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Exterior Dirichlet Problem for Hessian Equations on a Non-Convex Ring

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Dedicated to Gang Tian on his 65th birthday with friendship and respect.

Abstract. In this paper, we prove the existence of a solution for the exterior Dirichlet problem for Hessian equations on a non-convex ring. Moreover, the solution we obtained is smooth. This extends the result of [Bao-Li-Li, "On the exterior Dirichlet problem for Hessian equations" Trans. Amer. Math. Soc. 366(2014)].

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1 Introduction

In this paper, D^2u denotes the Hessian of u, and the k-th elementary symmetric function $\sigma_k(A)$ of a symmetric matrix A is defined by

$$\sigma_k(A)=\sigma_k(\lambda(A))=\sum_{1\le i_1<\cdots<i_k\le n}\lambda_{i_1}\cdots\lambda_{i_k},$$

where $\lambda(A)=(\lambda_1,\cdots,\lambda_n)$ are the eigenvalues of A. Let Γ_k be the Garding's cone

$$\Gamma_k=\{\lambda\in\mathbb{R}^n:\sigma_m(\lambda)>0, m=1,\cdots,k\}.$$

It is well known that Γ_k is a convex symmetric cone with vertex at the origin.

In this paper, we consider the solvability of the following exterior Dirichlet problem for Hessian equations

$$\begin{cases}\sigma_k\big(\lambda\big(D^2u\big)\big)=1 & \text{in } \mathbb{R}^n\setminus\bar{D},\\ u=\varphi & \text{on } \Gamma:=\partial D.\end{cases}\tag{1.1}$$

Before stating our main theorems, we will need the following definitions.

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Definition 1.1. *A C^2 regular hypersurface $\mathcal{M}\subset\mathbb{R}^n$ is called* ***strictly k-convex*** *(k-convex) if its principal curvature vector $\kappa(X)\in\Gamma_k$ $(\kappa(X)\in\bar{\Gamma}_k)$ for all $X\in\mathcal{M}$. We say that a domain D is strictly k-convex (k-convex) if ∂D is strictly k-convex (k-convex).*

Definition 1.2. *For any open set $U\subset\mathbb{R}^n$, a function $v\in C^2(U)$ is called a* ***strictly k-convex*** *(k-convex) function in U if $\sigma_j\big(D^2v(x)\big)>0$, $\big(\sigma_j\big(D^2v(x)\big)\geq 0\big)$ for all $j=1,\cdots,k$ and $x\in U$.*

Definition 1.3. *A function $u\in C^0(\mathbb{R}^n\setminus\bar{D})$ is said to be a viscosity subsolution of* (1.1) *in $\mathbb{R}^n\setminus\bar{D}$, if for any k-convex function $\psi\in C^2(\mathbb{R}^n\setminus\bar{D})$ and $\bar{x}\in\mathbb{R}^n\setminus\bar{D}$ satisfying*

$$\psi(\bar{x})=u(\bar{x})\quad \textit{and}\quad \psi\geq u \textit{ on } \mathbb{R}^n\setminus\bar{D},$$

we have

$$\sigma_k\big(\lambda\big(D^2\psi(\bar{x})\big)\big)\geq 1.$$

Similarly, $u\in C^0(\mathbb{R}^n\setminus\bar{D})$ is a viscosity supersolution of (1.1) *in $\mathbb{R}^n\setminus\bar{D}$, if for any k-convex function $\psi\in C^2(\mathbb{R}^n\setminus\bar{D})$ and $\bar{x}\in\mathbb{R}^n\setminus\bar{D}$ satisfying*

$$\psi(\bar{x})=u(\bar{x})\quad \textit{and}\quad \psi\leq u \textit{ on } \mathbb{R}^n\setminus\bar{D},$$

we have

$$\sigma_k\big(\lambda\big(D^2\psi(\bar{x})\big)\big)\leq 1.$$

We say u is a viscosity solution of (1.1) *if u is both a viscosity subsolution and a viscosity supersolution of* (1.1).

Following [1] we denote

$$\mathcal{A}_k=\{A: A \text{ is a real } n\times n \text{ symmetric positive definite matrix with } \sigma_k(\lambda(A))=1\}.$$

Our main results are

Theorem 1.1. *Let D be a bounded, smooth, star-shaped, strictly $(k-1)$-convex domain in $\mathbb{R}^n$, $n\geq 3$, and let $\varphi\in C^\infty(\partial D)$. Then for any given $b\in\mathbb{R}^n$, $A\in\mathcal{A}_k$ with $2\leq k\leq n$, there exists some constant $c_*=c_*\big(n,k,b,A,D,|\varphi|_{C^2(\partial D)}\big)$, such that for every $c>c_*$, there exists a unique strictly k-convex solution $u\in C^\infty(\mathbb{R}^n\setminus D)$ of* (1.1) *satisfying*

$$\limsup_{|x|\to\infty}|x|^{n-2}\left[u-\left(\frac{1}{2}x^TAx+b\cdot x+c\right)\right]<\infty. \tag{1.2}$$

Theorem 1.2. *Let $u\in C^\infty(\mathbb{R}^n\setminus D)$ be the solution of* (1.1) *obtained in Theorem* 1.1 *and denote*

$$E(x):=u-\left(\frac{1}{2}x^TAx+b\cdot x+c\right).$$

Then $E(x)$ satisfies

$$\limsup_{|x|\to\infty}|x|^{n-2+m}|D^mE(x)|<\infty \tag{1.3}$$

for any integer $m\geq 1$.

Remark 1.1. For $k=n$, our result extends [4, Theorem 1.5] by showing that the solution u belongs to $C^\infty(\mathbb{R}^n\setminus D)$, whereas [4] established $u\in C^\infty(\mathbb{R}^n\setminus \bar{D})$.

By the discussions on [1, page 4], we know that we can always assume $A\in\mathcal{A}_k$ to be diagonal. Now, let $s=\frac{1}{2}\sum_{i=1}^n a_i x_i^2$, then for $A=\mathrm{diag}(a_1,\cdots,a_n)\in\mathcal{A}_k$ we denote

$$E_\lambda:=\left\{x\in\mathbb{R}^n: s\equiv\frac{1}{2}\sum_{i=1}^n a_i x_i^2<\lambda,\lambda>0\right\}.$$

Moreover, in this paper, without loss of generality, we will always assume $b=\vec{0}$, $\bar{D}\subset E_1$, and D is star-shaped with respect to the origin. When $k=n$, D is a strictly convex domain, which is automatically star-shaped with respect to any interior point.

The Dirichlet problem on exterior domains is closely related to the asymptotic behavior of solutions defined on entire $\mathbb{R}^n$. In [4], Caffarelli and the first named author proved that if u is a convex viscosity solution of

$$\det(D^2u)=1 \tag{1.4}$$

outside a bounded subset of $\mathbb{R}^n$, then

$$\limsup_{|x|\to\infty}|x|^{n-2}\left[u-\left(\frac{1}{2}x^TAx+b\cdot x+c\right)\right]<\infty,\quad A\in\mathcal{A}_n.$$

Moreover, with such prescribed asymptotic behavior near infinity, they also established an existence and uniqueness theorem for solutions of (1.4). Since then, the solvability of the exterior Dirichlet problem for fully nonlinear equations has been studied intensively, see for example [6, 8–10] and references therein. We remark that all these earlier works require that the domain D is strictly convex. Moreover, the solutions found in these earlier works are viscosity solutions (except for the Monge–Ampère case). While in this paper, the solutions we obtain are smooth and the domain D is only required to be strictly $(k-1)$-convex.

The organization of the paper is as follows. In Section 2, we construct a viscosity subsolution for (1.1). In Section 3, by solving (1.1) on bounded domains we prove the existence part of Theorem 1.1. In Section 4, we study the asymptotic behavior of the solution near infinity. More precisely, we derive inequality (1.2) and prove Theorem 1.2.

2 Construction of the subsolution of (1.1)

In this section, we will combine ideas in [1, 4], and [11] to construct subsolutions for (1.1). More precisely, we will first apply the techniques developed in [11, Theorem 1.5] to construct subsolutions of (1.1) in the region $E_1\setminus D$. Then we will use the generalized symmetric subsolutions, which were studied in [1, Section 2], to construct subsolutions of (1.1) in the region $\mathbb{R}^n \setminus E_1$. Finally, we use the idea from [4] to glue the two subsolutions together and obtain a subsolution of (1.1).

We want to note that, in this section, we will only explicitly construct the subsolution of (1.1) for $2 \le k < n$. When $k = n$, the subsolution can be constructed in the same way but much easier. In particular, when $k = n$, in the region $E_1 \setminus D$, since the domain D is strictly convex, we can use the distance function, more precisely, $\mathrm{dist}(x,\partial D)+1$, instead of $\mathfrak{b}$ defined in (2.6); in the region $\mathbb{R}^n \setminus E_1$ we can use the symmetric subsolution that was studied in [4] instead of the generalized symmetric subsolution.

2.1 Subsolutions in $E_1 \setminus D$

Recall that $\Gamma := \partial D$ is star-shaped and strictly $(k-1)$-convex, we can parametrize Γ as a graph of the radial function $\rho(\theta) : \mathbb{S}^{n-1} \to \mathbb{R}$, i.e.,

$$\Gamma = \left\{ \rho(\theta)\theta : \theta \in \mathbb{S}^{n-1} \right\}.$$

Then the second fundamental form of Γ is ([2] for example)

$$h_{ij} = \frac{\rho}{w}\left(\delta_{ij} + 2\frac{\rho_i \rho_j}{\rho^2} - \frac{\rho_{i,j}}{\rho}\right),$$

where $w = \sqrt{1+|\nabla\rho|^2/\rho^2}$, $\rho_{i,j} = \nabla_{ij}\rho$, and ∇ denotes the Levi-Civita connection on $\mathbb{S}^{n-1}$.

