

# Exterior Dirichlet Problem for Hessian Equations on a Non-Convex Ring

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**Dedicated to Gang Tian on his 65th birthday with friendship and respect.**

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**Abstract.** In this paper, we prove the existence of a solution for the exterior Dirichlet problem for Hessian equations on a non-convex ring. Moreover, the solution we obtained is smooth. This extends the result of [Bao-Li-Li, “On the exterior Dirichlet problem for Hessian equations” *Trans. Amer. Math. Soc.* 366(2014)].

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## 1 Introduction

In this paper,  $D^2u$  denotes the Hessian of  $u$ , and the  $k$ -th elementary symmetric function  $\sigma_k(A)$  of a symmetric matrix  $A$  is defined by

$$\sigma_k(A) = \sigma_k(\lambda(A)) = \sum_{1 \leq i_1 < \dots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k},$$

where  $\lambda(A) = (\lambda_1, \dots, \lambda_n)$  are the eigenvalues of  $A$ . Let  $\Gamma_k$  be the Garding's cone

$$\Gamma_k = \{\lambda \in \mathbb{R}^n : \sigma_m(\lambda) > 0, m = 1, \dots, k\}.$$

It is well known that  $\Gamma_k$  is a convex symmetric cone with vertex at the origin.

In this paper, we consider the solvability of the following exterior Dirichlet problem for Hessian equations

$$\begin{cases} \sigma_k(\lambda(D^2u)) = 1 & \text{in } \mathbb{R}^n \setminus \bar{D}, \\ u = \varphi & \text{on } \Gamma := \partial D. \end{cases} \quad (1.1)$$

Before stating our main theorems, we will need the following definitions.

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**Definition 1.1.** A  $C^2$  regular hypersurface  $\mathcal{M} \subset \mathbb{R}^n$  is called **strictly  $k$ -convex** ( $k$ -convex) if its principal curvature vector  $\kappa(X) \in \Gamma_k$  ( $\kappa(X) \in \bar{\Gamma}_k$ ) for all  $X \in \mathcal{M}$ . We say that a domain  $D$  is strictly  $k$ -convex ( $k$ -convex) if  $\partial D$  is strictly  $k$ -convex ( $k$ -convex).

**Definition 1.2.** For any open set  $U \subset \mathbb{R}^n$ , a function  $v \in C^2(U)$  is called a **strictly  $k$ -convex** ( $k$ -convex) function in  $U$  if  $\sigma_j(D^2v(x)) > 0$ , ( $\sigma_j(D^2v(x)) \geq 0$ ) for all  $j = 1, \dots, k$  and  $x \in U$ .

**Definition 1.3.** A function  $u \in C^0(\mathbb{R}^n \setminus \bar{D})$  is said to be a viscosity subsolution of (1.1) in  $\mathbb{R}^n \setminus \bar{D}$ , if for any  $k$ -convex function  $\psi \in C^2(\mathbb{R}^n \setminus \bar{D})$  and  $\bar{x} \in \mathbb{R}^n \setminus \bar{D}$  satisfying

$$\psi(\bar{x}) = u(\bar{x}) \quad \text{and} \quad \psi \geq u \text{ on } \mathbb{R}^n \setminus \bar{D},$$

we have

$$\sigma_k(\lambda(D^2\psi(\bar{x}))) \geq 1.$$

Similarly,  $u \in C^0(\mathbb{R}^n \setminus \bar{D})$  is a viscosity supersolution of (1.1) in  $\mathbb{R}^n \setminus \bar{D}$ , if for any  $k$ -convex function  $\psi \in C^2(\mathbb{R}^n \setminus \bar{D})$  and  $\bar{x} \in \mathbb{R}^n \setminus \bar{D}$  satisfying

$$\psi(\bar{x}) = u(\bar{x}) \quad \text{and} \quad \psi \leq u \text{ on } \mathbb{R}^n \setminus \bar{D},$$

we have

$$\sigma_k(\lambda(D^2\psi(\bar{x}))) \leq 1.$$

We say  $u$  is a viscosity solution of (1.1) if  $u$  is both a viscosity subsolution and a viscosity supersolution of (1.1).

Following [1] we denote

$$\mathcal{A}_k = \{A : A \text{ is a real } n \times n \text{ symmetric positive definite matrix with } \sigma_k(\lambda(A)) = 1\}.$$

Our main results are

**Theorem 1.1.** Let  $D$  be a bounded, smooth, star-shaped, strictly  $(k-1)$ -convex domain in  $\mathbb{R}^n$ ,  $n \geq 3$ , and let  $\varphi \in C^\infty(\partial D)$ . Then for any given  $b \in \mathbb{R}^n$ ,  $A \in \mathcal{A}_k$  with  $2 \leq k \leq n$ , there exists some constant  $c_* = c_*(n, k, b, A, D, |\varphi|_{C^2(\partial D)})$ , such that for every  $c > c_*$ , there exists a unique strictly  $k$ -convex solution  $u \in C^\infty(\mathbb{R}^n \setminus D)$  of (1.1) satisfying

$$\limsup_{|x| \rightarrow \infty} |x|^{n-2} \left[ u - \left( \frac{1}{2} x^T A x + b \cdot x + c \right) \right] < \infty. \tag{1.2}$$

**Theorem 1.2.** Let  $u \in C^\infty(\mathbb{R}^n \setminus D)$  be the solution of (1.1) obtained in Theorem 1.1 and denote

$$E(x) := u - \left( \frac{1}{2} x^T A x + b \cdot x + c \right).$$

Then  $E(x)$  satisfies

$$\limsup_{|x| \rightarrow \infty} |x|^{n-2+m} |D^m E(x)| < \infty \tag{1.3}$$

for any integer  $m \geq 1$ .