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Tangent Flows of Symplectic Mean Curvature Flows

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Abstract. We prove that the tangent cone at the first blow-up time of the mean curvature flow of a closed symplectic surface in a compact Kähler-Einstein surface consists of a finite union of planes in $\mathbf{R}^4$. Furthermore, when the flow develops a Type I* singularity at (X_0,T), then the tangent cone is a holomorphic cone.

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1 Introduction

One possible way of constructing minimal surfaces is to deform a given surface by its mean curvature vector. So the surface evolves in the gradient flow of the area functional by the first variational formula of area, and such a flow is the so-called mean curvature flow. In general a mean curvature flow develops singularity after finite time. It is therefore desirable to understand behavior of the flow near the singular points ([3, 5, 14–20, 23–25] and so on). In this paper, we consider compact symplectic surfaces moving by a mean curvature flow in a Kähler-Einstein surface. Since a symplectic surface remains symplectic along the flow ([5, 8, 24]), one hopes to produce holomorphic curves, possibly with singularities, by deforming symplectic surfaces via the mean curvature flow. There are positive evidences for this approach when $C_1 > 0$ ([7, 10, 12] and [24]),

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where C_1 is the first Chern class of the ambient K-E surface. In the case that $C_1 < 0$, Arezzo ([2]) pointed out that a symplectic minimal surface may not be holomorphic even it represents a $(1,1)$-type class, by constructing examples of symplectic minimal surfaces which are not holomorphic.

Suppose that M is a compact Kähler surface. Let ω be the Kähler form on M and let J be a complex structure compatible with ω. The Riemannian metric $\langle,\rangle$ on M is defined by

$$\langle U,V\rangle = \omega(U,JV).$$

For a compact oriented real surface Σ which is smoothly immersed in M, one defines, following [9], the Kähler angle α of Σ in M by

$$\omega|_\Sigma = \cos\alpha d\mu_\Sigma \tag{1.1}$$

where $d\mu_\Sigma$ is the area element of Σ of the induced metric from $\langle,\rangle$. As a function on Σ, α is continuous everywhere and is smooth possibly except at the complex or anti-complex points of Σ, i.e., where $\alpha = 0$ or π. We say that Σ is a *holomorphic curve* if $\cos\alpha \equiv 1$, Σ is a *Lagrangian surface* if $\cos\alpha \equiv 0$ and Σ is a *symplectic surface* if $\cos\alpha > 0$.

Given an immersion $F_0 : \Sigma \to M$, we consider a one-parameter family of immersions $F_t = F(\cdot,t) : \Sigma \to M$, and denote the image surfaces by $\Sigma_t = F_t(\Sigma)$. The immersed surfaces Σ_t satisfy a mean curvature flow if

$$\begin{cases} \dfrac{d}{dt}F(x,t) = \mathbf{H}(x,t); \\ F(x,0) = F_0(x), \end{cases} \tag{1.2}$$

where $\mathbf{H}(x,t)$ is the mean curvature vector of Σ_t at $F(x,t)$ in M.

The standard parabolic theory implies that the mean curvature flow (1.2) has a smooth solution for short time. More precisely, there exists $T > 0$ such that (1.2) has a smooth solution in the time interval $[0,T)$. Huisken proved that if the second fundamental form $|\mathbf{A}|^2$ on Σ_t is bounded uniformly in t near T, then the solution can be extended smoothly to $[0,T+\epsilon)$ for some $\epsilon > 0$ ([14]). However, in general $\max_{\Sigma_t}|\mathbf{A}|^2$ becomes unbounded as $t \to T$. In this case we say that the mean curvature flow blows up at T; moreover, one ([15]) can classify the singularities of mean curvature flows, according to the blowing up rate of $|\mathbf{A}|$: Assume that

$$\lim_{t\to T^-}\max_{\Sigma_t}|\mathbf{A}|^2 = \infty.$$

Then it can be shown that ([5,15]) there is a constant $c > 0$ such that

$$\limsup_{t\to T^-}\left((T-t)\max_{\Sigma_t}|\mathbf{A}|^2\right) \geq c.$$

If there exists a positive constant C such that

$$\limsup_{t\to T^-}\left((T-t)\max_{\Sigma_t}|\mathbf{A}|^2\right) \leq C,$$

then the mean curvature flow F has a *Type I singularity at T*; otherwise it has a *Type II singularity at T*.

In the codimension one case, singularities of mean curvature flows have been studied in depth ([14–20] and so on). For higher co-dimensions, if the initial compact surface is symplectic in a compact Kähler-Einstein surface, the motion of the mean curvature flow preserves symplecticity of Σ_t as long as the smooth solution exists; and furthermore the flow does not develop any Type I singularities ([5,8,24]).

In this paper, we shall study the Type II singularities of the mean curvature flow of a compact symplectic surface in a compact Kähler-Einstein surface. Especially, we shall focus on the structure of tangent cones of the mean curvature flow where a singularity occurs at the first singular time $T<\infty$.

To describe the tangent cones, suppose (X_0,T) is a singular point of the flow (1.2), i.e., $|\mathbf{A}(x,t)|$ becomes unbounded when $(x,t)\to(X_0,T)$. For an arbitrary sequence of numbers $\lambda\to\infty$ and any $t<0$, if $T+\lambda^{-2}t>0$ we set

$$F_\lambda(x,t)=\lambda\big(F\big(x,T+\lambda^{-2}t\big)-X_0\big).$$

We denote the scaled surface by $\big(\Sigma_t^\lambda,d\mu_t^\lambda\big)$. If the initial surface is symplectic, it is proved in Lemma 2.2 that there is a subsequence $\lambda_i\to\infty$ such that for any $t<0$, $(\Sigma_t^{\lambda_i},d\mu_t^{\lambda_i})$ converges to $(\Sigma^\infty,d\mu^\infty)$ in the sense of measures; the limit Σ^∞ which is the support of $d\mu^\infty$, is called a *tangent cone at* (X_0,T). This tangent cone is independent of t as shown in Lemma 2.2. The first result of this paper is

Theorem 1.1. *Let M be a compact Kähler-Einstein surface. If the initial compact surface is symplectic and $T>0$ is the first blow-up time of the mean curvature flow, then the tangent cone Σ^∞ of the mean curvature flow at (X_0,T) is a finite union of planes.*

A priori, the Kähler angles on different planes may be different. We would like to know when the finite planes has the same constant Kähler angle so that the tangent flow is a holomorphic flat cone. For this purpose, we introduce the definition of Type I* singularity at finite time.

Definition 1.1. *Let T be the first singular time of the symplectic mean curvature flow. We say that the flow develops **Type I* singularity** at T if*

$$\limsup_{t\to T}\Big((T-t)\max_{\Sigma_t}|\nabla\cos\alpha|^2\Big)\leq\Lambda$$

for some constant Λ.

Remark 1.1. From (2.3), we see that a Type I* singularity does not need to be a Type I singularity in general.

The next result is as follows:

Theorem 1.2. *Let M be a compact Kähler-Einstein surface. If the initial compact surface is symplectic and $T>0$ is the first blow-up time of the mean curvature flow such that the flow develops a Type I* singularity at (X_0,T), then the tangent cone Σ^∞ arising from connected components of the rescaling of the mean curvature flow at (X_0,T) is a cone consisting of finitely many planes which is holomorphic with respect to some complex structure on $\mathbb{C}^2$.*

The subsequent sections of this paper are organized as follows: in Section 2, we provide necessary estimates followed from the monotonicity formula; in Section 3, we prove the first result that the tangent cone consists of finite union of planes; in Section 4, we prove the second theorem that the limiting planes have the same Kähler angle so that the tangent cone is holomorphic with respect to some complex structure if the singularity is of Type I* and the tangent cone arises from connected components of the rescaling of the mean curvature flow.