We denote $\Phi = \log\rho$, it is clear that the second fundamental form of Γ can be expressed as follows.

$$h_{ij} = \frac{\rho}{w}\left(\delta_{ij} + \Phi_i \Phi_j - \Phi_{i,j}\right),$$

where $w = \sqrt{1+|\nabla\Phi|^2}$. By a direct calculation, we obtain

$$g_{ij} = \rho^2(\delta_{ij} + \Phi_i\Phi_j), \quad g^{ij} = \frac{1}{\rho^2}\left(\delta_{ij} - \frac{\Phi_i\Phi_j}{w^2}\right) \quad \text{and} \quad \gamma^{ij} = \frac{1}{\rho}\left(\delta_{ij} - \frac{\Phi_i\Phi_j}{w(1+w)}\right). \tag{2.1}$$

Here, (g_{ij}) is the metric on Γ, (g^{ij}) is the inverse of (g_{ij}), and γ^{ij} is the square root of g^{ij}, i.e., $\sum_k \gamma^{ik}\gamma^{kj} = g^{ij}$. Let $a_{ij} = \gamma^{ik} h_{kl} \gamma^{lj}$, then the eigenvalues of $(a_{ij})_{1\le i,j\le n-1}$, denoted by $\kappa[a_{ij}] = (\kappa_1, \cdots, \kappa_{n-1})$ are the principal curvatures of Γ.

The following calculation can be found in [11, Section 3], for readers convenience, we include it here.

2.1.1 Hessian in spherical coordinates

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a scalar function, then f can also be expressed as a function of $(\theta, r) \in \mathbb{S}^{n-1} \times \mathbb{R}$. Note that the Euclidean metric is $g_E = r^2 dz^2 + dr^2$, where dz^2 is the standard metric on $\mathbb{S}^{n-1}$. In the following, we will denote the standard connection in $\mathbb{R}^n$ by D. Now, we choose a local orthonormal frame $\{e_1, \cdots, e_{n-1}\}$ on the unit sphere $\mathbb{S}^{n-1}$. Let $\tau_a = e_a/r$, $1 \le a \le n-1$, which is the orthonormal frame on the sphere with radius r, and we also let $\tau_r = \frac{\partial}{\partial r}$. Then a direct calculation yields the Hessian of f in spherical coordinates is

$$D^2_{ab}f = D^2 f(\tau_a,\tau_b) = \frac{1}{r^2} f_{ab} + \frac{1}{r} f_r \delta_{ab}, \tag{2.2}$$

$$D^2_{ar}f = D^2 f(\tau_a,\tau_r) = \frac{1}{r} f_{ar} - \frac{1}{r^2} f_a, \tag{2.3}$$

$$D^2_{rr}f = D^2 f(\tau_r,\tau_r) = f_{rr}. \tag{2.4}$$

Here $1 \le a,b \le n-1$, $f_{ab} = e_b e_a f$, $f_{ar} = \tau_r e_a f$ and $f_{rr} = \tau_r \tau_r f$.

2.1.2 Construction of subsolutions in $E_1 \setminus D$

Now, we fix an arbitrary point $p \in \mathbb{S}^{n-1}$, let $\{e_1,\cdots,e_{n-1}\}$ be the normal coordinates at p, then the Christoffel symbols vanish at p. This implies at this point we get $\nabla_{ij}\rho = e_i e_j \rho$. In the following, for any function f defined in a small neighborhood of p, we denote $f_{ij} = e_i e_j f$, then we have $\rho_{i,j} = \rho_{ij}$ at p. Moreover, we may rotate the coordinates such that $|\nabla \rho(p)| = \rho_1$ and $\rho_{\alpha\beta}(p) = \rho_{\alpha\alpha}\delta_{\alpha\beta}$ for $2 \le \alpha,\beta \le n-1$. Then at the point $\hat{p} = \rho(p)p \in \Gamma$, in view of (2.1) we have

$$\begin{cases} \gamma^{11} = \frac{1}{\rho}\left(1 - \frac{w^2-1}{w(1+w)}\right) = \frac{1}{\rho w}, & \\ \gamma^{1\alpha} = 0, & 2 \le \alpha \le n-1, \\ \gamma^{\alpha\beta} = \frac{1}{\rho}\delta_{\alpha\beta}, & 2 \le \alpha,\beta \le n-1, \end{cases}$$

and

$$\begin{cases} a_{11} = \gamma^{1k} h_{kl} \gamma^{l1} = \gamma^{11} h_{11} \gamma^{11} = \frac{h_{11}}{\rho^2 w^2}, & \\ a_{1\alpha} = \gamma^{1k} h_{kl} \gamma^{l\alpha} = \frac{h_{1\alpha}}{\rho^2 w}, & 2 \le \alpha \le n-1, \\ a_{\alpha\beta} = \gamma^{\alpha\alpha} h_{\alpha\beta} \gamma^{\beta\beta} = \frac{1}{\rho^2} h_{\alpha\beta}, & 2 \le \alpha,\beta \le n-1. \end{cases} \tag{2.5}$$

Now, let us consider the function

$$\mathfrak{b} = \frac{r}{\rho(\theta)}. \tag{2.6}$$

By a straight forward calculation we obtain that at the point $(p,r) \in (\mathbb{S}^{n-1} \times R) \setminus \{0\}$ the Hessian of $\mathfrak{b}$ is (for details see [11, Subsection 3.2])

$$\mathrm{Hessian}(\mathfrak{b}) = \begin{bmatrix} \frac{w^3}{r} a_{11} & \frac{w^2}{r} a_{12} & \cdots & \frac{w^2}{r} a_{1n-1} & 0 \\ \frac{w^2}{r} a_{12} & \frac{w}{r} a_{22} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{w^2}{r} a_{1n-1} & 0 & \cdots & \frac{w}{r} a_{n-1n-1} & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

We want to point out that $\kappa[a_{ij}]$ are the principal curvatures of Γ at $\hat{p}=\rho(p)p$.

Next, we will consider the function $\phi=\phi(\mathfrak{b})$, where ϕ is a function defined on $\mathbb{R}$. We will compute the Hessian of ϕ at (p,r). Denote

$$\phi'|_{(p,r)}=\frac{d\phi}{d\mathfrak{b}}|_{(p,r)}=M,\quad \phi''|_{(p,r)}=\frac{d^2\phi}{d\mathfrak{b}^2}|_{(p,r)}=B,$$

we get at this point, for $1\le i,j\le n$,

$$D^2_{ij}\phi=MD^2_{ij}\mathfrak{b}+B(\tau_i\mathfrak{b})(\tau_j\mathfrak{b}).$$

Here, $\tau_a=\frac{e_a}{r}$ for $1\le a\le n-1$, $\tau_n:=\tau_r=\frac{\partial}{\partial r}$, $\{e_1,\cdots,e_{n-1}\}$ be the normal coordinates at p chosen above, and $\{\tau_1,\cdots,\tau_n\}$ forms an orthonormal frame of $\mathbb{R}^n$ at (p,r). We denote $\rho_a:=e_a\rho$, then at (p,r) we have

$$\text{Hessian}(\phi)=\begin{bmatrix} \frac{Mw^3}{r}a_{11}+B\rho^{-4}\rho_1^2 & \frac{Mw^2}{r}a_{12} & \cdots & \frac{Mw^2}{r}a_{1n-1} & -B\rho^{-3}\rho_1 \\ \frac{Mw^2}{r}a_{12} & \frac{Mw}{r}a_{22} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ \frac{Mw^2}{r}a_{1n-1} & 0 & \cdots & \frac{Mw}{r}a_{n-1n-1} & 0 \\ -B\rho^{-3}\rho_1 & 0 & \cdots & 0 & B\rho^{-2} \end{bmatrix}.$$

Therefore, following the calculation on [11, Subsection 3.2] we get for any $2\le m\le k$

$$\begin{aligned}\sigma_m(D^2\phi)=&\frac{M^m w^{m+2}}{r^m}\left[\sigma_m\left(a_{\hat{i}\hat{j}}\right)-\sigma_m(a_{\alpha\beta})\right]\\ &+B\rho^{-2}\frac{M^{m-1}w^{m+1}}{r^{m-1}}\sigma_{m-1}\left(a_{\hat{i}\hat{j}}\right)+\left(\frac{Mw}{r}\right)^m\sigma_m(a_{\alpha\beta}),\end{aligned}\tag{2.7}$$

where $2\le\alpha,\beta\le n-1$ and $1\le\hat{i},\hat{j}\le n-1$. Moreover, we have

$$\sigma_1(D^2\phi)=\frac{Mw^3}{r}a_{11}+\frac{Mw}{r}\sigma_1(a_{\alpha\beta})+B\rho^{-2}+B\rho^{-4}\rho_1^2.\tag{2.8}$$

This yields for any $1\le m\le k$,

$$\sigma_m\left(D^2\phi\right)>-c_0\left(\frac{M}{r}\right)^m+c_1\frac{B}{\rho^2}\left(\frac{M}{r}\right)^{m-1}=\frac{M^{m-1}}{r^{m-1}}\left(\frac{c_1B}{\rho^2}-c_0\frac{M}{r}\right),\tag{2.9}$$

where

$$c_1=\min_{\hat{q}\in\Gamma}\sigma_{m-1}(\kappa(\hat{q}))>0\quad\text{and}\quad c_0=c_0(|\rho|_{C^2})>0$$

are two positive constants that only depend on Γ. We note that in this paper we use the convention $\sigma_0=1$, and when $m=1$ we let $c_1=1$.

