map $\widetilde{\Phi}:S\times\mathbb{P}^1(1,\ell)\dashrightarrow\hat{\mathcal{Y}}$ is well-defined near $S\times\{\infty\}$ and the induced map $\tilde{\bar{\Phi}}:\mathcal{Z}\dashrightarrow\hat{\mathcal{Y}}$ near $\sigma(\mathcal{M})$ is a covering map. Therefore the orbifold normal bundle of $\sigma(\mathcal{M})$ is isomorphic to $\mathcal{O}_{\mathbb{P}_{\bar{\mathbf{w}}}}(-\ell)$. As a consequence, we have $\mathcal{Z}=\mathbb{P}(\mathcal{O}_{\mathbb{P}_{\bar{\mathbf{w}}}}(-\ell)\oplus\mathcal{O})$. Then by blowing down $\sigma(\mathcal{M})$ in $\bar{\Phi}$, we get an orbifold morphism $\mathbb{P}(\bar{\mathbf{w}},\ell)\to\mathcal{Y}$ that is smooth and representable, and hence an orbifold covering.



□

3 Relation with moduli space in symplectic field theory

In [14, Proposition 3.4], we also derived the mld in terms of the lSFT invariant from symplectic geometry

$$\mathrm{mld}(o,X)=\frac{1}{2}\inf_{\gamma}\mathrm{lSFT}(\gamma)+1, \tag{3.1}$$

where γ ranges over all Reeb orbits of the link of $x\in X$ with respect to any conic contact form. We refer to [14, §2.3] for Conley-Zehnder indices and SFT degrees.

Let $\mathfrak{g}=(g_1,\cdots,g_k)$ be a sequence of elements in $\pi_0(I\mathcal{Y})$. We define $\overline{\mathcal{M}}_{0,k}(\mathcal{Y},A,\mathfrak{g})$ to be the compactified moduli spaces of orbifold $\mathbb{P}^1$ curves with k marked points (only marked points can be orbifold points) to $\mathcal{Y}$ with homology class A, such that the ith marked point is mapped to $\mathcal{Y}_{g_i}$. The order of the ith marked point is the order of g_i, hence the associated orbifold map $(\mathbb{P}^1,Q)\to\mathcal{Y}$ is representable as in [1]. Then the complex virtual dimension of $\overline{\mathcal{M}}_{0,k}(\mathcal{Y},A,\mathfrak{g})$ is given by [7, Theorem A]

$$\langle c_1(T\mathcal{Y}),A\rangle+\dim_{\mathbb{C}}(\mathcal{Y})+k-3-\sum_{i=1}^{k}\mathrm{age}(g_i).$$

This is equivalent to RHS of (2.4) with $k-3$ from the freedom of marked points and automorphisms and without $\sum_{i=1}^{k}\dim_{\mathbb{C}}\mathcal{Y}_{g_i}$ as we do not require marked points to go through fixed points in $\mathcal{Y}_{g_i}$. As in §2, we will be mainly interested in the case with one marked point. We use $\overline{\mathcal{M}}^{\bullet}_{0,1}(\mathcal{Y},A,g)$ to denote the compactified moduli space of orbifold $\mathbb{P}^1(1,\ell)$ in $\mathcal{Y}$ of homology class A with the order ℓ marked point mapped to a chosen point on $\mathcal{Y}_g$, where the order of g is ℓ. This is the compactification of the Deligne–Mumford stack $\mathrm{Mor}(\mathbb{P}^1(1,\ell),\mathcal{Y};f|_\infty)/\mathrm{Aut}(\mathbb{C})$. The virtual dimension is

$$\langle c_1(T\mathcal{Y}),A\rangle+\dim_{\mathbb{C}}\mathcal{Y}-2-\mathrm{age}(g)-\dim_{\mathbb{C}}\mathcal{Y}_g,$$

which is smaller than (2.5) by 2 from the automorphism of $\mathbb{P}(1,\ell)$.