2 Monotonicity formula and integral estimates

Let $\Sigma_t=F(\Sigma,t)$ be the family of immersed surfaces, which are determined by the mean curvature flow F, in the 4-dimensional manifold M. Denote the Riemannian metric on M by $\langle\cdot,\cdot\rangle$. In a normal coordinate chart around a point in Σ_t, the induced metric on Σ_t from $\langle\cdot,\cdot\rangle$ is given by $g_{ij}=\langle\partial_iF,\partial_jF\rangle$, where ∂_i $(i=1,2)$ are the partial derivatives with respect to the local coordinates on Σ. In the sequel, we denote by Δ and ∇ the Laplace operator and covariant derivative for the induced metric on Σ_t respectively. We choose a local field of orthonormal frames e_1,e_2, v_1,v_2 of M along Σ_t such that e_1,e_2 are tangent vectors of Σ_t and v_1,v_2 are in the normal bundle over Σ_t. The second fundamental form $\mathbf{A}$ and the mean curvature vector $\mathbf{H}$ of Σ_t can be expressed, in the local frame, as

$$\mathbf{A}=A^\alpha v_\alpha,\quad \mathbf{H}=-H^\alpha v_\alpha$$

where and throughout this paper all repeated indices are summed over suitable range. For each α, the coefficient A^α is a 2×2 matrix (h^α_{ij}). By Weingarten's equation, we have

$$h^\alpha_{ij}=\langle\partial_i v_\alpha,\partial_jF\rangle=\langle\partial_j v_\alpha,\partial_iF\rangle=h^\alpha_{ji}.$$

The trace and the norm of the second fundamental form of Σ_t in M are:

$$H^\alpha=g^{ij}h^\alpha_{ij}=h^\alpha_{ii},\quad |\mathbf{A}|^2=\sum_\alpha|A^\alpha|^2=g^{ij}g^{kl}h^\alpha_{ik}h^\alpha_{jl}=h^\alpha_{ik}h^\alpha_{ik}.$$

The area element of the induced metric g_{ij} on Σ_t is $\sqrt{\det(g_{ij})}dxdy$. Along the mean curvature flow, it is well known that

$$\frac{d}{dt}\sqrt{\det(g_{ij})}=-|\mathbf{H}|^2\sqrt{\det(g_{ij})}.$$

Logarithmic integration implies that F remains immersed as long as the smooth solution of (1.2) exists.

Let J_{Σ_t} be an almost complex structure in a tubular neighborhood of Σ_t on M with

$$\begin{cases} J_{\Sigma_t}e_1 = e_2 \\ J_{\Sigma_t}e_2 = -e_1 \\ J_{\Sigma_t}v_1 = v_2 \\ J_{\Sigma_t}v_2 = -v_1. \end{cases} \tag{2.1}$$

It is not difficult to verify ([8], [5]), with $\overline{\nabla}$ being the covariant derivative of the metric $\langle\cdot,\cdot\rangle$ on M, that

$$\begin{aligned} |\overline{\nabla} J_{\Sigma_t}|^2 =& |h_{11}^2+h_{12}^1|^2+|h_{21}^2+h_{22}^1|^2+|h_{12}^2-h_{11}^1|^2+|h_{22}^2-h_{21}^1|^2 \\ =& \frac{1}{2}|\mathbf{H}|^2+\frac{1}{2}\Big(((h_{11}^1+h_{22}^1)+2(h_{12}^2-h_{22}^1))^2+(h_{11}^2+h_{22}^2+2h_{21}^1-2h_{11}^2)^2\Big) \\ \geq& \frac{1}{2}|\mathbf{H}|^2. \end{aligned} \tag{2.2}$$

We also have ([11])

$$|\nabla\alpha|^2 = |h_{11}^2+h_{12}^1|^2+|h_{21}^2+h_{22}^1|^2, \tag{2.3}$$

so that

$$|\overline{\nabla} J_\Sigma|^2 \geq |\nabla\alpha|^2. \tag{2.4}$$

Let $H(\mathbf{X},\mathbf{X}_0,t)$ be the backward heat kernel on $\mathbf{R}^4$. Define

$$\rho(\mathbf{X},\mathbf{X}_0,t,t_0) = 4\pi(t_0-t)H(\mathbf{X},\mathbf{X}_0,t) = \frac{1}{4\pi(t_0-t)}\exp\left(-\frac{|\mathbf{X}-\mathbf{X}_0|^2}{4(t_0-t)}\right)$$

for $t<t_0$. Let i_M be the injective radius of M^4. We choose a cut off function $\phi\in C_0^\infty(B_{2r}(\mathbf{X}_0))$ with $\phi\equiv 1$ in $B_r(\mathbf{X}_0)$, where $\mathbf{X}_0\in M$, $0<2r<i_M$. Choose a normal coordinate in $B_{2r}(\mathbf{X}_0)$ and express F, by the coordinates (F^1,F^2,F^3,F^4), as a surface in $\mathbf{R}^4$. We define

$$\Phi(\mathbf{X}_0,t_0,t) = \int_{\Sigma_t}\phi(F)\rho(F,\mathbf{X}_0,t,t_0)d\mu_t. \tag{2.5}$$

Huisken derived the following monotonicity formula in [15]: there are positive constants c_1 and c_2 depending only on M^4, F_0 and r where r is the constant in the definition

$$\begin{aligned} &\frac{\partial}{\partial t}\left(e^{c_1\sqrt{t_0-t}}\Phi(\mathbf{X}_0,t_0,t)\right) \\ \leq& -e^{c_1\sqrt{t_0-t}}\int_{\Sigma_t}\phi\rho(F,\mathbf{X}_0,t,t_0)\left|\mathbf{H}+\frac{(F-\mathbf{X}_0)^\perp}{2(t_0-t)}\right|^2 d\mu_t + c_2e^{c_1\sqrt{t_0-t}}. \end{aligned} \tag{2.6}$$

Note that, c_1 and c_2 are zero when M is a Euclidean space.

When M is a Kähler-Einstein surface with scalar curvature R and Σ_t evolves under the mean curvature flow, the Kähler angle α of Σ_t in M satisfies the parabolic equation

$$\left(\frac{\partial}{\partial t}-\Delta\right)\cos\alpha=|\overline{\nabla}J_{\Sigma_t}|^2\cos\alpha+\frac{R}{4}\sin^2\alpha\cos\alpha, \tag{2.7}$$

Suppose that the initial surface is symplectic, i.e., $\cos\alpha(\cdot,0)$ has a positive lower bound. Then by applying the parabolic maximum principle to the evolution equation (2.7), one concludes that $\cos\alpha$ remains positive as long as the mean curvature flow has a smooth solution ([5,8,24]).

Let $R_0=\max\{0,-\frac{R}{4}\}$ and set

$$v(x,t)=e^{R_0t}\cos\alpha(x,t).$$

By (2.7), we have

$$\left(\frac{\partial}{\partial t}-\Delta\right)\frac{1}{v}\le-|\overline{\nabla}J_{\Sigma_t}|^2\frac{1}{v}-\frac{2}{v^3}|\nabla v|^2. \tag{2.8}$$

Along the flow, we introduce a function

$$\Psi(\mathbf{X}_0,t_0,t)=\int_{\Sigma_t}\frac{1}{v}\phi\rho(F,\mathbf{X}_0,t,t_0)d\mu_t. \tag{2.9}$$

The following weighted monotonicity formula in [5] (also see [6]) will play a crucial role in this paper.

Proposition 2.1 (Weighted Monotonicity Formula)**.** *If the initial compact surface Σ_0 is symplectic in a Kähler-Einstein surface M and Σ_t evolves under the mean curvature flow* (1.2) *, then*

$$\begin{aligned}\frac{\partial}{\partial t}\left(e^{c_1\sqrt{t_0-t}}\Psi(X_0,t_0,t)\right)\le-e^{c_1\sqrt{t_0-t}}\Bigg(&\int_{\Sigma_t}\frac{1}{v}\phi\rho(F,\mathbf{X}_0,t,t_0)\left|\mathbf{H}+\frac{(F-\mathbf{X}_0)^\perp}{2(t_0-t)}\right|^2d\mu_t\\&+\int_{\Sigma_t}\frac{1}{2v}\phi\rho(F,\mathbf{X}_0,t,t_0)\left|\overline{\nabla}J_{\Sigma_t}\right|^2d\mu_t\\&+\int_{\Sigma_t}\frac{2}{v^3}|\nabla v|^2\phi\rho(F,\mathbf{X}_0,t,t_0)d\mu_t\Bigg)+c_2e^{c_1\sqrt{t_0-t}}.\end{aligned} \tag{2.10}$$

Here the positive constants c_1 and c_2 depend on M, F_0 and r where r is the constant in the definition of ϕ.