Proposition 2.1. *Let D be a smooth, star-shaped, strictly $(k-1)$-convex domain in $\mathbb{R}^n$ for $n\geq 3$, and let $\varphi\in C^\infty(\partial D)$. Then given any $A\in\mathcal{A}_k$ with $2\leq k\leq n-1$, there exists $N_1=N_1(\varphi,\Gamma,A)>0$ such that when $N\geq N_1$, $\underline{\phi}=\mathfrak{b}^N-1+\varphi$ is strictly k-convex in $E_1\setminus\bar{D}$. Moreover, $\underline{\phi}$ satisfies $\sigma_k(\lambda(D^2\underline{\phi}))>1$ in $E_1\setminus\bar{D}$ and $\underline{\phi}=\varphi$ on Γ.*

Proof. Since Γ is star-shaped, it is clear that φ can be viewed as a function defined on $\mathbb{S}^{n-1}$. We may extend the domain of definition of φ to $\mathbb{R}^n\setminus\{0\}$ by setting $\varphi(r,\theta):=\varphi(\theta)$. Without causing any confusions, we will denote the extension of φ by φ.

Claim. Let $\phi=\mathfrak{b}^N$ and denote $\lambda\big(D^2(\phi)\big)=(\lambda_1,\cdots,\lambda_n)$ to be the eigenvalues of Hessian(ϕ), then $\lambda_a=O(M)$ for $a\leq n-1$ and $\lambda_n=B(\rho^{-4}|\nabla\rho|^2+\rho^{-2})+O(M)$. Here, $M=N\mathfrak{b}^{N-1}$ and $B=N(N-1)\mathfrak{b}^{N-2}$.

Proof of the claim. Following the proof of [5, Lemma 1.2], let's consider the eigenvalues of the following matrix

$$Q=\begin{bmatrix} d_1+aX^2 & a_2 & \cdots & a_{n-1} & -aXY\\ a_2 & d_2 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \\ a_{n-1} & 0 & \cdots & d_{n-1} & 0\\ -aXY & 0 & \cdots & 0 & aY^2 \end{bmatrix}$$

with $d_1,\cdots,d_{n-1}$, X,Y, and $a_2,\cdots,a_{n-1}$ being fixed. Denote

$$f_a(\lambda):=\det\begin{bmatrix} \frac{d_1}{a}+X^2-\frac{\lambda}{a} & a_2 & \cdots & a_{n-1} & -aXY\\ \frac{a_2}{a} & d_2-\lambda & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \\ \frac{a_{n-1}}{a} & 0 & \cdots & d_{n-1}-\lambda & 0\\ -XY & 0 & \cdots & 0 & aY^2-\lambda \end{bmatrix},$$

the eigenvalues of Q satisfies $f_a(\lambda)=0$. A direct calculation yields

$$\begin{aligned} f_a(\lambda)=&\left[\left(X^2+\frac{d_1}{a}-\frac{\lambda}{a}\right)(aY^2-\lambda)-aX^2Y^2\right]\prod_{\alpha=2}^{n-1}(d_\alpha-\lambda)\\ &-\sum_{\beta=2}^{n-1}\frac{a_\beta^2}{d_\beta-\lambda}\prod_{\alpha=2}^{n-1}(d_\alpha-\lambda)\cdot\left(Y^2-\frac{\lambda}{a}\right). \end{aligned}$$

For $a=\infty$, $f_\infty(\lambda)$ is a polynomial of degree $n-1$ with coefficients only depending on $d_1,\cdots,d_{n-1}$, X,Y, and $a_2,\cdots,a_{n-1}$. Therefore, it has $n-1$ roots $\tilde{d}_1,\cdots,\tilde{d}_{n-1}$ that are depending on $d_1,\cdots,d_{n-1}$, X,Y, and $a_2,\cdots,a_{n-1}$. By continuity of the roods it follows that $\lambda_a=\tilde{d}_a+o(1)$ for $a\leq n-1$.

To find the last eigenvalue set $\lambda = a\mu$. Then μ satisfies

$$\det \begin{bmatrix} \frac{d_1}{a}+X^2-\mu & \frac{a_2}{a} & \cdots & \frac{a_{n-1}}{a} & -XY \\ \frac{a_2}{a} & \frac{d_2}{a}-\mu & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ \frac{a_{n-1}}{a} & 0 & \cdots & \frac{d_{n-1}}{a}-\mu & 0 \\ -XY & 0 & \cdots & 0 & Y^2-\mu \end{bmatrix} = 0.$$

For $a=\infty$, we see that $\mu = X^2+Y^2$ is a simple root. By the implicity function theorem it follows that for $a>0$ large there is a root $\mu = X^2+Y^2+O(\frac{1}{a})$, i.e., $\lambda_n = a\left(X^2+Y^2+O(\frac{1}{a})\right)$. □

Now, recall that $\mathfrak{b}$ is uniformly bounded in $E_1\setminus\bar{D}$, we have $B/M=(N-1)/\mathfrak{b}\to\infty$ as $N\to\infty$. Therefore, we can apply the conclusion from above discussions to the matrix $\frac{1}{M}$Hessian(ϕ). In this case, correspondingly we have

$$d_1 = \frac{w^3}{r}a_{11}, \quad d_i = \frac{w}{r}a_{ii}, \quad a_i = \frac{w^2}{r}a_{1i} \quad \text{for } 2\le i\le n-1,$$

and

$$a = \frac{B}{M}, \quad X = \rho^{-2}\rho_1, \quad Y = \rho^{-1}.$$

Note that, if λ is an eigenvalue of $\frac{1}{M}$Hessian(ϕ), then $M\lambda$ is an eigenvalue of Hessian(ϕ). Moreover, by our choice of coordinates we have $|\nabla\rho(p)| = \rho_1$. The claim follows immediately. By the Claim we can see that for all $1\le m\le k$

$$\sigma_m\left(\lambda\left(D^2\underline{\phi}\right)\right) \ge \sigma_m\left(\lambda\left(D^2\phi\right)\right) - CM^{m-2}B \quad \text{in } E_1\setminus\bar{D},$$

where $C=C(\Gamma,A,\varphi)>0$. In view of inequality (2.9) and equality (2.8) we obtain there exists $N_1 = N_1(\varphi,\Gamma,A)>0$ such that when $N\ge N_1$, $\underline{\phi}$ satisfies

$$\lambda(D^2\underline{\phi})\in\Gamma_k \ \text{ and } \ \sigma_k\left(\lambda\left(D^2\underline{\phi}\right)\right) > 1 \quad \text{on } E_1\setminus\bar{D}.$$

This completes the proof of the proposition.

2.2 Subsolutions in $\mathbb{R}^n\setminus E_1$

In the following, we will assume $N\ge N_1$ to be a fixed constant, we will also denote $\tilde{\varphi} := \underline{\phi}|_{\partial E_1}$. Here, N_1 and $\underline{\phi}$ are the same as the ones in Proposition 2.1.

First, recall that $s=\frac{1}{2}\sum a_i x_i^2$, let

$$\omega_{\alpha,\beta}(s) = \int_1^s \left(1+\alpha t^{-\beta}\right)^{\frac{1}{k}} dt, \quad \text{where } \alpha>0 \text{ and } \beta>1 \text{ to be determined.}$$

In the following, when there is no confusion, we will drop the subscript α,β and write ω instead of $\omega_{\alpha,\beta}$. A direct calculation yields

$$\omega'(s)=\left(1+\alpha s^{-\beta}\right)^{\frac{1}{k}},\qquad \omega''(s)=\frac{1}{k}\left(1+\alpha s^{-\beta}\right)^{\frac{1}{k-1}}\left(-\alpha\beta s^{-\beta-1}\right)=-\frac{\alpha\beta\omega'(s)}{k(s^{\beta+1}+\alpha s)}.$$

By [1, page 5] we have

$$\begin{aligned}&D_i\omega(x)=\omega'(s)a_ix_i=\left(1+\alpha s^{-\beta}\right)^{\frac{1}{k}}a_ix_i\\&D_{ij}\omega(x)=\omega'(s)a_i\delta_{ij}+\omega''(s)(a_ix_i)(a_jx_j)=\omega'(s)\left[a_i\delta_{ij}-\frac{\alpha\beta}{ks(s^\beta+\alpha)}(a_ix_i)(a_jx_j)\right].\end{aligned}\tag{2.10}$$

In view of [1, Proposition 1.2] we obtain

$$\sigma_m(\lambda(D_{ij}\omega))=[\omega'(s)]^m\left\{\sigma_m(a)-\frac{\alpha\beta}{ks(s^\beta+\alpha)}\sum_{i=1}^n(a_ix_i)^2\sigma_{m-1}(a|i)\right\},\tag{2.11}$$

where $a=(a_1,\cdots,a_n)$ and $\sigma_{m-1}(a|i)=\sigma_{m-1}(a)|_{a_i=0}$. Assume $a_{i_0}=\max_i\{a_1,\cdots,a_n\}$, following [1], let

$$\begin{aligned}h_m:=&\max_{1\le i\le n}A_m^i(a)=\max_{1\le i\le n}\sigma_{m-1}(a|i)a_i\\=&\max_{1\le i\le n}\{\sigma_m(a)-\sigma_m(a|i)\}=\sigma_m(a)-\sigma_m(a|i_0).\end{aligned}$$

Then we have

$$\sigma_m(\lambda(D_{ij}\omega))\ge[\omega'(s)]^m\left\{\sigma_m(a)-\frac{2\alpha\beta}{k(s^\beta+\alpha)}h_m\right\}.\tag{2.12}$$