We have a correspondence between $\pi_0(I\mathcal{Y})$ and the connected components of the Morse-Bott family of Reeb orbits with period at most the principal orbit. We may rescale the contact form such that the principal orbit (the simple Reeb orbit over a non-singular point of M/S^1) has period 1. For $g\in\pi_0(I\mathcal{Y})$, we use γ_g to represent the corresponding Reeb orbit. We use $\gamma_{g,l}$ to denote the Reeb orbit γ_g followed by $l\in\mathbb{N}$ principal orbits. They form all the Reeb orbits on M. Let W be a strong symplectic (orbifold) filling of M, we use $\overline{\mathcal{M}}_{\mathrm{SFT}}(W,A,\gamma_{g,l})$ to denote the compactified moduli spaces of holomorphic planes in the completion $\widehat{W}$ of homology class A and one positive puncture asymptotic to $\gamma_{g,l}$, see e.g., [10, §1.5]. The *real* virtual dimension of $\overline{\mathcal{M}}_{\mathrm{SFT}}(W,A,\gamma_{g,l})$ is given by [10, Proposition 1.7.1] (strictly speaking, we are considering the Morse-Bott case with the holomorphic curve asymptotic to a fixed orbit in the family, therefore we use μ_{LCZ})

$$\mu^A_{\mathrm{LCZ}}(\gamma_{g,l})+\dim_{\mathbb{C}}W-3, \tag{3.2}$$

where $\mu^A_{\mathrm{LCZ}}(\gamma_{g,l})$ is the lower semi-continuous Conley-Zehnder index of $\gamma_{g,l}$ using trivializations of $\det_{\mathbb{C}}u^*TW$ such that $u:\mathbb{D}\to W$ is a continuous (orbifold) map with boundary mapped to $\gamma_{g,l}$ with homology class A. In our case, the unit disk bundle $D(\mathcal{L}^{-1})$ in $\mathcal{L}^{-1}$ is naturally a symplectic orbifold filling of M, and we have the following.

Proposition 3.1. *Let γ be a Reeb orbit and u a disk in $D(\mathcal{L}^{-1})$ with boundary $\gamma_{g,l}$, whose intersection number with the zero section is zero. Then we have*

$$\mathrm{lSFT}(\gamma_{g,l})=\mathrm{vdim}_{\mathbb{R}}\overline{\mathcal{M}}_{\mathrm{SFT}}\big(D\big(\mathcal{L}^{-1}\big),[u],\gamma_{g,l}\big).$$

Proof. By definition

$$\mathrm{lSFT}(\gamma_{g,l})=\mu^{\mathbb{Q}}_{\mathrm{LCZ}}(\gamma_{g,l})+\dim_{\mathbb{C}}D(\mathcal{L}^{-1})-3,$$

where $\mu^{\mathbb{Q}}_{\mathrm{LCZ}}(\gamma_{g,l})$ is computed using a trivialization of $\det_{\mathbb{C}}\oplus^N\xi$ for the contact structure ξ of M and some $N\in\mathbb{Z}_{>0}$. By [14, Lemma 3.1], as $c_1(D(\mathcal{L}^{-1}))$ viewed in $H^2\big(D(\mathcal{L}^{-1}),M;\mathbb{Q}\big)$ is Lefschetz dual to a multiple of the fundamental class of the zero section, we have $\big\langle c_1\big(D(\mathcal{L}^{-1})\big),[u]\big\rangle=0$ and

$$\mathrm{lSFT}(\gamma_{g,l})=\mu^u_{\mathrm{LCZ}}(\gamma_{g,l})+\dim_{\mathbb{C}}D(\mathcal{L}^{-1})-3=\mathrm{vdim}_{\mathbb{R}}\overline{\mathcal{M}}_{\mathrm{SFT}}\big(D\big(\mathcal{L}^{-1}\big),[u],\gamma_{g,l}\big). \qquad\square$$

We restate (2.6) as follows and reprove it using (3.1) as the definition of mld. The proof is disguised as degenerating the SFT curves in $D(\mathcal{L}^{-1})$ into fiber holomorphic disks and holomorphic curves contained in the zero section $\mathcal{Y}_0$, hence we can relate their virtual dimensions. However, as we only care about the virtual dimension, we do not require such degeneration to actually happen. One can verify the formulae in the proof directly using the virtual dimension formulae without appealing to the degeneration picture.