Suppose that (X_0,T) is a singular point of the mean curvature flow (1.2). We now describe the rescaling process around (X_0,T). We choose normal coordinates centered at X_0 with radius r ($0<r<i_M/2$), using the exponential map. We express F in its coordinates functions. For any $t<0$, we set

$$F_\lambda(x,t)=\lambda\big(F(x,T+\lambda^{-2}t)-X_0\big),$$

where λ are positive constants which go to infinity. The scaled surface is denoted by $\Sigma_t^\lambda = F_\lambda(\Sigma,t)$ on which $d\mu_t^\lambda$ is the area element obtained from $d\mu_t$. For any $R>0$, let $B_R(0)$ be a ball in $\mathbf{R}^4$ with radius R in the Euclidean metric and center 0. Then

$$\Sigma_t^\lambda \cap B_R(0) = \{|F_\lambda(x,t)| \le R\}.$$

It is clear that for $\lambda^{-1}R<r$ the surface Σ_t^λ is defined on $B_R(0)$ because

$$\exp_{X_0}\left(\lambda^{-1}\{|F_\lambda(x,t)|\le R\}\right) \subset B_{\lambda^{-1}R}(X_0) \subset B_r(X_0),$$

where $B_r(X_0)$ is a metric ball in M with $0<r<i_M/2$. Moreover, for any fixed $R>0$ and any $\lambda^{-1}R<r$, we pull back the metric on $B_{\lambda^{-1}R}(X_0)\subset M$ via $\exp_{X_0}$ so that we get a metric h_R^λ on the Euclidean ball $B_{\lambda^{-1}R}(0)$. Then

$$h_{\lambda,R}(x) = h_R^\lambda(\lambda^{-1}x)$$

is a metric on $B_R(0)$ and with respect to which Σ_t^λ evolves along MCF, which will be derived as follows.

If g^λ is the metric on Σ_t^λ, it is clear that

$$g_{ij}^\lambda = \lambda^2 g_{ij}, \quad (g^\lambda)^{ij} = \lambda^{-2} g^{ij}.$$

It is easy to check that

$$\frac{\partial F_\lambda}{\partial t} = \lambda^{-1}\frac{\partial F}{\partial t}, \quad \mathbf{H}_\lambda = \lambda^{-1}\mathbf{H}, \quad |\mathbf{A}_\lambda|^2 = \lambda^{-2}|\mathbf{A}|^2.$$

It follows that the scaled surface also evolves by a mean curvature flow

$$\frac{\partial F_\lambda}{\partial t} = \mathbf{H}_\lambda. \tag{2.11}$$

For each fixed $R>0$, $h_{\lambda,R}$ converges uniformly in $B_R(0)$ to the Euclidean metric as $\lambda\to\infty$, and the Christoffel symbols $\left(\overline{\Gamma}^\lambda\right)$ of $h_{\lambda,R}$ converges uniformly in $B_R(0)$ to 0 as $\lambda\to\infty$. Let $\mathbf{A}_\lambda^0$ and $\mathbf{H}_\lambda^0$ be the second fundamental form and the mean curvature vector of Σ_t^λ in the Euclidean metric on $B_R(0)$ respectively. Let Γ_t^λ and $\Gamma_t^{0\lambda}$ be the Christoffel symbols of Σ_t^λ for the metric $h_{\lambda,R}$ and the Euclidean metric on $B_R(0)$ respectively. Let $\{\nu_{t\alpha}^\lambda : \alpha=1,2\}$ and $\{\nu_{t\alpha}^{0\lambda} : \alpha=1,2\}$ be bases of the normal space of Σ_t^λ with respect to the metric $h_{\lambda,R}$ and the Euclidean metric on $B_R(0)$ respectively. It is clear that F_λ is an isometric immersion in $(B_R(0),h_{\lambda,R})$ with respect to the induced metric, hence by the Gauss equation we have

$$(\mathbf{A}_\lambda)_{ij} = \sum_{\alpha=1,2}(h_\lambda)_{ij}^\alpha \nu_{t\alpha}^\lambda = -\partial_{ij}^2 F_\lambda + \sum_{k=1,2}(\Gamma_t^\lambda)_{ij}^k \partial_k F_\lambda - \sum_{\alpha,\beta,\gamma=1}^{4}(\overline{\Gamma}^\lambda)_{\beta\gamma}^\alpha \partial_i F_\lambda^\beta \partial_j F_\lambda^\gamma e_\alpha \tag{2.12}$$

where e_α are the local coordinate frames in $B_r(0)$ extended to $B_R(0)$ with

$$\langle e_\alpha, e_\beta\rangle_{h_{\lambda,R}}(x) = \langle e_\alpha, e_\beta\rangle_{h_R^\lambda}(\lambda^{-1}x)$$

for all $x\in B_R(0)$ and $\lambda^{-1}R<r$; and similarly, considering F_λ as an isometric immersion in $B_R(0)$ with Euclidean metric, we have

$$(\mathbf{A}^0_\lambda)_{ij} = \sum_{\alpha=1,2}(h_0)^\alpha_{ij}(\nu_t^{0\lambda})_\alpha = -\partial^2_{ij}F_\lambda + \sum_{k=1,2}(\Gamma_t^{0\lambda})^k_{ij}\partial_k F_\lambda. \tag{2.13}$$

Note that the induced metric on Σ_t^λ from $h_{\lambda,R}$ is given by $\langle\partial F_\lambda,\partial F_\lambda\rangle_{h_{\lambda,R}}$, so it holds

$$|\partial F_\lambda|^2_{h_{\lambda,R}} = 2,$$

which in turn implies that for λ sufficiently large and R fixed $|\partial F_\lambda^\beta|$ and $|e_\alpha|$ are uniformly bounded in $B_R(0)$ with the Euclidean metric.

Using the Euclidean metric on $B_R(0)$, we decompose the tangent bundle of $B_R(0)$ along Σ_t^λ into the tangential component $T\Sigma_t^\lambda$ and the normal component $T^\perp\Sigma_t^\lambda$. Let

$$\mathbf{A}^\perp_\lambda : T\Sigma_t^\lambda\times T\Sigma_t^\lambda \to T^\perp\Sigma_t^\lambda$$

be the normal component of $\mathbf{A}_\lambda$. Notice that $\mathbf{A}^\perp_\lambda - \mathbf{A}^0_\lambda$ lies in $T^\perp\Sigma_t^\lambda$ and $\partial_k F_\lambda$ is in $T\Sigma_t^\lambda$, it follows from (2.12) and (2.13) that

$$\sup_{B_R(0)}\left|\mathbf{A}^\perp_\lambda - \mathbf{A}^0_\lambda\right| \le C\sup_{B_R(0)}\left|\overline{\Gamma}^\lambda\right| \to 0$$

as $\lambda\to\infty$ for any $R>0$. From the uniform convergence of the metrics $h_{\lambda,R}$ to the Euclidean metric,

$$\left|\mathbf{A}^\perp_\lambda\right| \le |\mathbf{A}_\lambda| \le 2|\mathbf{A}_\lambda|_{h_{\lambda,R}}$$

for any fixed $R>0$ and sufficiently large λ. Hence, there exist positive constants $\delta_{\lambda,R}$ which tend to 0 as $\lambda\to\infty$ such that

$$\left|\mathbf{A}^0_\lambda\right| = \left|\mathbf{A}^\perp_\lambda\right| + \delta_{\lambda,R} \le 2|\mathbf{A}_\lambda|_{h_{\lambda,R}} + \delta_{\lambda,R} \tag{2.14}$$

for all λ and any fixed $R>0$; and similarly there exist constants $\delta'_{\lambda,R}>0$ with $\delta'_{\lambda,R}\to 0$ as $\lambda\to\infty$ such that

$$|\mathbf{H}^0_\lambda| \le 2|\mathbf{H}_\lambda|_{h_{\lambda,R}} + \delta'_{\lambda,R} \tag{2.15}$$

for all sufficiently large λ and any given $R>0$. We may therefore assume that the metric in $B_R(0)$ is Euclidean. In fact, this assumption (more precisely, the estimation (2.14) and (2.15)) is not needed in this section, it is only needed when we use Allard's compactness theorem (for the Euclidean ambient space) in next section.