In particular, when $m=k$ we get

$$\sigma_k(\lambda(D_{ij}\omega))\ge\left(1+\frac{\alpha}{s^\beta}\right)\left[1-\frac{2\alpha\beta h_k}{k(s^\beta+\alpha)}\right].$$

According to equation (2.15) of [1], we know that for $n\ge3$ and $2\le k\le n-1$, we have

$$\frac{k}{2}<\frac{k}{2h_k(a)}\le\frac{n}{2}.\tag{2.13}$$

In the following, we will set $\boldsymbol{\beta=\frac{k}{2h_k}-\eta}$, where $\eta>0$ is an arbitrary constant such that $\beta>k/2$. In this case we have

$$\begin{aligned}\sigma_k(\lambda(D_{ij}\omega))\ge&\left(1+\frac{\alpha}{s^\beta}\right)\left[1-\frac{2\alpha h_k}{k(s^\beta+\alpha)}\left(\frac{k}{2h_k}-\eta\right)\right]\\=&\left(1+\frac{\alpha}{s^\beta}\right)\left[1-\frac{\alpha}{s^\beta+\alpha}+\frac{2\alpha h_k\eta}{k(s^\beta+\alpha)}\right]\\=&\frac{\alpha+s^\beta}{s^\beta}\left[\frac{s^\beta}{s^\beta+\alpha}+\frac{2\alpha h_k\eta}{k(s^\beta+\alpha)}\right]=1+\frac{2\alpha h_k\eta}{ks^\beta}.\end{aligned}\tag{2.14}$$

Next, we will show that the $w_{\alpha,\beta}(s)$ constructed above is a strictly k-convex function in $\mathbb{R}^n\setminus E_1$. In view of (2.14), we can see that we only need to show $\lambda\left(D^2_{ij}\omega\right)\in\Gamma_m$ in $\mathbb{R}^n\setminus E_1$ for $1\le m<k$. Note that

$$\begin{aligned}
&\sigma_m(a)-\frac{\alpha\beta}{ks(s^\beta+\alpha)}\sum_{i=1}^n\sigma_{m-1}(a|i)(a_ix_i)^2\\
=&\sigma_m(a)-\frac{\alpha}{ks(s^\beta+\alpha)}\sum_{i=1}^n\sigma_{m-1}(a|i)(a_ix_i)^2\left(\frac{k}{2h_k}-\eta\right)\\
=&\sigma_m(a)-\frac{\alpha}{2h_ks(s^\beta+\alpha)}\sum_{i=1}^n\sigma_{m-1}(a|i)(a_ix_i)^2+III\\
\ge&\sigma_m(a)-\frac{\alpha}{2s(s^\beta+\alpha)}\sum_{i=1}^n\frac{\sigma_{m-1}(a|i)(a_ix_i^2)}{a_i\sigma_{k-1}(a|i)}+III,
\end{aligned}$$

where $III:=\frac{\alpha\eta}{ks(s^\beta+\alpha)}\sum\limits_{i=1}^n\sigma_{m-1}(a|i)(a_ix_i)^2$. By virtue of (2.22) of [1] we know

$$\frac{\sigma_{m-1}(a|i)}{\sigma_{k-1}(a|i)}\le\sigma_m(a)\quad\text{for } 1\le i\le n \text{ and } 1\le m\le k-1.$$

Therefore, we obtain

$$\begin{aligned}
&\sigma_m(a)-\frac{\alpha\beta}{ks(s^\beta+\alpha)}\sum_{i=1}^n\sigma_{m-1}(a|i)(a_ix_i)^2\\
\ge&\sigma_m(a)-\frac{\alpha}{2s(s^\beta+\alpha)}\sum_{i=1}^n\sigma_m(a)a_ix_i^2+III\\
=&\sigma_m(a)\left(1-\frac{\alpha}{s^\beta+\alpha}\right)+III.
\end{aligned}\tag{2.15}$$

Denote $\underline{h}_m=\min_{1\le i\le n}\sigma_{m-1}(a|i)a_i=\sigma_m(a)-\sigma_m(a|i_1)$, where $a_{i_1}=\min\{a_1,\cdots,a_n\}$. Then

$$III=\frac{\alpha\eta}{ks(s^\beta+\alpha)}\sum_{i=1}^n\sigma_{m-1}(a|i)(a_ix_i)^2\ge\frac{2\alpha\eta\underline{h}_m}{k(s^\beta+\alpha)}.$$

Combining with (2.15) we have

$$\sigma_m(a)-\frac{\alpha\beta}{ks(s^\beta+\alpha)}\sum_{i=1}^n\sigma_{m-1}(a|i)(a_ix_i)^2\ge\sigma_m(a)\frac{s^\beta}{s^\beta+\alpha}+\frac{2\alpha\eta\underline{h}_m}{k(s^\beta+\alpha)}.\tag{2.16}$$

Thus, from (2.11) we get $\lambda(D_{ij}\omega)\in\Gamma_k$. We conclude

Proposition 2.2. *For $n\geq 3$, $2\leq k\leq n-1$, and $A\in\mathcal{A}_k$, let $\omega_{\alpha,\beta}(x)=\int_1^s(1+\alpha t^{-\beta})^{1/k}dt$ for $s=\frac{1}{2}x^TAx$. Then when $\alpha>0$, $\frac{k}{2}<\beta<\frac{k}{2h_k}$, $\omega_{\alpha,\beta}$ is a smooth strictly k-convex subsolution of $\sigma_k(\lambda(D^2u))=1$ in $\mathbb{R}^n\setminus\{0\}$. Moreover, $\omega_{\alpha,\beta}$ satisfies*

$$\omega_{\alpha,\beta}=\frac{1}{2}x^TAx+\mu(\alpha,\beta)+O(s^{1-\beta}),\ \ \text{as } s\to\infty. \tag{2.17}$$

Here,

$$\mu(\alpha,\beta)=\int_1^\infty\left[(1+\alpha t^{-\beta})^{\frac{1}{k}}-1\right]dt-1.$$

Proof. From (2.14) and (2.16) we know that $\omega_{\alpha,\beta}$ is a smooth strictly k-convex subsolution of $\sigma_k(\lambda(D^2u))=1$ in $\mathbb{R}^n\setminus\{0\}$. To prove this proposition, we only need to prove (2.17). A straightforward calculation yields

$$\begin{aligned}\omega_{\alpha,\beta}&=\int_1^s(1+\alpha t^{-\beta})^{\frac{1}{k}}dt\\&=\int_1^s\left[(1+\alpha t^{-\beta})^{\frac{1}{k}}-1\right]dt+s-1\\&=\int_1^\infty\left[(1+\alpha t^{-\beta})^{\frac{1}{k}}-1\right]dt-\int_s^\infty\left[(1+\alpha t^{-\beta})^{\frac{1}{k}}-1\right]dt+s-1\\&=s+\mu(\alpha,\beta)+O(s^{1-\beta}).\end{aligned}$$

Here, $\beta=\frac{k}{2h_k}-\eta>\frac{k}{2}$ and

$$\mu(\alpha,\beta)=\int_1^\infty\left[(1+\alpha t^{-\beta})^{\frac{1}{k}}-1\right]-1<\infty.$$

□

Remark 2.1. We want to point out that for any fixed $\frac{k}{2}<\beta<\frac{k}{2h_k}$, there exists $\alpha^*=\alpha^*(\beta,A,\varphi,D)>0$ such that when $\alpha\geq\alpha^*$, we have $\frac{1}{2}x^TAx+\mu(\alpha,\beta)\geq\varphi$ on Γ. □

Finally, we will use $\omega_{\alpha,\beta}$ to construct a subsolution of $\sigma_k(\lambda(D^2u))=1$ in $\mathbb{R}^n\setminus\bar{E}_1$.

Recall that we have set $\tilde{\varphi}=\underline{\phi}|_{\partial E_1}$. We may express $\tilde{\varphi}$ as a function of $\mathbb{S}^{n-1}$, i.e., $\tilde{\varphi}$ is independent of r. Then set $\Psi=s^{-\Lambda}\tilde{\varphi}$, where $\Lambda>0$ is a constant to be determined. Same as before, we choose a local orthonormal frame $\{e_1,\cdots,e_{n-1}\}$ on the unit sphere $\mathbb{S}^{n-1}$ and let $\tau_r=\frac{\partial}{\partial r}$. It is easy to see that for $1\leq a,b\leq n-1$

$$s_a=O(s),\quad s_{ab}=O(s),\quad s_{ar}=O(s^{\frac{1}{2}}),\quad s_r=O(s^{\frac{1}{2}}),\quad \text{and}\ \ s_{rr}=O(1).$$

This implies

$$\begin{aligned}
&\Psi_a=-\Lambda s^{-\Lambda-1}\tilde{\varphi}s_a+s^{-\Lambda}\tilde{\varphi}_a=O(s^{-\Lambda}),\\
&\Psi_{ab}=\Lambda(\Lambda+1)s^{-\Lambda-2}\tilde{\varphi}s_as_b-\Lambda s^{-\Lambda-1}\tilde{\varphi}_bs_a\\
&\qquad-\Lambda s^{-\Lambda-1}\tilde{\varphi}s_{ab}-\Lambda s^{-\Lambda-1}\tilde{\varphi}_as_b+s^{-\Lambda}\tilde{\varphi}_{ab}=O(s^{-\Lambda}),\\
&\Psi_{ar}=-\Lambda s^{-\Lambda-1}\tilde{\varphi}s_{ar}+\Lambda(\Lambda+1)s^{-\Lambda-2}\tilde{\varphi}s_as_r-\Lambda s^{-\Lambda-1}\tilde{\varphi}_as_r=O\left(s^{-\Lambda-\frac{1}{2}}\right),\\
&\Psi_r=-\Lambda s^{-\Lambda-1}s_r\tilde{\varphi}=O\left(s^{-\Lambda-\frac{1}{2}}\right),\\
&\Psi_{rr}=\Lambda(\Lambda+1)s^{-\Lambda-2}(s_r)^2\tilde{\varphi}-\Lambda s^{-\Lambda-1}s_{rr}\tilde{\varphi}=O(s^{-\Lambda-1}).
\end{aligned}$$

