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Pluriclosed Flow on Oeljeklaus-Toma Manifolds

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The authors warmly dedicate this article to Professor Gang Tian on the occasion of his 65th birthday.

Abstract. We establish global existence of the pluriclosed flow with arbitrary initial data on Oeljeklaus-Toma manifolds, and Gromov-Hausdorff convergence of blowdown limits to a torus under natural conjectural bounds on the flow at infinity. In the case of generalized Kähler-Ricci flow we prove refined a priori estimates in support of these conjectural bounds.

AMS subject classifications: 53E30

Key words: Pluriclosed flow, Oeljeklaus-Toma manifolds, Long-time existence, Parabolic estimate.

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1 Introduction

In recent years the pluriclosed flow [27, 29] and generalized Kähler-Ricci flow [3, 25, 28] have been developed as a tool for understanding the geometry of complex, especially non-Kähler, manifolds [5–7, 10–12, 32]. A natural class of non-Kähler manifolds are the Oeljeklaus-Toma (OT) manifolds [22], whose geometry is linked to the structure of number fields, and which are natural higher dimensional generalizations of Inoue surfaces [18]. In [11] a complete description of the pluriclosed flow with left-invariant initial data on OT manifolds was obtained, in particular showing that the solution exists for all time and collapses after blowdown to a torus in the Gromov-Hausdorff sense. Moreover the blowdown on the universal cover converges in the Cheeger-Gromov sense to a soliton. It is natural to conjecture that these statements hold for arbitrary initial data (Conjecture 3.1). In this work we confirm some aspects of this conjecture.

The first main result is to establish the global existence of the flow:

Theorem 1.1. *Fix (M^{2n}, J) an Oeljeklaus-Toma manifold and g_0 a pluriclosed metric on M. The solution to pluriclosed flow with initial data g_0 exists on $[0,\infty)$.*

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The proof exploits natural class of background metrics arising from the homogeneous structure on OT-manifolds. In the case of Inoue surfaces these are known as Tricerri metrics [30]. In the next theorem we give some refined estimates in the special case of generalized Kähler-Ricci flow (GKRF) [28]. Oeljeklaus-Toma manifolds admit natural classes of generalized Kähler structures with split tangent bundle, and for such metrics the GKRF reduces to a scalar parabolic flow of Monge-Ampère type [25].

Theorem 1.2. *For generalized Kähler-Ricci flow on an Oeljeklaus-Toma manifold*

1. *The scalar potential ϕ satisfies*

$$-C \le \phi \le Ce^{-t}(1+t).$$

2. *Assuming there exists a Tricerri-type metric in $[\omega_0]$, we have*

$$-Ce^{-t}(1+t) \le \phi \le Ce^{-t}(1+t).$$

3. *On Inoue surfaces of type S_M, the estimate of item (2) holds. In addition:*

$$-C \le \dot{\phi} \le C.$$

4. *On Inoue surfaces of type S_M, we have:*

$$\omega(t) \ge C\omega_h(t),$$

where $\omega_h(t)$ is the model flow with initial data the Tricerri metric h.

2 Geometry of Oeljeklaus-Toma manifolds

2.1 Definition

In this section, we recall the family of compact non-Kähler complex manifolds constructed by Oeljeklaus-Toma [22]. First, we need some facts from algebraic number theory. Let K be an algebraic number field, i.e., $K \simeq \mathbb{Q}[x]/(f)$, where $f \in \mathbb{Q}[x]$ is a monic irreducible polynomial of degree $s+2t = [K:\mathbb{Q}]$, where

$$s = \#\text{ real roots}, \quad 2t = \#\text{ complex roots},$$

which for arbitrary given s and t exists by ([22] Remark 1.1).

Now, consider the embedding, $K \hookrightarrow \mathbb{Q}$, given by the roots of f: let $a_1, \cdots, a_s \in \mathbb{R}$ be the real roots of f, and let $a_{s+t+1} = \bar{a_{s+1}}, \cdots, a_{s+2t} = \bar{a_{s+t}} \in \mathbb{C}$ be the complex roots of f. Define the embedding:

$$\sigma_i : K \to \mathbb{C}, \quad x \mapsto a_i.$$

Thus, $\sigma_1,\cdots,\sigma_s:K\to\mathbb{R}$, and $\sigma_{s+1},\cdots,\sigma_{s+2t}:K\to\mathbb{C}$, where $\sigma_{s+i}=\bar{\sigma}_{s+t+i}$. There exist $n:=s+2t$ different embeddings.