The weighted monotonicity formula leads to the following integral estimates.

Proposition 2.2. *Let M be a Kähler-Einstein surface. If the initial compact surface is symplectic, then for any $R>0$ and any $-\infty<s_1<s_2<0$, we have*

$$\int_{s_1}^{s_2}\int_{\Sigma_t^\lambda\cap B_R(0)}\left|\overline{\nabla}J_{\Sigma_t^\lambda}\right|^2 d\mu_t^\lambda dt\to 0\quad\text{as}\ \lambda\to\infty, \tag{2.16}$$

$$\int_{s_1}^{s_2}\int_{\Sigma_t^\lambda\cap B_R(0)}|\nabla\cos\alpha_\lambda|^2 d\mu_t^\lambda dt\to 0\quad\text{as}\ \lambda\to\infty, \tag{2.17}$$

$$\int_{s_1}^{s_2}\int_{\Sigma_t^\lambda\cap B_R(0)}\left|\mathbf{H}_\lambda\right|^2 d\mu_t^\lambda dt\to 0\quad\text{as}\ \lambda\to\infty, \tag{2.18}$$

$$\int_{s_1}^{s_2}\int_{\Sigma_t^\lambda\cap B_R(0)}|F_\lambda^\perp|^2 d\mu_t^\lambda dt\to 0\quad\text{as}\ \lambda\to\infty. \tag{2.19}$$

Proof. For any $R>0$, we choose a cut-off function $\phi_R\in C_0^\infty(B_{2R}(0))$ with $\phi_R\equiv 1$ in $B_R(0)$, where $B_\rho(0)$ is the metric ball centered at 0 with radius ρ in $\mathbf{R}^4$. For any fixed $t<0$, the mean curvature flow (1.2) has a smooth solution near $T+\lambda^{-2}t<T$ for sufficiently large λ, because $T>0$ is the first blow-up time of the flow. It is clear that

$$\begin{aligned}&\int_{\Sigma_t^\lambda}\frac{1}{v_\lambda}\frac{1}{0-t}\phi_R(F_\lambda)\exp\left(-\frac{|F_\lambda|^2}{4(0-t)}\right)d\mu_t^\lambda\\ =&\int_{\Sigma_{T+\lambda^{-2}t}}\frac{1}{v_\lambda}\phi(F_\lambda)\frac{1}{T-(T+\lambda^{-2}t)}\exp\left(-\frac{|F(x,T+\lambda^{-2}t)-X_0|^2}{4(T-(T+\lambda^{-2}t))}\right)d\mu_t,\end{aligned}$$

where ϕ is the function defined in the definition of Φ. Note that $T+\lambda^{-2}t\to T$ for any fixed t as $\lambda\to\infty$. By (2.10),

$$\frac{\partial}{\partial t}\left(e^{c_1\sqrt{t_0-t}}\Psi\right)\le c_2 e^{c_1\sqrt{t_0-t}},$$

and it then follows that $\lim_{t\to t_0}e^{c_1\sqrt{t_0-t}}\Psi$ exists. This implies, by taking $t_0=T$ and $t=T+\lambda^{-2}s$, that for any fixed s_1 and s_2 with $-\infty<s_1<s_2<0$,

$$\begin{aligned}&e^{c_1\sqrt{T-(T+\lambda^{-2}s_2)}}\int_{\Sigma_{s_2}^\lambda}\frac{1}{v_\lambda}\phi_R\frac{1}{0-s_2}\exp\left(-\frac{|F_\lambda|^2}{4(0-s_2)}\right)d\mu_{s_2}^\lambda\\ &\quad-e^{c_1\sqrt{T-(T+\lambda^{-2}s_1)}}\int_{\Sigma_{s_1}^\lambda}\frac{1}{v_\lambda}\phi_R\frac{1}{0-s_1}\exp\left(-\frac{|F_\lambda|^2}{4(0-s_1)}\right)d\mu_{s_1}^\lambda\to 0\quad\text{as}\ \lambda\to\infty.\end{aligned} \tag{2.20}$$

Integrating

$$
\begin{aligned}
&-e^{c_1\sqrt{-\lambda^{-2}s_2}}\int_{\Sigma^\lambda_{s_2}}\frac{1}{v_\lambda}\phi_R\frac{1}{0-s_2}\exp\left(-\frac{|F_\lambda|^2}{4(0-s_2)}\right)d\mu^\lambda_{s_2}\\
&\quad+e^{c_1\sqrt{-\lambda^{-2}s_1}}\int_{\Sigma^\lambda_{s_1}}\frac{1}{v_\lambda}\phi_R\frac{1}{0-s_1}\exp\left(-\frac{|F_\lambda|^2}{4(0-s_1)}\right)d\mu^\lambda_{s_1}\\
&\geq\int_{s_1}^{s_2}e^{c_1\sqrt{-\lambda^{-2}t}}\int_{\Sigma^\lambda_t}\frac{1}{v_\lambda}\phi_R\rho(F_\lambda,t)\left|\mathbf{H}_k+\frac{(F_k)^\perp}{2(t_0-t)}\right|^2 d\mu^\lambda_t\\
&\quad+\int_{s_1}^{s_2}e^{c_1\sqrt{-\lambda^{-2}t}}\int_{\Sigma^\lambda_t}\frac{1}{2v_\lambda}\phi_R\rho(F_k,t)|\overline{\nabla}J_{\Sigma^\lambda_t}|^2 d\mu^\lambda_t\\
&\quad+\int_{s_1}^{s_2}e^{c_1\sqrt{-\lambda^{-2}t}}\int_{\Sigma^\lambda_t}\frac{2}{v_\lambda^3}|\nabla v_\lambda|^2\phi_R\rho(F_\lambda,t)d\mu^\lambda_t\\
&\quad-c_2\lambda^{-2}(s_2-s_1)e^{c_1\lambda^{-1}\sqrt{-s_1}}.
\end{aligned}
\tag{2.21}
$$

Putting (2.20) and (2.21) together, we have

$$\lim_{\lambda\to\infty}\int_{s_1}^{s_2}\int_{\Sigma^\lambda_t}\phi_R\rho(F_k,t)\left|\overline{\nabla}J_{\Sigma^\lambda_t}\right|^2 d\mu^\lambda_t=0,$$

and

$$\lim_{\lambda\to\infty}\int_{s_1}^{s_2}\int_{\Sigma^\lambda_t}|\nabla v_\lambda|^2\phi_R\rho(F_\lambda,t)d\mu^\lambda_t=0,$$

which yield (2.16) and (2.17) respectively, and

$$\lim_{\lambda\to\infty}\int_{s_1}^{s_2}\int_{\Sigma^\lambda_t}\phi_R\rho(F_\lambda,t)\left|\mathbf{H}_\lambda+\frac{(F_\lambda)^\perp}{2(t_0-t)}\right|^2 d\mu^\lambda_t=0. \tag{2.22}$$

Finally, (2.2) and (2.16) imply (2.18), and (2.18) and (2.22) imply (2.19). □

Lemma 2.1. *For any $R>0$ and any $t<0$, for sufficiently large λ,*

$$\mu^\lambda_t\left(\Sigma^\lambda_t\cap B_R(0)\right)\leq CR^2, \tag{2.23}$$

where $B_R(0)$ is a metric ball in $\mathbf{R}^4$ and $C>0$ is independent of λ.