Consider

$$\underline{\phi}^1=\int_1^s(1+\alpha t^{-\beta})^{\frac{1}{k}}dt+\Psi=\omega_{\alpha,\beta}+\Psi,$$

then by (2.2)-(2.4) and (2.10) we obtain

$$D_{ij}\underline{\phi}^1=w'(s)\left[a_i\delta_{ij}-\frac{\alpha\beta}{ks(s^\beta+\alpha)}(a_ix_i)(a_jx_j)\right]+\frac{b_{ij}}{S^{\Lambda+1}},$$

where $|b_{ij}|_{1\le i,j\le n}$ are uniformly bounded by some constant $C=C(|\tilde{\varphi}|_{C^2},A,\Lambda)>0$. Now let's fix some Λ such that $\Lambda+1\ge\beta>k/2$, by virtue of (2.11) we can see that

$$\begin{aligned}
\sigma_m\left(\lambda\left(D^2\underline{\phi}^1\right)\right)&=\sigma_m\left(\lambda\left(D^2\omega_{\alpha,\beta}+D^2\Psi\right)\right)\\
&\ge\sigma_m\left(\lambda\left(D^2\omega_{\alpha,\beta}\right)\right)-\sum_{i=1}^m\frac{C_i}{s^{(\Lambda+1)i}}(\omega'(s))^{m-i}\\
&=\sigma_m\left(\lambda\left(D^2\omega_{\alpha,\beta}\right)\right)-\sum_{i=1}^m\left(1+\frac{\alpha}{s^\beta}\right)^{\frac{m-i}{k}}\frac{C_i}{s^{(\Lambda+1)i}},
\end{aligned}\tag{2.18}$$

where $C_i=C_i(A,\Lambda,|\tilde{\varphi}|_{C^2})>0$. Denote $y:=\frac{\alpha}{s^\beta}$ and $\frac{\Lambda+1}{\beta}:=1+\gamma$, when $m=k$ plugging (2.14) into (2.18) gives

$$\begin{aligned}
\sigma_k\left(\lambda\left(D^2\underline{\phi}^1\right)\right)&\ge1+\frac{2h_k\eta}{k}y-\sum_{i=1}^k(1+y)^{\frac{k-i}{k}}C_i\left(\frac{y}{\alpha}\right)^{(1+\gamma)i}\\
&\ge1+y\left[\frac{2h_k\eta}{k}-\sum_{i=1}^k(1+y)^{\frac{k-i}{k}}C_i\left(\frac{y}{\alpha}\right)^{(1+\gamma)i-1}\frac{1}{\alpha}\right]\\
&\ge1+y\left[\frac{2h_k\eta}{k}-\sum_{i=1}^k(1+\alpha)^{\frac{k-i}{k}}C_i\frac{1}{\alpha}\right].
\end{aligned}$$

Here, we have used the fact that in $\mathbb{R}^n\setminus\bar{E}_1$ we have $0<y<\alpha$. Therefore, there exists $\alpha_0=\alpha_0(A,\Lambda,|\tilde{\varphi}|_{C^2},\beta,k)>0$ such that for any $\alpha>\alpha_0$

$$\sigma_k\left(\lambda\left(D^2\underline{\phi}^1\right)\right)>1\quad\text{in }\mathbb{R}^n\setminus\bar{E}_1.$$

When $1\le m\le k-1$, since we have shown $\lambda(D^2\omega_{\alpha,\beta})\in\Gamma_k$, by Maclaurin's inequality we have

$$\frac{\sigma_m\big(\lambda\big(D^2\omega_{\alpha,\beta}\big)\big)}{C_n^m}\ge\left(\frac{\sigma_k\big(\lambda\big(D^2\omega_{\alpha,\beta}\big)\big)}{C_n^k}\right)^{\frac{m}{k}}.$$

Combining with (2.14) we obtain

$$\sigma_m\big(\lambda\big(D^2\omega_{\alpha,\beta}\big)\big)\ge\frac{C_n^m}{(C_n^k)^{\frac{m}{k}}}\left(1+\frac{2h_k\eta}{k}y\right)^{\frac{m}{k}}.$$

It follows that

$$\begin{aligned}\sigma_m\big(\lambda\big(D^2\underline{\phi}^1\big)\big)&\ge y^{\frac{m}{k}}\left[\frac{C_n^m}{(C_n^k)^{\frac{m}{k}}}\left(\frac{1}{y}+\frac{2h_k\eta}{k}\right)^{\frac{m}{k}}-\sum_{i=1}^m(1+y)^{\frac{m-i}{k}}C_i\left(\frac{y}{\alpha}\right)^{(1+\gamma)i-\frac{m}{k}}\frac{1}{\alpha^{\frac{m}{k}}}\right]\\&\ge y^{\frac{m}{k}}\left[\frac{C_n^m}{(C_n^k)^{\frac{m}{k}}}\left(\frac{1}{y}+\frac{2h_k\eta}{k}\right)^{\frac{m}{k}}-\sum_{i=1}^m(1+\alpha)^{\frac{m-i}{k}}C_i\frac{1}{\alpha^{\frac{m}{k}}}\right].\end{aligned}$$

Therefore, there exists $\alpha_0=\alpha_0(A,\Lambda,|\tilde{\varphi}|_{C^2},\beta,k)>0$ such that for any $\alpha>\alpha_0$

$$\sigma_m\big(\lambda\big(D^2\underline{\phi}^1\big)\big)>0\quad\text{in }\mathbb{R}^n\setminus\bar{E}_1.$$

We conclude this subsection with the following proposition.

Proposition 2.3. *For $n\ge3$, $2\le k\le n-1$, and $A\in\mathcal{A}_k$, denote*

$$\omega_{\alpha,\beta}(x)=\int_1^s(1+\alpha t^{-\beta})^{\frac{1}{k}}dt$$

where $s=\frac{1}{2}x^TAx$. Then given any $\frac{k}{2}<\beta<\frac{k}{2h_k}$, $\Lambda\ge\beta-1$, and $\tilde{\varphi}\in C^\infty(S^{n-1})$, there exists $\alpha_0=\alpha_0(A,\Lambda,|\tilde{\varphi}|_{C^2},\beta,k)>0$ such that when $\alpha>\alpha_0$

$$\underline{\phi}^1=\omega_{\alpha,\beta}+s^{-\Lambda}\tilde{\varphi}\tag{2.19}$$

is a smooth strictly k-convex subsolution of $\sigma_k\big(\lambda\big(D^2u\big)\big)=1$ in $\mathbb{R}^n\setminus E_1$ satisfying $\underline{\phi}^1=\tilde{\varphi}$ on ∂E_1.

2.3 Subsolutions of (1.1) in $\mathbb{R}^n\setminus D$

In this subsection, we will follow the idea of [4] to glue $\underline{\phi}$ and $\underline{\phi}^1$ together to obtain a subsolution of (1.1) in $\mathbb{R}^n\setminus D$.

Recall that $\underline{\phi}=\mathfrak{b}^N-1+\varphi$, following earlier notations, a direct calculation gives

$$e_a\underline{\phi}=N\mathfrak{b}^{N-1}\mathfrak{b}_a+\varphi_a,\quad\tau_r\underline{\phi}=N\mathfrak{b}^{N-1}\mathfrak{b}_r,$$

this implies

$$|D\underline{\phi}|\le C_0=C_0(N,|\varphi|_{C^1},\Gamma,A)\quad \text{on } \partial E_1.$$

Next, we compute $|D\underline{\phi}^1|$ on ∂E_1

$$e_a\underline{\phi}^1=\frac{\partial\underline{\phi}^1}{\partial s}s_a+s^{-\Lambda}\tilde{\varphi}_a=\left[(1+\alpha)^{\frac{1}{k}}-\Lambda\tilde{\varphi}\right]s_a+\tilde{\varphi}_a,$$

and

$$\tau_r\underline{\phi}^1=\left[(1+\alpha)^{\frac{1}{k}}-\Lambda\tilde{\varphi}\right]s_r,$$

where $\tilde{\varphi}=\underline{\phi}|_{\partial E_1}$. Hence, when $\alpha>\alpha_1=\alpha_1(C_0,\Lambda,|\tilde{\varphi}|_{C^1},A)>0$ we have

$$|D\underline{\phi}^1|>|D\underline{\phi}|\quad \text{on } \partial E_1.$$

Let ν be the outward unit normal of ∂E_1, i.e., pointing into $\mathbb{R}^n\setminus\bar{E}_1$, since $\underline{\phi}^1=\underline{\phi}$ on ∂E_1, we have

$$D_\nu\underline{\phi}^1>D_\nu\underline{\phi} \tag{2.20}$$

on ∂E_1. We will denote

$$\underline{u}:=\begin{cases}\underline{\phi} & x\in\bar{E}_1\setminus D,\\ \underline{\phi}^1 & x\in\mathbb{R}^n\setminus E_1\end{cases}. \tag{2.21}$$

It is clear that $\underline{u}$ is continuous on $\mathbb{R}^n\setminus D$. In next section we will show that $\underline{u}$ is a viscosity subsolution of (1.1).