Let $\mathcal{O}_K\subset K$ be the ring of algebraic integers. Note $\mathcal{O}_K$ is a finitely generated free abelian group of rank $n=s+2t$, i.e., $\mathcal{O}_k\simeq\mathbb{Z}^n$. (The rank of $\mathcal{O}_k$ as a free $\mathbb{Z}$-module is $[K:\mathbb{Q}]=s+2t$). Moreover, let $\mathcal{O}_K^*\subseteq\mathcal{O}_K$ be the multiplicative group of units of $\mathcal{O}_K$. By the Dirichlet unit theorem, for $s\geq 1$ we have $\mathcal{O}_K^*\cong\{\pm1\}\times\mathbb{Z}^{t+s-1}$. We define

$$\mathcal{O}_K^{*,+}:=\{a\in\mathcal{O}_K^*\mid\sigma_i(a)>0,\ 1\leq i\leq s\}.$$

By definition, $\mathcal{O}_K^{*,+}$ is a finite index subgroup of $\mathcal{O}_K^*$. Now, let $\mathbb{H}=\{z\in\mathbb{C}:\mathrm{Im}z>0\}\}$. We define two kinds of actions on $\mathbb{H}^s\times\mathbb{C}^t$:

$$\begin{aligned}&T:\mathcal{O}_K\to\mathrm{Aut}(\mathbb{H}^s\times\mathbb{C}^t)\\&T(a)=[(w_1,\cdots,w_s,z_{s+1},\cdots,z_{s+t})\mapsto(w_1+\sigma_1(a),\cdots,z_{s+t}+\sigma_{s+t}(a))],\end{aligned}$$

and

$$\begin{aligned}&R:\mathcal{O}_K^{*,+}\to\mathrm{Aut}(\mathbb{H}^s\times\mathbb{C}^t)\\&R(u)=[(w_1,\cdots,w_s,z_{s+1},\cdots,z_{s+t})\mapsto(w_1\cdot\sigma_1(u),\cdots,z_{s+t}\cdot\sigma_{s+t}(u))].\end{aligned}$$

Let

$$\sigma:K\to\mathbb{C}^{s+t},\quad a\mapsto(\sigma_1(a),\cdots,\sigma_{s+t}(a)).$$

Note $\sigma(\mathcal{O}_K)$ is a lattice of rank $s+2t$ in $\mathbb{C}^{s+t}$. Thus, we have $\mathrm{rank}(T(\mathcal{O}_K))=s+2t$. Note $R(u)\sigma(\mathcal{O}_K)=\sigma(\mathcal{O}_K)$, by the property of the algebraic integer ring structure. Then:

$$\mathcal{O}_K^{*,+}\ltimes\mathcal{O}_K$$

acts on $\mathbb{H}^s\times\mathbb{C}^t$. By [22] there exists a subgroup $U\leq\mathcal{O}_K^{*,+}$, $\mathrm{rank}(U)=s$, such that $U\ltimes\mathcal{O}_K$ is cocompact and acts properly discontinuously on $\mathbb{H}^s\times\mathbb{C}^t$. The compact quotient:

$$X(K,U)=\mathbb{H}^s\times\frac{\mathbb{C}^t}{U\ltimes\mathcal{O}_K}$$

is called an *Oeljeklaus-Toma manifold* of type (s,t). Note:

$$\mathbb{H}^s\times\frac{\mathbb{C}^t}{\sigma(\mathcal{O}_K)}\cong(\mathbb{R}_{>0})^s\times(S^1)^{2t+s}$$

and U acts on $(\mathbb{R}_{>0})^s$ preserves the fiber by the property of the algebraic integer structure. Thus, we may regard $X(K,U)$ as a torus $\mathbb{T}^t$ bundle over $\mathbb{T}^s$.

When $s=t=1$, it is the famous *Inoue surface* of type S_M [18]. The original S_M is constructed in a more direct way: consider $M\in\mathrm{SL}(3,\mathbb{Z})$, $M=(m_{ik})$. Denote the eigenvalues:

$\lambda,\mu,\bar{\mu}$ with the relation: $\lambda\cdot|\mu|^2=1$. In addition, we require $\mu\neq\bar{\mu}$ and $\lambda>1$. The corresponding unit eigenvectors: (a_1,a_2,a_3), (b_1,b_2,b_3) and $(\bar{b_1},\bar{b_2},\bar{b_3})$. Define the actions:

$$g_0:(z,w)\mapsto(\mu z,\lambda w)$$
$$g_i:(z,w)\mapsto(z+b_i,w+a_i)$$

for $i=1,2,3$. Note that

$$\left\{\begin{pmatrix}a_i\\b_i\end{pmatrix}\right\}$$

are linearly independent and satisfy:

$$\begin{pmatrix}\lambda a_i\\ \mu b_i\end{pmatrix}=\sum_{k=1}^{3}m_{ik}\begin{pmatrix}a_k\\b_k\end{pmatrix}.$$

Then, the quotient $M:=\mathbb{C}\times\mathbb{H}/\langle g_0,g_1,g_2,g_3\rangle$ is a compact complex surface, which is the desired Inoue surface S_M. The relation between two constructions can be seen in this way: choose f as the characteristic polynomial of $SL(3,\mathbb{Z})$. Note that when $s=t=1$, then $\mathcal{O}_K^{*,+}\simeq\mathbb{Z}$. Choose a generator $u\in\mathcal{O}_K^{*,+}$, by Dirichlet unit theorem, $\sigma_1(u)\cdot|\sigma_2(u)|^2=1$. Then, any choice of the basis of $\mathcal{O}_K\simeq\mathbb{Z}^3\subset\mathbb{R}\times\mathbb{C}$ will correspond to an element of $SL(3,\mathbb{Z})$.