Proof. We shall use C below for uniform positive constants which are independent of R and λ. Straightforward computation shows

$$
\begin{aligned}
\mu^\lambda_t(\Sigma^\lambda_t\cap B_R(0))&=\lambda^2\int_{\Sigma_{T+\lambda^{-2}t}\cap B_{\lambda^{-1}R}(X_0)}d\mu_t\\
&=R^2\left(\lambda^{-1}R\right)^{-2}\int_{\Sigma_{T+\lambda^{-2}t}\cap B_{\lambda^{-1}R}(X_0)}d\mu_t\\
&\leq CR^2\int_{\Sigma_{T+\lambda^{-2}t}\cap B_{\lambda^{-1}R}(X_0)}\frac{1}{4\pi(\lambda^{-1}R)^2}e^{-\frac{|X-X_0|^2}{4(\lambda^{-1}R)^2}}d\mu_t\\
&=CR^2\Phi\left(X_0,T+\left(\lambda^{-1}R\right)^2+\lambda^{-2}t,T+\lambda^{-2}t\right).
\end{aligned}
$$

By the monotonicity inequality (2.6), we have

$$\mu_t^\lambda(\Sigma_t^\lambda \cap B_R(0)) \leq CR^2\left(\Phi\left(X_0, T+(\lambda^{-1}R)^2+\lambda^{-2}t, \frac{T}{2}\right)+C\right)$$
$$\leq C\frac{R^2}{T}\left(\mu_{\frac{T}{2}}\left(\Sigma_{\frac{T}{2}}\right)+C\right).$$

Since

$$\frac{\partial}{\partial t}\mu_t(\Sigma_t) = -\int_{\Sigma_t}|\mathbf{H}|^2 d\mu_t,$$

we can now conclude (2.23)

$$\mu_t^\lambda(\Sigma_t^\lambda \cap B_R(0)) \leq CR^2. \qquad \square$$

Fixed $t_0<0$. By (2.23), for any $R>0$, we see that the total measure of $(\Sigma_{t_0}^\lambda \cap B_R(0), \mu_{t_0}^\lambda)$ is bounded from above by CR^2, the compactness theorem of the measures ([22], 4.4) implies that there is a subsequence $\lambda_i(R)\to\infty$ of λ such that,

$$\left(\Sigma_{t_0}^{\lambda_i(R)} \cap B_R(0), \mu_{t_0}^{\lambda_i(R)}\right) \to (\Sigma_{t_0}^\infty \cap B_R(0), \mu_{t_0}^\infty)$$

in the sense of measure. Using a diagonal subsequence argument, we conclude that, there is a subsequence $\lambda_k\to\infty$ such that $\left(\Sigma_{t_0}^{\lambda_k}, \mu_{t_0}^{\lambda_k}\right) \to (\Sigma_{t_0}^\infty, \mu_{t_0}^\infty)$ in the sense of measures.

We now show that, for any $t<0$, the subsequence λ_k which we have chosen above satisfies $\left(\Sigma_t^{\lambda_k}, \mu_t^{\lambda_k}\right) \to (\Sigma_{t_0}^\infty, \mu_{t_0}^\infty)$ in the sense of measure. And consequently the limiting surface $(\Sigma_{t_0}^\infty, \mu_{t_0}^\infty)$ is independent of t_0.

Lemma 2.2. *For any $t<0$, the sequence $\lambda_k\to\infty$ we chosen above satisfies that $\left(\Sigma_t^{\lambda_k}, \mu_t^{\lambda_k}\right) \to (\Sigma^\infty, \mu^\infty)$ in the sense of measure, where $(\Sigma^\infty, \mu^\infty)$ is independent of t. The multiplicity of Σ^∞ is finite.*

Proof. Recall that the following standard formula for mean curvature flows

$$\frac{d}{dt}\int_{\Sigma_t^\lambda}\phi d\mu_t^\lambda = -\int_{\Sigma_t^\lambda}\left(\phi|\mathbf{H}_\lambda|^2 + \nabla\phi\cdot\mathbf{H}_\lambda\right)d\mu_t^\lambda \tag{2.24}$$

is valid for any test function $\phi\in C_0^\infty(M)$ (cf. (1) in [3,20, Section 6] in the varifold setting).

Then for any given $t<0$ integrating (2.24) yields

$$\int_{\Sigma_t^{\lambda_k}}\phi d\mu_t^{\lambda_k} - \int_{\Sigma_{t_0}^{\lambda_k}}\phi d\mu_{t_0}^{\lambda_k} = \int_t^{t_0}\int_{\Sigma_t^{\lambda_k}}\left(\phi|\mathbf{H}_{\lambda_k}|^2 + \nabla\phi\cdot\mathbf{H}_{\lambda_k}\right)d\mu_t^{\lambda_k}dt$$
$$\to 0 \quad \text{as } k\to\infty \text{ by } (2.18). \tag{2.25}$$

So, for any fixed $t<0$, $\left(\Sigma_t^{\lambda_k}, \mu_t^{\lambda_k}\right) \to (\Sigma_{t_0}^\infty, \mu_{t_0}^\infty)$ in the sense of measures as $k\to\infty$. We denote $(\Sigma_{t_0}^\infty, \mu_{t_0}^\infty)$ by $(\Sigma^\infty, \mu^\infty)$, which is independent of t_0.

The inequality (2.23) yields a uniform upper bound on $R^{-2}\mu_t^{\lambda_k}\left(\Sigma_t^{\lambda_k}\cap B_R(0)\right)$, which yields finiteness of the multiplicity of Σ^∞. $\square$

Definition 2.1. Let (X_0,T) be a singular point of the mean curvature flow of a closed symplectic surface Σ_0 in a compact Kähler-Einstein surface M. We call $(\Sigma^\infty, d\mu^\infty)$ obtained in Lemma 2.2 *a tangent cone of the mean curvature flow* Σ_t *at* (X_0,T).

3 Flatness of the tangent cones

In this section, we prove that the tangent cones are flat. And for simplicity in notation, we write $\Sigma_t^{\lambda_k}$ as Σ_t^k.

A k-varifold is a Radon measure on $G^k(M)$, where $G^k(M)$ is the Grassmann bundle of all k-planes tangent to M. Allard's compactness theorem for rectifiable varifolds (6.4 in [1], also see 1.9 in [20] and [22, Theorem 42.7]) can be stated as follows.

Theorem 3.1 (Allard's compactness theorem). *Let* (V_i,μ_i) *be a sequence of rectifiable* k*-varifolds in* M *with*

$$\sup_{i\geq 1}(\mu_i(U)+|\delta V_i|(U))<\infty \quad \text{for each } U\subset\subset M.$$

Then there is a rectifiable varifold (V,μ) *of locally bounded first variation and a subsequence, which we also denote by* (V_i,μ_i)*, such that*

(i) Convergence of measures: $\mu_i\to\mu$ *as Radon measures on* M*,*

(ii) Convergence of tangent planes: $V_i\to V$ *as Radon measures on* $G^k(M)$*,*

(iii) Convergence of first variations: $\delta V_i\to\delta V$ *as* TM*-valued Radon measures,*

(iv) Lower semi-continuity of total first variations: $|\delta V|\leq \liminf_{i\to\infty}|\delta V_i|$ *as Radon measures.*

We first show that the tangent cone is rectifiable and stationary.

Proposition 3.1. *Let* M *be a compact Kähler-Einstein surface. If the initial compact surface is symplectic, then the tangent cone* Σ^∞ *is rectifiable and stationary.*

Proof. We set

$$A_R=\left\{t\in(-\infty,0)\,\middle|\,\liminf_{k\to\infty}\int_{\Sigma_t^k\cap B_R(0)}\left(|\mathbf{H}_k|^2+|\nabla\cos\alpha_k|^2+|\overline{\nabla}J_{\Sigma_t^k}|^2\right)d\mu_t^k\neq 0\right\},$$

and

$$A=\bigcup_{R>0}A_R.$$

Denote the measures of A_R and A by $|A_R|$ and $|A|$ respectively. It is clear from (2.16)-(2.18) that $|A_R|=0$ for any $R>0$. So $|A|=0$.