3 Proof of the existence part of Theorem 1.1

In this section we will prove the existence part of Theorem 1.1. We will assume $N,\alpha,\frac{k}{2}<\beta<\frac{k}{2h_k}$, and $\Lambda>\beta-1$ are fixed constants such that the following conditions are satisfied:

(a) $\underline{\phi}=\mathfrak{b}^N-1+\varphi$ is strictly k-convex and satisfies $\sigma_k(\lambda(D^2\underline{\phi}))>1$ in $E_1\setminus\bar{D}$;

(b) $\frac{1}{2}x^TAx+\mu(\alpha,\beta)\ge\varphi$ on Γ;

(c) $\underline{\phi}^1=\omega_{\alpha,\beta}+s^{-\Lambda}\tilde{\varphi}$ satisfies Proposition 2.3 and equation (2.20).

Here $\tilde{\varphi}=\underline{\phi}|_{\partial E_1}$ is a fixed smooth function determined by N,φ,A and Γ.

Let $\underline{u}$ be the function defined by (2.21), then in view of of Proposition 2.2 we get

$$\underline{u}=\underline{\phi}^1=s+\mu(\alpha,\beta)+O\left(s^{1-\beta}\right)\text{ as } s\to\infty.$$

Now, let $\bar{u}_R=\frac{1}{2}x^TAx+\mu(\alpha,\beta)+\bar{C}R^{1-\beta}$, where $\bar{C}=\bar{C}(\alpha,\beta)>0$ is chosen such that $\bar{u}_R>\underline{u}$ on ∂E_R. Here and in the following, without loss of generality, we will always assume

$R\geq R_0\gg 1$ is an arbitrarily large constant. We will also assume $\bar{C}>0$ is a fixed constant. Moreover, in view of condition (b) we have $\bar{u}_R>\underline{u}=\varphi$ on Γ.

Now, let us consider the following Dirichlet problem

$$\begin{cases}\sigma_k\big(\lambda\big(D^2u\big)\big)=1 & \text{in } \Omega_R:=E_R\setminus\bar{D};\\ u=\varphi & \text{on } \Gamma:=\partial D;\\ u=\bar{u}_R & \text{on } \partial E_R.\end{cases} \tag{3.1}$$

We will show that for any $R\geq R_0\gg 1$ there exists a smooth, strictly k-convex solution $u_R\in C^\infty(E_R\setminus\bar{D})$ of (3.1).

3.1 C^0 estimates

Lemma 3.1. *Let u_R be the strictly k-convex solution of* (3.1), *then u_R satisfies*

$$\underline{u}<u_R<\bar{u}_R \quad \textit{in } E_R\setminus\bar{D}.$$

Proof. Since $u_R\leq\bar{u}_R$ on $\partial\Omega_R$ and $\sigma_k(\lambda(D^2u_R))=\sigma_k(\lambda(D^2\bar{u}_R))=1$. By the strong maximum principle we obtain the second inequality.

Now, we consider the first inequality. We know that for some $\delta<0$ small, we have

$$u_R\geq\underline{u}+\delta,\quad x\in\bar{\Omega}_R. \tag{3.2}$$

Suppose $\bar{\delta}$ is the largest number for which inequality (3.2) holds, we will show $\bar{\delta}=0$. If not, then $\bar{\delta}<0$ and there exists $\bar{x}\in\Omega_R$ such that

$$u_R(\bar{x})=\underline{u}(\bar{x})+\bar{\delta}.$$

Since on $(E_R\setminus\bar{E}_1)\cup(E_1\setminus\bar{D})$ we get

$$\sigma_k\big(\lambda\big(D^2\underline{u}\big)\big)>\sigma_k\big(\lambda\big(D^2u_R\big)\big).$$

By the standard maximum principle we know $\bar{x}\notin(E_R\setminus\bar{E}_1)\cup(E_1\setminus\bar{D})$. We conclude that $\bar{x}\in\partial E_1$. However, this is impossible. Since by (2.20) and (2.21) we have,

$$\lim_{x\to\partial E_1^-}D_\nu\underline{u}<\lim_{x\to\partial E_1^+}D_\nu\underline{u}$$

while u_R is smooth. This implies $\bar{\delta}=0$ and the lemma follows from the strong maximum principle. □

Remark 3.1. With a small modification of the proof of Lemma 3.1, one can show that $\underline{u}$ is a viscosity subsolution of (1.1).

3.2 C^1 estimates

In order to prove the C^1 estimate of u_R on Γ we need to construct an upper barrier on Γ first.

3.2.1 Construction of the upper barrier on Γ

Recall that Γ is a smooth $(n-1)$-dimensional manifold, we know that there exists $\eta_0>0$ such that for any $\hat{p}\in\Gamma$ and $\eta\leq\eta_0$ there exists $z_{\hat{p},\eta}\in D$ satisfying $\bar{B}_\eta(z_{\hat{p},\eta})\cap\Gamma=\hat{p}$. Now, fix an arbitrary $\hat{p}\in\Gamma$, we may choose a new coordinate $\{\hat{x}_1,\cdots,\hat{x}_n\}$ of $\mathbb{R}^n$ such that $\hat{p}$ is the origin. We also let $\hat{x}_n$ axis be the unit normal of Γ at $\hat{p}$ pointing into D. We will first restrict ourselves to a small neighborhood of $\hat{p}$. Denote $U_\delta=\{\hat{x}'=(\hat{x}_1,\cdots,\hat{x}_{n-1})\in\mathbb{R}^{n-1}:|\hat{x}'|<\delta\}$ for some fixed small constant $\delta>0$, then near $\hat{p}$ the boundary Γ can be written as a graph over U_δ

$$\gamma(\hat{x}')=\frac{1}{2}\sum_{\beta=1}^{n-1}\kappa_\beta\hat{x}_\beta^2+O\big(|\hat{x}'|^3\big),$$

where $(\kappa_1,\cdots,\kappa_{n-1})$ are the principle curvature vector of Γ at $\hat{p}$. Furthermore, the function φ can be written as a function over U_δ:

$$\varphi(\hat{x}',\gamma(\hat{x}'))=\varphi(0)+\sum_{\alpha=1}^{n-1}\varphi_\alpha(0)\hat{x}_\alpha+O(|\hat{x}'|^2). \tag{3.3}$$

Let

$$\hat{u}=-C|\hat{x}-z_{\hat{p},\eta}|^{-(n-2)}+C\eta^{-(n-2)}+\varphi(0)+\sum_{\alpha=1}^{n-1}\varphi_\alpha(0)\hat{x}_\alpha, \tag{3.4}$$

where $C>0$ will be determined later. We note that $\hat{u}$ is a smooth function defined on $\mathbb{R}^n\setminus\{z_{\hat{p},\eta}\}$.

Lemma 3.2. *If $C=C(\varphi,\Gamma)>0$ is chosen to be large enough and $\eta=\eta(\Gamma)\leq\eta_0$ is chosen to be small enough, then $\hat{u}>\varphi$ on $\Gamma\setminus\{0\}$ and $\hat{u}(0)=\varphi(0)$.*

Proof. It is clear that $\hat{u}(0)=\varphi(0)$, we only need to show that for properly chosen constants $C>0$ large and $\eta>0$ small, we have $\hat{u}>\varphi$ on $\Gamma\setminus\{0\}$. Note that, by our choice of coordinates, $z_{\hat{p},\eta}=(0,\cdots,0,\eta)$. In this proof, we will always assume $\delta\leq\frac{\eta}{10}$ is a fixed small number. Then on Γ when $|\hat{x}'|<\delta$,

$$\begin{aligned}\hat{u}&=-C\big(|\hat{x}'|^2+\big(\gamma(\hat{x}')-\eta\big)^2\big)^{-\frac{n-2}{2}}+C\eta^{-(n-2)}+\varphi(0)+\sum_{\alpha=1}^{n-1}\varphi_\alpha(0)\hat{x}_\alpha\\&=C\eta^{-(n-2)}\left[1-\frac{\eta^{n-2}}{(|\hat{x}'|^2+\gamma^2-2\eta\gamma+\eta^2)^{\frac{n-2}{2}}}\right]+\varphi(0)+\sum_{\alpha=1}^{n-1}\varphi_\alpha(0)\hat{x}_\alpha.\end{aligned}$$

Since

$$\gamma(\hat{x}')=\frac{1}{2}\sum_{\beta=1}^{n-1}\kappa_\beta\hat{x}_\beta^2+O(|\hat{x}'|^3),$$

we can see that when $\eta=\eta(\Gamma)>0$ small, we have

$$|\hat{x}'|^2+\gamma^2-2\eta\gamma>\lambda|\hat{x}'|^2 \quad \text{for some } \tfrac{1}{2}<\lambda<1.$$

This gives

$$\begin{aligned}\hat{u}&>C\eta^{-(n-2)}\left[1-\frac{\eta^{n-2}}{(\lambda|\hat{x}'|^2+\eta^2)^{\frac{n-2}{2}}}\right]+\varphi(0)+\sum_{\alpha=1}^{n-1}\varphi_\alpha(0)\hat{x}_\alpha\\&=C\eta^{-(n-2)}\left[1-\frac{1}{\left(1+\frac{\lambda|\hat{x}'|^2}{\eta^2}\right)^{\frac{n-2}{2}}}\right]+\varphi(0)+\sum_{\alpha=1}^{n-1}\varphi_\alpha(0)\hat{x}_\alpha\\&>\frac{C}{(\eta^2+\lambda|\hat{x}'|^2)^{\frac{n-2}{2}}}\cdot\frac{n-2}{4}\cdot\frac{\lambda|\hat{x}'|^2}{\eta^2}+\varphi(0)+\sum_{\alpha=1}^{n-1}\varphi_\alpha(0)\hat{x}_\alpha.\end{aligned}$$

By virtue of (3.3), it is clear that when $C=C(\varphi,\Gamma)>0$ large and $\eta=\eta(\Gamma)>0$ small, we have $\hat{u}\geq\varphi$ on $|\hat{x}'|<\delta$.

