J. Math. Study
doi: 10.4208/jms.v59n1.26.03

Vol. **59**, No. 1, pp. 40-59
March 2026

Tangent Flows of Symplectic Mean Curvature Flows

Jingyi Chen[1], Xiaoli Han[2], Jiayu Li[3,*] and Jun Sun[4]

[1]*Department of Mathematics, University of British Columbia, Vancouver, BC V6T 1Z2, Canada;*
[2]*Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China;*
[3]*School of Mathematical Sciences, University of Science and Technology of China Hefei 230026 & AMSS CAS, Beijing 100190, China;*
[4]*School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China.*

Received October 9, 2025; Accepted December 29, 2025;
Published online March 15, 2026.

Dedicated to Professor Gang Tian for his 65th birthday.

Abstract. We prove that the tangent cone at the first blow-up time of the mean curvature flow of a closed symplectic surface in a compact Kähler-Einstein surface consists of a finite union of planes in $\mathbf{R}^4$. Furthermore, when the flow develops a Type I* singularity at (X_0, T), then the tangent cone is a holomorphic cone.

AMS subject classifications: 53C42, 53E10

Key words: Symplectic mean curvature flow, tangent flow, Type I* singularity, holomorphic curve.

---

1 Introduction

One possible way of constructing minimal surfaces is to deform a given surface by its mean curvature vector. So the surface evolves in the gradient flow of the area functional by the first variational formula of area, and such a flow is the so-called mean curvature flow. In general a mean curvature flow develops singularity after finite time. It is therefore desirable to understand behavior of the flow near the singular points ([3, 5, 14–20, 23–25] and so on). In this paper, we consider compact symplectic surfaces moving by a mean curvature flow in a Kähler-Einstein surface. Since a symplectic surface remains symplectic along the flow ([5, 8, 24]), one hopes to produce holomorphic curves, possibly with singularities, by deforming symplectic surfaces via the mean curvature flow. There are positive evidences for this approach when $C_1 > 0$ ([7, 10, 12] and [24]),

*Corresponding author. *Email addresses:* `jychen@math.ubc.ca` (Chen J), `hanxiaoli@mail.tsinghua.edu.cn` (Han X), `jiayuli@ustc.edu.cn` (Li J), `sunjun@whu.edu.cn` (Sun J)

where C_1 is the first Chern class of the ambient K-E surface. In the case that $C_1<0$, Arezzo ([2]) pointed out that a symplectic minimal surface may not be holomorphic even it represents a $(1,1)$-type class, by constructing examples of symplectic minimal surfaces which are not holomorphic.

Suppose that M is a compact Kähler surface. Let ω be the Kähler form on M and let J be a complex structure compatible with ω. The Riemannian metric $\langle,\rangle$ on M is defined by

$$\langle U,V\rangle=\omega(U,JV).$$

For a compact oriented real surface Σ which is smoothly immersed in M, one defines, following [9], the Kähler angle α of Σ in M by

$$\omega|_\Sigma=\cos\alpha d\mu_\Sigma \tag{1.1}$$

where $d\mu_\Sigma$ is the area element of Σ of the induced metric from $\langle,\rangle$. As a function on Σ, α is continuous everywhere and is smooth possibly except at the complex or anti-complex points of Σ, i.e., where $\alpha=0$ or π. We say that Σ is a *holomorphic curve* if $\cos\alpha\equiv 1$, Σ is a *Lagrangian surface* if $\cos\alpha\equiv 0$ and Σ is a *symplectic surface* if $\cos\alpha>0$.

Given an immersion $F_0:\Sigma\to M$, we consider a one-parameter family of immersions $F_t=F(\cdot,t):\Sigma\to M$, and denote the image surfaces by $\Sigma_t=F_t(\Sigma)$. The immersed surfaces Σ_t satisfy a mean curvature flow if

$$\begin{cases}\dfrac{d}{dt}F(x,t)=\mathbf{H}(x,t);\\ F(x,0)=F_0(x),\end{cases} \tag{1.2}$$

where $\mathbf{H}(x,t)$ is the mean curvature vector of Σ_t at $F(x,t)$ in M.

The standard parabolic theory implies that the mean curvature flow (1.2) has a smooth solution for short time. More precisely, there exists $T>0$ such that (1.2) has a smooth solution in the time interval $[0,T)$. Huisken proved that if the second fundamental form $|\mathbf{A}|^2$ on Σ_t is bounded uniformly in t near T, then the solution can be extended smoothly to $[0,T+\epsilon)$ for some $\epsilon>0$ ([14]). However, in general $\max_{\Sigma_t}|\mathbf{A}|^2$ becomes unbounded as $t\to T$. In this case we say that the mean curvature flow blows up at T; moreover, one ([15]) can classify the singularities of mean curvature flows, according to the blowing up rate of $|\mathbf{A}|$: Assume that

$$\lim_{t\to T^-}\max_{\Sigma_t}|\mathbf{A}|^2=\infty.$$

Then it can be shown that ([5,15]) there is a constant $c>0$ such that

$$\limsup_{t\to T^-}\left((T-t)\max_{\Sigma_t}|\mathbf{A}|^2\right)\geq c.$$

If there exists a positive constant C such that

$$\limsup_{t\to T^-}\left((T-t)\max_{\Sigma_t}|\mathbf{A}|^2\right)\leq C,$$