Choose $t\notin A$. Let V_t^k be the varifold defined by Σ_t^k. It is explained in the previous section, that V_t^k is well defined in $B_R(0)\subset\mathbf{R}^4$ for any $R>0$ when k sufficiently large. To apply the Allard's compactness theorem, we should use the Euclidean metric in $B_R(0)$.

However, it does not make difference between using the Euclidean metric and using the Riemannian metric $h_{\lambda,R}$ by (2.14) and (2.15). For simplicity in notation, we still use the Riemannian metric $h_{\lambda,R}$ in $B_R(0)$ in this section. By the definition of varifolds, we have

$$V_t^k(\psi)=\int_{\Sigma_t^k}\psi(x,T\Sigma_t^k)d\mu_t^k$$

for any $\psi\in C_0^0(G^2(\mathbf{R}^4),R)$, where $G^2(\mathbf{R}^4)$ is the Grassmannian bundle of all 2-planes tangent to Σ_t^∞ in $\mathbf{R}^4$. For each smooth surface Σ_t^k, the first variation δV_t^k of V_t^k ([1], (39.4) in [22] and (1.7) in [20]) is that, for any smooth vector field X with support in $B_R(0)$,

$$\delta V_t^k(X)=-\int_{\Sigma_t^k\cap B_R(0)}X\cdot\mathbf{H}_k d\mu_t^k,$$

so

$$|\delta V_t^k(X)|\le CR\|X\|_{L^\infty(B_R(0))}\Big(\int_{\Sigma_t^k\cap B_R(0)}|\mathbf{H}_k|^2d\mu_t^k\Big)^{\frac{1}{2}} \tag{3.1}$$

We therefore have that, for any $R>0$,

$$\mu_t^k(B_R(0))+|\delta V_t^k|(B_R(0))\le C(R). \tag{3.2}$$

By Allard's compactness theorem, there exists a subsequence which we also denote by (V_t^k,μ_t^k) such that $(V_t^k,\mu_t^k)\to(V_t^\infty,\mu_t^\infty)$ satisfying the four facts in Theorem 3.1 in $B_R(0)$. By a diagonal subsequence argument, there exists a subsequence which we also denote by (V_t^k,μ_t^k) such that $(V_t^k,\mu_t^k)\to(V_t^\infty,\mu_t^\infty)$ satisfying the four facts in Theorem 3.1 in $\mathbf{R^4}$.

Because $t\notin A$, by (3.1), we see that $\delta V_t^k\to 0$ at t as $k\to\infty$ and Σ^∞ is rectifiable by applying Theorem 3.1. Furthermore by (iii) in Theorem 3.1, we have that

$$-\mu^\infty\lfloor\mathbf{H}_\infty=\delta V^\infty=\lim_{k\to\infty}\delta V_t^k=0.$$

Therefore Σ^∞ is stationary. □

Now we can prove the Theorem 1.1. For the reader's convenience, we restate the theorem here.

Theorem 3.2. *Let M be a compact Kähler-Einstein surface. If the initial compact surface is symplectic and $T>0$ is the first blow-up time of the mean curvature flow, then the tangent cone Σ^∞ of the mean curvature flow at (X_0,T) is a finite union of planes.*

Proof. Without loss of any generality, we may assume $0\in\Sigma^\infty$ where 0 is the origin of $\mathbf{R}^4$. In fact, if not, Σ^∞ would move to infinity, then we would have

$$\Phi(F,X_0,T,T-\lambda_k^{-2}r^2)=\Phi(F_k,0,0,0-r^2)\to 0 \quad \text{as } k\to\infty.$$

But White's regularity theorem ([25]) then implies that (X_0,T) is a regular point. This is impossible.

There is a sequence of points $X_k\in\Sigma_t^k$ satisfying $X_k\to 0$ as $k\to\infty$. By Proposition 2.2, for any s_1 and s_2 with $-\infty<s_1<s_2<0$ and any $R>0$, we have

$$\int_{s_1}^{s_2}\int_{\Sigma_t^k\cap B_R(0)}\left|F_k^{\perp}\right|^2 d\mu_t^k dt\to 0 \quad \text{as } k\to\infty.$$

Thus, by (2.23)

$$\begin{aligned}&\lim_{k\to\infty}\int_{s_1}^{s_2}\int_{\Sigma_t^k\cap B_R(0)}\left|(F_k-X_k)^{\perp}\right|^2 d\mu_t^k dt\\ \leq&2\lim_{k\to\infty}\int_{s_1}^{s_2}\int_{\Sigma_t^k\cap B_R(0)}\left|F_k^{\perp}\right|^2 d\mu_t^k dt+C(s_2-s_1)R^2\lim_{k\to\infty}|X_k|^2=0.\end{aligned}$$

Let us denote the tangent spaces of Σ_t^k at the point $F_k(x,t)$ and of Σ^∞ at the point $F^\infty(x,t)$ by $T\Sigma_t^k$ and $T\Sigma^\infty$ respectively. It is clear that

$$|(F_k-X_k)^{\perp}|=\text{dist}\left(X_k,T\Sigma_t^k\right),$$

and

$$|(F_\infty)^{\perp}|=\text{dist }(0,T\Sigma^\infty).$$

By Allard's compactness theorem, i.e., Theorem 3.1 (ii), we have

$$\begin{aligned}\int_{s_1}^{s_2}\int_{\Sigma^\infty\cap B_R(0)}\left|(F_\infty)^{\perp}\right|^2 d\mu^\infty dt&=\int_{s_1}^{s_2}\int_{\Sigma^\infty\cap B_R(0)}|\text{dist }(0,T\Sigma^\infty)|^2 d\mu^\infty dt\\&=\lim_{k\to\infty}\int_{s_1}^{s_2}\int_{\Sigma_t^k\cap B_R(0)}\left|\text{dist }(X_k,T\Sigma_t^k)\right|^2 d\mu_t^k dt\\&=\lim_{k\to\infty}\int_{s_1}^{s_2}\int_{\Sigma_t^k\cap B_R(0)}\left|(F_k-X_k)^{\perp}\right|^2 d\mu_t^k dt=0.\end{aligned}$$

By [19, Theorem 1], we know that Σ^∞ is smooth outside a discrete set of points $\mathcal{S}$. So outside $\mathcal{S}$, we have

$$\langle F_\infty,v_\alpha\rangle=0.$$

Note that the above inner product is taken in $\mathbf{R}^4$, and differentiating in $\mathbf{R}^4$ then yields

$$0=\langle\partial_i F_\infty,v_\alpha\rangle+\langle F_\infty,\partial_i v_\alpha\rangle=\langle F_\infty,\partial_i v_\alpha\rangle.$$

Because $\partial_i F_\infty$ is tangential to Σ^∞, by Weingarten's equation we observe

$$(h_\infty)_{ij}^\alpha\langle F_\infty,e_j\rangle=0 \quad \text{for all } \alpha,\ i=1,2.$$

So for $\alpha=1,2$, we have

$$\det((h_\infty)_{ij}^\alpha)=0.$$

Since $\mathbf{H}=0$, for $\alpha=1,2$ we also have

$$tr((h_\infty)_{ij}^\alpha)=0.$$

It then follows immediately that the symmetric matrix $((h_\infty)_{ij}^\alpha)$ is in fact the zero matrix, for all $i,j,\alpha=1,2$, which obviously yields $|\mathbf{A}_\infty|\equiv 0$. □

4 Uniqueness of the Kähler angle of the tangent flow

In the last section, we proved that the tangent cone of the symplectic mean curvature flow at the first singular time consists of a finite union of holomorphic planes in $\mathbb{C}^2$. A priori, the Kähler angles on different planes may be different. We would like to examine when the finite planes have the same constant Kähler angle so that the tangent flow is a single holomorphic flat cone. Some ideas in [21] for Lagrangian MCF can be applied to the symplectic MCF with Type I* singularity. Now we can prove the Theorem 1.3. For the reader's convenience, we restate the theorem here.