Now, we consider the set $\Gamma_\delta^c:=\{x\in\mathbb{R}^n : x\in\Gamma\setminus\{(\hat{x}',\gamma(\hat{x}')) : |\hat{x}'|<\delta\}\}$. It is easy to see that when $x\in\Gamma_\delta^c$, we have $|x-z_{\hat{p},\eta}|\geq\eta+\delta_1$ for some small constant $\delta_1=\delta_1(\eta,\Gamma)>0$. Then

$$\begin{aligned}\hat{u}&\geq C\eta^{-(n-2)}-C(\eta+\delta_1)^{-(n-2)}+\varphi(0)+\sum_{\alpha=1}^{n-1}\varphi_\alpha(0)\hat{x}_\alpha\\&>\frac{C(n-2)\delta_1}{2\eta^{n-1}}+\varphi(0)+\sum_{\alpha=1}^{n-1}\varphi_\alpha(0)\hat{x}_\alpha.\end{aligned}$$

Therefore, when $C=C(\varphi,\Gamma)>0$ large and $\eta=\eta(\Gamma)>0$ small the claim holds. □

Lemma 3.3. *Let u_R be the strictly k-convex solution of* (3.1)*, then u_R satisfies*

$$|Du_R|<C_\Gamma \quad \text{on } \Gamma, \tag{3.5}$$

$$|Du_R|<C_R \quad \text{on } \partial E_R, \tag{3.6}$$

where $C_\Gamma=C_\Gamma(\Gamma,\varphi)$ is independent of R and $C_R=O(R^{1/2})$.

Proof. By Lemma 3.1 we have $\underline{u}<u_R$ in Ω_R and $u_R=\underline{u}$ on Γ. Therefore, we get $D_\nu u_R>D_\nu\underline{u}$ on Γ, where ν is the inward normal of Γ, that is, pointing into Ω_R. Now, for any $\hat{p}\in\Gamma$, we consider $\hat{u}$ that is given in (3.4). It is easy to verify that

$$\sigma_1\big(\lambda\big(D^2\hat{u}\big)\big)=0<\sigma_1\big(\lambda\big(D^2u_R\big)\big).$$

Futhermore, we can always choose $C=C(\varphi,\Gamma,A)>0$ large and $\eta=\eta(\Gamma,\varphi,A)>0$ small such that

$$\hat{u}>\bar{u}_R>u_R \quad \text{on } \partial E_1.$$

In view of Lemma 3.2 and the maximum principle we obtain $\hat{u}\geq u_R$ in $\overline{E_1\setminus D}$. Moreover, at $\hat{p}$ we have $\hat{u}=u_R$. Thus $D_\nu\hat{u}>D_\nu u_R$ at $\hat{p}$. Since $\hat{p}\in\Gamma$ is arbitrary we conclude (3.5). On E_R, let

$$\underline{u}_R=\lambda x^TAx-(2\lambda R-R-\mu(\alpha,\beta)-\bar{C}R^{1-\beta}),$$

where $\lambda=\lambda(\mu(\alpha,\beta),R_0,\varphi)>1/2$ such that for any $R>R_0$, $\underline{u}_R<\varphi$ on Γ. Applying the maximum principle we get

$$\underline{u}_R<u_R<\bar{u}_R \quad \text{in } E_R\setminus\bar{D}.$$

Moreover, on ∂E_R it is clear that $\underline{u}_R=u_R=\bar{u}_R$. Therefore, we have

$$D_\nu\underline{u}_R<D_\nu u_R<D_\nu\bar{u}_R,$$

where ν is the inward unit normal to ∂E_R, that is, pointing into Ω_R. This proves (3.6). □

Lemma 3.4. *Let u_R be the strictly k-convex solution of* (3.1), *then u_R satisfies*

$$\max_{x\in\bar{\Omega}_R}|Du_R|=\max_{x\in\partial\Omega_R}|Du_R|. \tag{3.7}$$

Proof. Differentiating (3.1) with respect to x_l gives

$$\sigma_k^{ij}u_{ijl}=0.$$

Therefore, we have

$$\sigma_k^{ij}\left(|Du_R|^2\right)_{ij}=2\sigma_k^{ij}u_lu_{lij}+2\sigma_k^{ij}u_{li}u_{lj}=2\sigma_k^{ij}u_{li}u_{lj}\geq 0.$$

By virtue of the maximum principle this lemma is proved. □

3.3 C^2 estimates

In this subsection, we will establish the C^2 estimates of u_R. The techniques are used here are the same as [5]. For readers' convenience, we include the argument here.

3.3.1 C^2 boundary estimates on Γ

Let $x_0\in\Gamma$ be an arbitrary point. Without loss of generality, we may choose local coordinates $\{\tilde{x}_1,\cdots,\tilde{x}_n\}$ in the neighborhood of x_0 such that x_0 is the origin and $\tilde{x}_n$ axis is the inward normal of Γ (pointing into Ω_R) at x_0. Then the boundary near x_0 can be expressed as

$$\tilde{x}_n=\gamma(\tilde{x}')=-\frac{1}{2}\sum_{\alpha=1}^{n-1}\kappa_\alpha\tilde{x}_\alpha^2+O\left(|\tilde{x}'|^3\right),$$

where $\kappa_1,\cdots,\kappa_{n-1}$ are the principal curvatures of Γ at x_0 and $\tilde{x}'=(\tilde{x}_1,\cdots,\tilde{x}_{n-1})$. Following [5], we may assume φ has been extended smoothly to $\bar{\Omega}_R$ with $\varphi(0)=0$. Then we get

$$u_{\tilde{\alpha}\tilde{\beta}}(0)=\varphi_{\tilde{\alpha}\tilde{\beta}}(0)-u_{\tilde{n}}(0)\kappa_\alpha\delta_{\tilde{\alpha}\tilde{\beta}} \quad \text{for } \alpha,\beta<n.$$

This gives

$$|u_{\tilde{\alpha}\tilde{\beta}}(0)|\le C, \quad \alpha,\beta<n \tag{3.8}$$

for some $C=C\big(\varphi,\Gamma,|Du|_{C^0(\Gamma)}\big)>0$ that is independent of R. Next, we estimate $|u_{\tilde{\alpha}\tilde{n}}(0)|$ for $\alpha<n$.

Lemma 3.5. *Let u_R be the strictly k-convex solution of* (3.1)*, then u_R satisfies*

$$|(u_R)_{\tau\nu}|<C \text{ on } \Gamma, \tag{3.9}$$

where τ is an arbitrary unit tangent vector of Γ, ν is the inward unit normal of Γ, and $C=C(\Gamma,\varphi,|Du|_{C^0(\Gamma)})>0$ is independent of R.

Proof. Notice that the boundary Γ near x_0 can be expressed as

$$\tilde{x}_n=\gamma(\tilde{x}')=-\frac{1}{2}\sum_{\alpha=1}^{n-1}\kappa_\alpha\tilde{x}_\alpha^2+O\big(|\tilde{x}'|^3\big).$$

Denote $W_R=u_R-\varphi$ and let

$$T:=\partial_{\tilde{\alpha}}-\kappa_\alpha(\tilde{x}_\alpha\partial_{\tilde{n}}-\tilde{x}_n\partial_{\tilde{\alpha}}),$$

it is clear that on Γ near x_0 we have

$$TW_R=(\partial_{\tilde{\alpha}}+\gamma_{\tilde{\alpha}}\partial_{\tilde{n}})W_R+O\big(|\tilde{x}'|^2\big)=O\big(|\tilde{x}'|^2\big). \tag{3.10}$$

Set $L:=\frac{\partial\sigma_k\left(\lambda\left(D^2u\right)\right)}{\partial u_{ij}}\partial_i\partial_j$, then we get

$$|L(TW_R)|\le C_0\sum\sigma_k^{ii} \tag{3.11}$$

for some $C_0=C_0(\varphi,\Gamma,|Du|_{C^0(\Gamma)})>0$. In view of Proposition 2.1, we know that $\lambda(D^2\underline{\phi})\in\Gamma_k$ in $E_1\setminus\bar{D}$. Moreover, we also have

$$\sigma_k\big(\lambda\big(D^2\underline{\phi}\big)\big)>1+\delta_0 \quad \text{in } B_\epsilon(x_0)\cap\Omega_R,$$

where $\epsilon>0$ is a fixed small number and $\delta_0=\delta_0(N)>0$ for some N chosen in Proposition 2.1. Therefore, there exists $\delta_1=\delta_1(\underline{\phi},\delta_0)>0$ satisfying

$$\sigma_k\big(\lambda\big(D^2\underline{\phi}-\delta_1|\tilde{x}|^2\big)\big)>1 \quad \text{in } B_\epsilon(x_0)\cap\Omega_R.$$

By the concavity of $\sigma_k^{1/k}$ we know

$$L\big(\underline{\phi}-\delta_1|\tilde{x}|^2\big)>k \quad \text{in } B_\epsilon(x_0)\cap\Omega_R,$$

which implies

$$L\left(u_R-\underline{\varphi}+\delta_1|\tilde{x}|^2\right)<0 \quad \text{in } B_\epsilon(x_0)\cap\Omega_R.$$

We will denote $h=u_R-\underline{\varphi}+\frac{\delta_1}{2}|\tilde{x}|^2$, then we have $Lh<-\delta_1\sum_i\sigma_k^{ii}$. Set $\tilde{W}=Bh\pm TW_R$, we may choose $B>0$ large such that $\tilde{W}\geq 0$ on $\partial(B_\epsilon\cap\Omega_R)$ and $L\tilde{W}<0$ in $B_\epsilon\cap\Omega_R$. By the maximum principle we conclude $\tilde{W}>0$ in $B_\epsilon\cap\Omega_R$. Thus, at x_0 we have $\tilde{W}_{\tilde{n}}>0$, which gives

$$|(u_R)_{\tilde{\alpha}\tilde{n}}|<C$$

for some $C=C\left(\Gamma,\varphi,|Du|_{C^0(\Gamma)}\right)>0$ that is independent of R. □

Finally, following the well-known argument of [5] on page 284 and 285 we obtain

Lemma 3.6. *Let u_R be the strictly k-convex solution of* (3.1), *then u_R satisfies*

$$|(u_R)_{\nu\nu}|<C \quad \text{on } \Gamma, \tag{3.12}$$

where ν is the inward unit normal of Γ, and $C=C\left(\Gamma,\varphi,|Du|_{C^0(\Gamma)}\right)>0$ is a constant that is independent of R.