Proposition 3.2. *Let (X,o) be an n-dimensional Fano cone singularity over a Fano orbifold $\mathcal{Y}$ with associated ample line bundle $\mathcal{L}$. If $\overline{\mathcal{M}}^{\bullet}_{0,1}(\mathcal{Y},A,g)\neq\emptyset$, we have*

$$\mathrm{mld}(o,X)\leq \mathrm{vdim}_{\mathbb{C}}\overline{\mathcal{M}}^{\bullet}_{0,1}(\mathcal{Y},A,g)+2.$$

Proof. We start with the seemingly simpler case where $g=1$ represents the identity component $\mathcal{Y}$ of $I\mathcal{Y}$, i.e. the marked point on $\mathbb{P}^1$ is not an orbifold point. Assume $u\in\overline{\mathcal{M}}^{\bullet}_{0,1}(D(\mathcal{L}^{-1}),A,1)$. Here we treat u as irreducible; the nodal case is the same. Then $u^*\mathcal{L}$ is an ample bundle over $\mathbb{P}^1$. We write $k=\langle c_1(\mathcal{L}),A\rangle>0$, let v be the fiber disk in $D(\mathcal{L}^{-1})$ over a smooth point in $\mathcal{Y}$ and v^k be the natural kth branched cover of v. Then we have

$$\mathrm{vdim}_{\mathbb{R}}\overline{\mathcal{M}}_{\mathrm{SFT}}\big(D\big(\mathcal{L}^{-1},[v^k],\gamma^k_1\big)\big)=2k-2.$$

This follows from that $\mu^{v^k}_{LCZ}(\gamma^k_1)=2k-\dim_{\mathbb{C}}\mathcal{Y}$ as the linearized flow of γ^k_1 using the induced trivialization from v^k is rotation by k rounds in the fiber direction and identity on the complement. Then the claim follows from (3.2). Since $v^k\#u$ has intersection number zero with $\mathcal{Y}_0$, by Proposition 3.1, we have

$$\begin{aligned}\mathrm{lSFT}(\gamma^k_{\mathrm{id}})&=\mathrm{vdim}_{\mathbb{R}}\overline{\mathcal{M}}_{\mathrm{SFT}}\big(D\big(\mathcal{L}^{-1}\big),v^k\#u,\gamma^k_1\big)\\&=\mathrm{vdim}_{\mathbb{R}}\overline{\mathcal{M}}_{\mathrm{SFT}}\big(D\big(\mathcal{L}^{-1}\big),[v^k],\gamma^k_1\big)+4+\mathrm{vdim}_{\mathbb{R}}\overline{\mathcal{M}}^{\bullet}_{0,1}\big(D\big(\mathcal{L}^{-1}\big),[u],1\big).\end{aligned}\tag{3.3}$$

To see the last equality, the nodal curve $v^k\cup u$ is contained in $\overline{\mathcal{M}}_{\mathrm{SFT}}\big(D\big(\mathcal{L}^{-1}\big),v^k\#u,\gamma^k_1\big)$, and those nodal curves form a stratum of $\overline{\mathcal{M}}_{\mathrm{SFT}}\big(D\big(\mathcal{L}^{-1}\big),v^k\#u,\gamma^k_1\big)$ of virtual codimension 2. On the other hand, those nodal curves form the fiber product

$$\overline{\mathcal{M}}_{\mathrm{SFT},1}\big(D\big(\mathcal{L}^{-1}\big),[v^k],\gamma^k_1\big)\times_{D(\mathcal{L}^{-1})}\overline{\mathcal{M}}_{0,1}\big(D\big(\mathcal{L}^{-1}\big),[u],1\big),$$