Theorem 4.1. *Let M be a compact Kähler-Einstein surface. If the initial compact surface is symplectic and $T>0$ is the first blow-up time of the mean curvature flow such that the flow develops a Type I* singularity at (X_0,T), then the tangent cone Σ^∞ arising from connected components of the rescaling of the mean curvature flow at (X_0,T) is a cone consisiting of finite planes which is holomorphic with respect some complex structure on $\mathbb{C}^2$.*

Proof. By Theorem 3.2, it suffices to show that there exists a rescaled flow $(\Sigma_s^i)_{s<0}$, each of which is connected, such that the Kähler angles on each plane of the limit are the same.

Suppose the flow develops Type I* singularity at T. We define a sequence $\lambda_i\to\infty$ as follows: (i) If

$$\limsup_{t\to T}\max_{\Sigma_t}|\nabla\cos\alpha|^2<\infty,$$

then we choose λ_i to be any sequence converging to infinity. (ii) if

$$\limsup_{t\to T}\max_{\Sigma_t}|\nabla\cos\alpha|^2=\infty,$$

then we choose

$$\lambda_i:=|\nabla\cos\alpha|(x_i,t_i)=\max_{t\le t_i}\max_{\Sigma_t}|\nabla\cos\alpha|\to\infty.$$

for some (x_i,t_i). It is clear that $t_i\to T$ in this second case. We consider the recaled flow defined by

$$\Sigma_s^i=\lambda_i\Sigma_{T+\lambda_i^{-2}s},\quad -\lambda_i^2T<s<0.$$

By Theorem 3.2, after passing to a subsequence, Σ_s^i converges weakly to a finite union of flat planes so that each of which is holomorphic with respect to some orthogonal complex struture in $\mathbb{C}^2$.

On the other hand, the choice of λ_i implies that

$$|\nabla \cos\alpha_{i,s}|^2 \leq 1$$

for $s \in [-\lambda_i^2 T, \lambda_i^2 (t_i - T)]$. The assumption that the singularity is of Type I* implies that

$$\lambda_i^2 (t_i - T) = -(T - t_i) \max_{\Sigma_{t_i}} |\nabla \cos\alpha|^2 \geq -\Lambda.$$

Combining with Proposition 2.2, we may choose $a < -\Lambda - 1$ such that

$$\lim_{i\to\infty} \int_{\Sigma_a^i \cap B_R(0)} \left(|F_i^\perp|^2 + |\mathbf{H}_i|^2 + |\nabla \cos\alpha_{i,a}|^2 \right) d\mu_s^i = 0. \tag{4.1}$$

and

$$|\nabla \cos\alpha_{i,a}|^2 \leq 1. \tag{4.2}$$

Notice that the condition that $\cos\alpha \geq \delta$ for some $\delta > 0$ holds on each slice of the rescaled flow, which implies the isoperimetric inequality holds on the sequence of surfaces.

Lemma 4.1. *Suppose* Σ *is a sympletic surface in a Kähler-Einstein surface* M *with* $\cos\alpha \geq \delta > 0$*, then there is a constant* D_1 *depending only on* δ *such that*

$$\left(\mathcal{H}^2(A)\right)^{\frac{1}{2}} \leq D_1 \mathcal{H}^1(\partial A) \tag{4.3}$$

where A *is any open subset of* Σ *with rectifiable boundary.*

Proof. The Isoperimetric Theorem ([22]) guarantees the existence of an integral current B with compact support such that $\partial B = \partial A$ and for which

$$\left(\mathcal{H}^2(B)\right)^{\frac{1}{2}} \leq C \mathcal{H}^1(\partial A)$$

for some absolute constant C. If T denotes the cone over the current $A - B$, then $\partial T = A - B$. Since the Kähler form ω is closed, by the definition of Kähler angle and the assumption, we have

$$\mathcal{H}^2(A) = \int_A \frac{1}{\cos\alpha}\omega \leq \frac{1}{\delta}\int_A \omega = \frac{1}{\delta}\left(\int_B \omega + \int_T d\omega\right) = \frac{1}{\delta}\int_B \omega \leq \frac{1}{\delta}\mathcal{H}^2(B) \leq \frac{1}{\delta}\left(C\mathcal{H}^1(\partial A)\right)^2.$$

This finishes the proof of the lemma. □

We will denote by Σ^i connected components of $B_{4R}(0) \cap \Sigma_a^i$ intersecting $B_R(0)$. By the assumption, we may assume that Σ^i converges weakly to an integer rectifiable stationary varifold Σ. To finish the proof of the theorem, it suffices to show that there is a real number $\overline{\alpha}$ such that, after passing to a subsequence, we have

$$\lim_{i\to\infty} \int_{\Sigma^i} f(\cos\alpha_i) \phi d\mu^i = m f(\cos\overline{\alpha}) \mu(\phi),$$

for every f in $C(\mathbb{R})$ and every smooth ϕ compactly supported in $B_{2R}(0)$, where μ and m denote the Radon measure of the support of Σ and its multiplicity, respectively.

For this purpose, we will find $\bar{\alpha}\in(0,\frac{\pi}{2}]$ and a sequence (ε_j) converging to zero such that, for some appropriate subsequence, we have for all $j\in\mathbb{N}$

$$\lim_{i\to\infty}\mathcal{H}^2(\{|\cos\alpha_i-\cos\bar{\alpha}|\le\varepsilon_j\}\cap B_R(0))=\mathcal{H}^2(\Sigma\cap B_R(0)). \tag{4.4}$$

Choose any sequence $(\mathbf{x_i})$ in $\Sigma^i\cap B_R(0)$. After passing to a subsequence, we have that

$$\lim_{i\to\infty}\mathbf{x}_i=\mathbf{x}_0,\quad\text{and}\quad\lim_{i\to\infty}\cos\alpha_i(x_i)=\cos\bar{\alpha},$$

for some $\mathbf{x}_0\in\overline{B_R(0)}$ and $\cos\bar{\alpha}\in[\delta,1]$. Notice that on Σ^i, we have by Lemma 2.1 that

$$\mathcal{H}^2(\Sigma^i\cap B_r(0))\le D_0r^2,\quad\forall\, r\in[0,4R), \tag{4.5}$$

and by the scaling-invariance of the Kähler angle that

$$\inf_{\Sigma^i}\cos\alpha_i\ge\delta>0.$$

In particular, the above isoperimetric inequality applies to each Σ^i, and (4.1)-(4.2) also hold with Σ^i_a replaced by Σ^i. Coarea formula implies that

$$\begin{aligned}&\lim_{i\to\infty}\int_0^1\mathcal{H}^1(\{\cos\alpha_i=s\}\cap B_{3R}(0))ds\\=&\lim_{i\to\infty}\int_{\Sigma^i\cap B_{3R}(0)}|\nabla\cos\alpha_i|d\mu^i\le\lim_{i\to\infty}\sqrt{9D_0R^2}\left(\int_{\Sigma^i\cap B_{3R}(0)}|\nabla\cos\alpha_i|^2d\mu^i\right)^{\frac{1}{2}}=0,\end{aligned}$$

which implies that we can choose a sequence (ε_j) converging to zero such that, for all $j\in\mathbb{N}$

$$\lim_{i\to\infty}\mathcal{H}^1(\{\cos\alpha_i=\cos\bar{\alpha}\pm\varepsilon_j\}\cap B_{3R}(0))=0,$$

Define

$$\Sigma^{i,\bar{\alpha},j}\equiv\{|\cos\alpha_i-\cos\bar{\alpha}|\le\varepsilon_j\}.$$