3.3.2 C^2 boundary estimates on ∂E_R

Now, we let $\tilde{u}_R=\frac{1}{R}u_R\left(\sqrt{R}x\right)$, then $\tilde{u}_R$ satisfies

$$\begin{cases} \sigma_k\left(\lambda\left(D^2\tilde{u}\right)\right)=1 & \text{in } \frac{1}{\sqrt{R}}\Omega_R, \\ \tilde{u}=\frac{\varphi}{R} & \text{on } \partial\frac{D}{\sqrt{R}}, \\ \tilde{u}=1+\frac{\mu(\alpha,\beta)}{R}+\bar{C}R^{-\beta} & \text{on } \partial E_1. \end{cases} \tag{3.13}$$

By Section 5 of [5] we have on ∂E_1

$$|D^2\tilde{u}_R|\leq C$$

for some constant $C=C(A)>0$ that is independent of R. This in turn implies that

$$\left|D^2u_R\right|\leq C \quad \text{on } \partial E_R.$$

We conclude

Lemma 3.7. *Let u_R be the strictly k-convex solution of* (3.1), *then u_R satisfies*

$$\left|D^2u_R\right|<C \quad \text{on } \partial\Omega_R, \tag{3.14}$$

for some constant $C=C\left(\Gamma,\varphi,|Du|_{C^0(\Gamma)},A\right)>0$ that is independent of R.

3.3.3 C^2 global estimates

Finally we prove.

Lemma 3.8. *Let u_R be the strictly k-convex solution of* (3.1), *then u_R satisfies*

$$|D^2u_R|<C \quad \text{in } \bar{\Omega}_R, \tag{3.15}$$

for some constant $C=C\big(A,\varphi,\Gamma,|Du|_{C^0(\Gamma)}\big)>0$ that is independent of R.

Proof. We denote $F(D^2u)=\sigma_k^{1/k}(\lambda(D^2u))$, then (3.1) can be written as

$$F(D^2u)=1 \quad \text{in } \Omega_R. \tag{3.16}$$

Now differentiating (3.16) twice we get

$$F^{ij}u_{ijkk}+F^{pq,rs}u_{pqk}u_{rsk}=0.$$

It is well known that F is a concave function. Therefore, we get

$$F^{ij}u_{ijkk}=F^{ij}(\Delta u)_{ij}\geq 0.$$

By the maximum principle we obtain

$$\max_{x\in\bar{\Omega}_R}\Delta u_R=\max_{x\in\partial\Omega_R}\Delta u. \tag{3.17}$$

Since

$$(\Delta u)^2-\sum_{i,j}|u_{ij}|^2=2\sigma_2(\lambda(D^2u))>0,$$

(3.15) follows from (3.17) and Lemma 3.7. □

3.4 Proof of the existence part of Theorem 1.1

In this subsection, we will complete the proof of the existence part of Theorem 1.1. First, by virtue of equation (3.5) and Lemma 3.8 it is easy to see that there exist some constants $C_0,C_1>0$ that are independent of R such that $|Du_R|\leq C_0+C_1\text{dist}(x,\partial D)$. Moreover, applying Evans-Krylov theorem and Schauder estimates, we can obtain higher order estimates of u_R that are independent of R. By the standard maximum principle we know that when $R_1<R_2$, for any compact set $K\subset\Omega_{R_1}\subset\Omega_{R_2}$, we have $u_{R_2}<u_{R_1}$ on K. Therefore, $\{u_R\}$ is decreasing in R. We conclude

$$u_R(x)\to u(x) \quad \text{in } C^\infty_{loc} \text{ topology.}$$

Moreover, $u(x)\in C^\infty(\mathbb{R}^n\setminus D)$ satisfies

$$\begin{cases}\sigma_k\big(\lambda\big(D^2u\big)\big)=1 & \text{in } \mathbb{R}^n\setminus\bar{D};\\ u=\varphi & \text{on } \partial D.\end{cases} \tag{3.18}$$

This completes the proof of the existence part of Theorem 1.1.

4 Asymptotic behavior near infinity

In this section, we will prove the second part of Theorem 1.1, that is, inequality (1.2) and Theorem 1.2. Recall Lemma 3.1 we have

$$\underline{u} < u_R < \bar{u}_R \quad \text{in } E_R \setminus \bar{D}.$$

By virtue of equations (2.19), (2.21), and Proposition 2.2 we obtain that in $E_R \setminus E_{R_0}$

$$s+\mu(\alpha,\beta)+C_1 s^{1-\beta} < u_R < s+\mu(\alpha,\beta)+\bar{C}R^{1-\beta} \leq s+\mu(\alpha,\beta)+\bar{C}s^{1-\beta}.$$

Here $C_1 = C_1(\alpha,\beta) < 0$ and $R_0 \gg 1$ is a large constant. Therefore, when $|x| > R_0^{1/2}$ is very large we have

$$|x|^{2\beta-2}|u-s-\mu(\alpha,\beta)| \leq C \tag{4.1}$$

for some $C = C(\alpha,\beta,A) > 0$.

Without loss of generality, in the following, we may assume $\mu(\alpha,\beta) = 0$. Now for any $x \in \mathbb{R}^n \setminus \bar{D}$ and $|x| = L > 2R_0^{1/2}$, let

$$u^L(y) := \left(\frac{4}{L}\right)^2 u\left(x+\frac{L}{4}y\right), \quad |y| \leq 2.$$

Then by Lemma 3.8 $u^L(y)$ satisfies

$$\sigma_k(\lambda(D^2u^L(y))) = 1 \quad \text{and} \quad |D^2u^L| \leq C \quad \text{in } B_2(0).$$

Applying Evans-Krylov theorem and Schauder estimates we immediately obtain for any $m \geq 3$

$$|D^m u^L| \leq C \,\text{in } B_1(0) \tag{4.2}$$

for some $C = C\left(m,n,k,|u^L|_{C^2(B_2(0))}\right) > 0$. Now, denote

$$E^L(y) := u^L(y) - 8\left(\frac{x}{L}+\frac{y}{4}\right)^T A\left(\frac{x}{L}+\frac{y}{4}\right).$$

By (4.1) we get, on $B_2(0)$

$$|E^L(y)| \leq \left(\frac{4}{L}\right)^2 \frac{C}{\left|x+\frac{L}{4}y\right|^{2(\beta-1)}} \leq \frac{C}{L^{2\beta}}.$$

Moreover, since $D^2E^L(y) = D^2u^L(y) - A$, we have $F(A+D^2E^L) = F(A) = 1$. Let

$$F(\xi) := \sigma_k^{\frac{1}{k}}(\lambda(\xi_{ij})), \quad F_{\xi_{ij}} := \frac{\partial F}{\partial \xi_{ij}},$$

and $\hat{a}_{ij}(y) := \int_0^1 F_{\xi_{ij}}(A + \tau D^2 E^L(y))d\tau$. Then we can see that E^L satisfies

$$\hat{a}_{ij}(y) D_{ij} E^L(y) = 0 \quad \text{on } B_1(0).$$

Note that by Lemma 3.8 and the estimate (4.2) we have on $B_1(0)$,

$$\frac{1}{\Lambda} I < \hat{a}_{ij}(y) < \Lambda I \quad \text{and} \quad |D^m \hat{a}_{ij}(y)| < C \quad \text{for } m \geq 1,$$

where $\Lambda = \Lambda\big(n,k,A,|u^L|_{C^2(B_1(0))}\big) > 0$ and $C = C\big(m,n,k,A,|u^L|_{C^2(B_2(0))}\big) > 0$. By Schauder estimates ([7]) we get for any $m \geq 1$

$$\big|D^m E^L(0)\big| \leq C \big|E^L(y)\big|_{C^0(B_1(0))} \leq \frac{C}{L^{2\beta}},$$

where $C = C\big(m,n,k,A,|u^L|_{C^2(B_2(0))}\big) > 0$. This yields for any $x \in \mathbb{R}^n \setminus \bar{D}$ and $|x| > 2R_0^{1/2}$, we have

$$|D^m E(x)| \leq \frac{C}{|x|^{2\beta - 2 + m}} \tag{4.3}$$

for $E(x) = u(x) - \frac{1}{2} x^T A x$ and $C = C(m,n,k,A,|u|_{C^2}) > 0$.

Now, we can follow the proof of [4, Lemma 3.6] (which is an estimate for linear elliptic equation) and obtain (1.2). This completes the proof of Theorem 1.1. Repeating the argument above by replacing β with $\frac{n}{2}$ gives Theorem 1.2.

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