where $\overline{\mathcal{M}}_{\mathrm{SFT},1}(D(\mathcal{L}^{-1}),[v^k],\gamma^k_1)$ is the SFT curves as in $\overline{\mathcal{M}}_{\mathrm{SFT}}(D(\mathcal{L}^{-1}),[v^k],\gamma^k_1)$ but with one marked point and the fiber product is taken over the evaluation maps on the marked points. Then we have

$$\mathrm{vdim}_{\mathbb{R}}\overline{\mathcal{M}}_{\mathrm{SFT},1}\big(D\big(\mathcal{L}^{-1}\big),[v^k],\gamma^k_1\big)\times_{D(\mathcal{L}^{-1})}\overline{\mathcal{M}}_{0,1}\big(D\big(\mathcal{L}^{-1}\big),[u],1\big)$$

is

$$\begin{aligned}&\mathrm{vdim}_{\mathbb{R}}\overline{\mathcal{M}}_{\mathrm{SFT},1}\big(D\big(\mathcal{L}^{-1}\big),[v^k],\gamma^k_1\big)+\mathrm{vdim}_{\mathbb{R}}\overline{\mathcal{M}}_{0,1}\big(D\big(\mathcal{L}^{-1}\big),[u],1\big)-\dim_{\mathbb{R}}D\big(\mathcal{L}^{-1}\big)\\=&\mathrm{vdim}_{\mathbb{R}}\overline{\mathcal{M}}_{\mathrm{SFT}}\big(D\big(\mathcal{L}^{-1}\big),[v^k],\gamma^k_1\big)+2+\mathrm{vdim}_{\mathbb{R}}\overline{\mathcal{M}}^{\bullet}_{0,1}\big(D\big(\mathcal{L}^{-1}\big),[u],1\big).\end{aligned}$$

Therefore (3.3) holds. Note that

$$\mathrm{vdim}_{\mathbb{R}}\overline{\mathcal{M}}^{\bullet}_{0,1}\big(D\big(\mathcal{L}^{-1}\big),[u],1\big)=2+\mathrm{vdim}_{\mathbb{R}}\overline{\mathcal{M}}^{*}_{0,1}(\mathcal{Y},[u],1)-2\langle c_1(\mathcal{L}),A\rangle-2.$$

The first 2 is from the fact that the point constraint in $\overline{\mathcal{M}}^{\bullet}_{0,1}(D(\mathcal{L}^{-1}),[u],1)$ is of real codimension $\dim_{\mathbb{R}}D(\mathcal{L}^{-1})$, while the point constraint in $\overline{\mathcal{M}}^{\bullet}_{0,1}(\mathcal{Y},[u],1)$ is of real codimension $\dim_{\mathbb{R}}\mathcal{Y}$. The $-2\langle c_1(\mathcal{L}),A\rangle-2$ comes from the index of the Cauchy-Riemann operator in the normal direction of $\mathcal{Y}_0\subset D(\mathcal{L}^{-1})$. Combining them together, we have

$$\mathrm{lSFT}(\gamma^k_1)=\mathrm{vdim}_{\mathbb{R}}\overline{\mathcal{M}}^{\bullet}_{0,1}(\mathcal{Y},[u],1)+2$$

In general, assume g is represented by $G\in S$ and multiplicity $k\neq 0\in G\backslash G^{-}\subset\mathbb{Z}/|G|$. The orbifold marked point on $\mathbb{P}^1$ then has order $|G|/\gcd(k,|G|)$. Then $u^*(\mathcal{L}^{-1})$ has a meromorphic section without zero and an order $-\langle c_1(\mathcal{L}^{-1}),A\rangle$ pole at the marked point. Then $-k/|G|+\langle c_1(\mathcal{L}^{-1}),A\rangle$ is a negative integer $-l$. We consider the Reeb orbit $\gamma_{g,l-1}$. Let v be the fiber orbifold disk over a generic point in M^G/S^1, with boundary $\gamma_{g,l-1}$ and an orbifold marked point asymptotic to $\mathcal{Y}_g\subset ID(\mathcal{L}^{-1})$, we use