The first variation formula yields for any vector field $\mathbf{Y}$ supported in $B_{2R}(0)$

$$\delta\Sigma^{i,\bar{\alpha},j}(\mathbf{Y})=-\int_{\Sigma^{i,\bar{\alpha},j}\cap B_{2R}(0)}\langle\mathbf{H},\mathbf{Y}\rangle d\mu^i+\oint_{\partial\Sigma^{i,\bar{\alpha},j}\cap B_{2R}(0)}\langle\mathbf{Y},\nu\rangle d\mathcal{H}^1,$$

where ν denotes the exterior unit normal. Hence, whenever the sup norm of $\mathbf{Y}$ satisfies $|\mathbf{Y}|_\infty\le1$, we get

$$|\delta\Sigma^{i,\bar{\alpha},j}(\mathbf{Y})|\le\sqrt{9D_0R^2}\left(\int_{\Sigma^{i,\bar{\alpha},j}\cap B_{2R}(0)}|\mathbf{H}|^2d\mu^i\right)^{\frac{1}{2}}+\mathcal{H}^1(\{\cos\alpha_i=\cos\bar{\alpha}\pm\varepsilon_j\}\cap B_{2R}(0))\to0.$$

Applying Allard compactness theorem and the standard diagonalization argument, we can find a subsequence which converges to an integer rectifiable stationary varifold $\Sigma^{\overline{\alpha},j}$ for every positive integer j.

The next lemma will be used in order to proceed further.

Lemma 4.2. *For all $j\in\mathbb{N}$, and $R\geq\varepsilon_j$, we have*

$$\mathcal{H}^2(\Sigma^{\overline{\alpha},j}\cap B_R(\mathbf{x_0}))\geq\frac{1}{16D_1^2}R^2,$$

where D_1 is the constant in the isoperimetric inequality in Lemma 4.1.

Proof. Set

$$\psi_i(s)\equiv\mathcal{H}^2(\{|\cos\alpha_i-\cos\alpha_i(\mathbf{x}_i)|\leq s\}\cap B_s(\mathbf{x}_i)).$$

Applying Sard's theorem to $\cos\alpha_i$ and $|\mathbf{x}-\mathbf{x}_i|$ and using the coarea formula, we have

$$\psi_i'(s)=\oint_{\partial B_s(x_i)\cap\{|\cos\alpha_i-\cos\alpha_i(\mathbf{x}_i)|\leq s\}}\frac{|\mathbf{x}-\mathbf{x}_i|}{|(\mathbf{x}-\mathbf{x}_i)^T|}d\mathcal{H}^1+\oint_{B_s(\mathbf{x}_i)\cap\partial\{|\cos\alpha_i-\cos\alpha_i(\mathbf{x}_i)|\leq s\}}\frac{1}{|\nabla\cos\alpha_i|}d\mathcal{H}^1$$

for almost all s. By (4.2), we compute

$$\begin{aligned}\psi_i'(s)\geq&\mathcal{H}^1(\partial B_s(\mathbf{x}_i)\cap\{|\cos\alpha_i-\cos\alpha_i(\mathbf{x}_i)|\leq s\})\\&+\mathcal{H}^1(B_s(x_i)\cap\partial\{|\cos\alpha_i-\cos\alpha_i(\mathbf{x}_i)|\leq s\})\\\geq&\mathcal{H}^1(\partial(B_s(\mathbf{x}_i)\cap\{|\cos\alpha_i-\cos\alpha_i(\mathbf{x}_i)|\leq s\})).\end{aligned}$$

The isoperimetric inequality (Lemma 4.1) implies that

$$(\psi_i(s))^{\frac{1}{2}}\leq D_1\mathcal{H}^1(\partial(B_s(\mathbf{x}_i)\cap\{|\cos\alpha_i-\cos\alpha_i(\mathbf{x}_i)|\leq s\}))\leq D_1\psi_i'(s),$$

for almost all $s\leq R$. Solving this differential ienquality yields

$$\mathcal{H}^2(\{|\cos\alpha_i-\cos\alpha_i(\mathbf{x}_i)|\leq s\}\cap B_s(\mathbf{x}_i))=\psi_i(s)\geq\frac{s^2}{4D_1^2}$$

for all $s\leq R$. Here we used the fact that $\psi_i(s)>0$ for $s>0$. On the other hand, we have

$$\left\{|\cos\alpha_i-\cos\alpha_i(\mathbf{x}_i)|\leq\frac{\varepsilon_j}{2}\right\}\cap B_{\frac{\varepsilon_j}{2}}(\mathbf{x}_i)\subset\{|\cos\alpha_i-\cos\overline{\alpha}|\leq\varepsilon_j\}\cap B_{\varepsilon_j}(\mathbf{x}_0)$$

for all i sufficiently large. Hence we have

$$\varepsilon_j^{-2}\mathcal{H}^2(\Sigma^{i,\overline{\alpha},j}\cap B_{\varepsilon_j}(\mathbf{x}_0))\geq\varepsilon_j^{-2}\mathcal{H}^2\left(\left\{|\cos\alpha_i-\cos\alpha_i(\mathbf{x}_i)|\leq\frac{\varepsilon_j}{2}\right\}\cap B_{\frac{\varepsilon_j}{2}}(\mathbf{x}_i)\right)\geq\frac{1}{16D_1^2},$$

for all i sufficiently large. Taking the limit when i goes to infinity and recalling that $\Sigma^{\overline{\alpha},j}$ is a stationary varifold we get, by the monotonicity formula, that for $R \geq \varepsilon_j$

$$R^{-2}\mathcal{H}^2(\Sigma^{\overline{\alpha},j} \cap B_R(\mathbf{x}_0)) \geq \varepsilon_j^{-2}\mathcal{H}^2(\Sigma^{\overline{\alpha},j} \cap B_{\varepsilon_j}(\mathbf{x}_0)) \geq \frac{1}{16D_1^2}.$$

This proves the lemma. □

Now, we can finish the proof of (4.4). We prove it by contradiction. Assume that the conclusion is false, then for some integer j we have

$$\lim_{i\to\infty}\mathcal{H}^2(\{|\cos\alpha_i - \cos\overline{\alpha}| \leq \varepsilon_j\} \cap B_R(0)) < \mathcal{H}^2(\Sigma \cap B_R(0)).$$

Repeating the same arguments, we can find $\mathbf{y_0}$ in $B_R(0)$ and a closed interval I disjoint from $[\cos\overline{\alpha} - \varepsilon_j, \cos\overline{\alpha} + \varepsilon_j]$ so that, after passing to a subsequence,

$$\lim_{i\to\infty}\mathcal{H}^2(\cos\alpha_i^{-1}(I) \cap B_R(\mathbf{y_0})) \geq \frac{1}{16D_1^2}R^2,$$

Given any positive integer p, pick disjoint closed intervals

$$I_1, \cdots, I_p$$

lying between I and $[\cos\overline{\alpha} - \varepsilon_j, \cos\overline{\alpha} - \varepsilon_j]$. The connectedness of

$$B_{2R}(0) \cap \Sigma_i^a$$

implies that all $\cos\alpha_i^{-1}(I_l) \cap B_{2R}(0)$ are nonempty for i sufficiently large. Hence, arguing as before, we find $\mathbf{y}_1, \cdots, \mathbf{y}_p$ in $B_{2R}(0)$ such that, after passing to a subsequence,

$$\lim_{i\to\infty}\mathcal{H}^2\big(\cos\alpha_i^{-1}(I_l) \cap B_R(\mathbf{y}_l)\big) \geq \frac{1}{16D_1^2}R^2,$$

for all l in $\{1, \cdots, p\}$. This implies that

$$\lim_{i\to\infty}\mathcal{H}^2(\Sigma^i \cap B_{2R}(0)) \geq \lim_{i\to\infty}\sum_{l=1}^{p}\mathcal{H}^2\big(\cos\alpha_i^{-1}(I_l) \cap B_R(\mathbf{y}_l)\big) \geq \frac{p}{16D_1^2}R^2.$$

Choosing p large enough, we obtain the desired contradiction with (4.5). Thus (4.4) holds and the proof of the theorem is completed. □

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