$$\mathrm{vdim}_{\mathbb{R}}\overline{\mathcal{M}}_{\mathrm{SFT},1}\big(D\big(\mathcal{L}^{-1}\big),[v],\gamma_{g,l-1},g\big)$$

to denote the moduli space of such v, then

$$\mathrm{vdim}_{\mathbb{R}}\overline{\mathcal{M}}_{\mathrm{SFT},1}\big(D\big(\mathcal{L}^{-1}\big),[v],\gamma_{g,l-1},g\big)=2l-2.$$

as this can be computed in the local model $\mathbb{C}\times\mathbb{C}^{n-1}/\frac{1}{|G|}(1,b_2,\cdots,b_n)$. Since we have

$$\begin{aligned}\mathrm{vdim}_{\mathbb{R}}\overline{\mathcal{M}}^{\bullet}_{0,1}(D(\mathcal{L}^{-1}),[u],g)&=\mathrm{vdim}_{\mathbb{R}}\overline{\mathcal{M}}^{\bullet}_{0,1}(\mathcal{Y},[u],g)+2+2\Big\langle c_1\big(\mathcal{L}^{-1}\big),[u]\Big\rangle-\frac{2k}{|G|}\\&=\mathrm{vdim}_{\mathbb{R}}\overline{\mathcal{M}}^{\bullet}_{0,1}(\mathcal{Y},[u],g)+2-2l,\end{aligned}$$

where $2+2\langle c_1(\mathcal{L}^{-1}),[u]\rangle-2k/|G|$ is the Fredholm index of the Cauchy-Riemann operator in the normal direction. Since $v\#u$ has zero intersection with $\mathcal{Y}_0$, by Proposition 3.1, we have

$$\begin{aligned}\mathrm{lSFT}(\gamma_{g,l-1})&=\mathrm{vdim}_{\mathbb{R}}\overline{\mathcal{M}}_{\mathrm{SFT}}\big(D\big(\mathcal{L}^{-1}\big),[v\#u],\gamma_{g,l-1}\big)\\&=2+\mathrm{vdim}_{\mathbb{R}}\overline{\mathcal{M}}_{\mathrm{SFT},1}\big(D\big(\mathcal{L}^{-1}\big),[v],\gamma_{g,l-1}\big)\times_{\mathcal{Y}_g}\overline{\mathcal{M}}_{0,1}\big(D\big(\mathcal{L}^{-1}\big),[u],g\big)\\&=2+\mathrm{vdim}_{\mathbb{R}}\overline{\mathcal{M}}_{\mathrm{SFT},1}\big(D\big(\mathcal{L}^{-1}\big),[v],\gamma_{g,l-1}\big)+\mathrm{vdim}\overline{\mathcal{M}}_{0,1}(D(\mathcal{L}^{-1}),[u],g)-\dim_{\mathbb{R}}\mathcal{Y}_g\\&=2+\mathrm{vdim}_{\mathbb{R}}\overline{\mathcal{M}}_{\mathrm{SFT},1}\big(D\big(\mathcal{L}^{-1}\big),[v],\gamma_{g,l-1}\big)+\mathrm{vdim}_{\mathbb{R}}\overline{\mathcal{M}}^{\bullet}_{0,1}\big(D\big(\mathcal{L}^{-1}\big),[u],g\big).\end{aligned}$$

Then $\mathrm{lSFT}(\gamma_{g,l-1})=2+\mathrm{vdim}_{\mathbb{R}}\overline{\mathcal{M}}^{\bullet}_{0,1}(\mathcal{Y},[u],g)$. Since $2\mathrm{mld}(X,o)\leq\mathrm{lSFT}(\gamma)+2$, we have $\mathrm{mld}(X,o)\leq\mathrm{vdim}_{\mathbb{C}}\overline{\mathcal{M}}^{\bullet}_{0,1}(\mathcal{Y},[u],g)+2$. □

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