2.2 Pluriclosed and generalized Kähler structures on Oeljeklaus-Toma manifolds

Consider the pluriclosed metrics on OT manifolds. Note that not all OT manifolds admit pluriclosed metrics. In fact:

Theorem 2.1 ([2,23]). *Let $X(K,U)$ be an OT manifold of type (s,t). It admits pluriclosed metrics if and only if $s=t$ and, for any $u\in U$,*

$$\sigma_j(u)|\sigma_{s+j}(u)|^2=1,\quad \textit{for any } i\in\{1,\cdots,s\}.$$

Note that for Inoue surfaces S_M, this is automatically true. From now on, we restrict ourselves to the case where $s=t$, i.e. the universal cover of the OT manifolds is $\mathbb{H}^s\times\mathbb{C}^s$. Some pluriclosed metrics can be constructed explicitly on OT manifolds of type (s,s):

$$\omega_h=\sum_{i=1}^{s}\sqrt{-1}\frac{dw_i\wedge d\bar{w}_i}{(\mathrm{Im}w_i)^2}+\sqrt{-1}\mathrm{Im}w_i dz_i\wedge d\bar{z}_i. \tag{2.1}$$

It is easy to verify $\partial\bar{\partial}\omega_h=0$, hence it is pluriclosed. Moreover, this metric is invariant under the group action $U\ltimes\mathcal{O}_K$, and therefore descends to the quotient. When $s=1$, the metric ω_h is the pluriclosed metric discovered by Tricerri [30] on the Inoue Surface.

More generally, for any given sequence $a=\{a_1,\cdots,a_s\}$, and $b=\{b_1,\cdots,b_s\}$, where each a_i, $b_i\in\mathbb{R}^+$. We can define the following metric:

$$\omega_h^{a,b}=\sum_{i=1}^{s}\sqrt{-1}a_i\frac{dw_i\wedge d\bar{w}_i}{(\mathrm{Im}w_i)^2}+\sqrt{-1}b_i\mathrm{Im}w_i dz_i\wedge d\bar{z}_i. \tag{2.2}$$

Note, the family of metrics $\omega_h^{a,b}$ are pluriclosed as well. Thus, we have a family of pluriclosed metrics on $X(K,U)$. We shall write ω_h for short if a_i, $b_i=1$. We record some curvature facts of these model metrics, the proof of which is a standard computation.

Proposition 2.1. *The metric $\omega_h^{a,b}$ satisfies:*

1. *The only nonvanishing components of the Chern curvature are*

$$\Omega^{Ch}_{w_i\bar{w}_iw_i\bar{w}_i}=-\frac{a_i}{2(\mathrm{Im}w_i)^4},\quad \Omega^{Ch}_{w_i\bar{w}_iz_i\bar{z}_i}=\frac{b_i}{4(\mathrm{Im}w_i)}.$$

2. *The Bismut Ricci form satisfies*

$$\rho_B(\omega_h)=-\frac{3}{4(\mathrm{Im}w_i)^2}dw_i\wedge d\bar{w}_i.$$

2.3 Generalized Kähler structures on Oeljeklaus-Toma manifolds

A *generalized Kähler structure (GK structure)* [14, 15] on a manifold M is a triple (g,I,J), where g is a Riemannian metric compatible with two integrable complex structures I and J, furthermore satisfying

$$d_I^c\omega_I=-d_J^c\omega_J,\quad dd_I^c\omega_I=-dd_J^c\omega_J=0.$$

Hitchin [17] shows that a GK structure has an associated Poisson tensor

$$\sigma=\tfrac{1}{2}g^{-1}[I,J].$$

We are interested here in the simplest case of GK manifolds, namely when σ vanishes. This is equivalent to

$$[I,J]=0,$$

and then we refer to the GK structure as *commuting type*. See [4] for further background on such structures. We define

$$\Pi:=IJ\in\mathrm{End}(TM)$$

where $\Pi^2=\mathrm{Id}$. Then, we can decompose TM with respect to the eigenvalue ±1:

$$TM=T_+M\oplus T_-M.$$

Now, we introduce the GK-structure of commuting type on OT manifolds. On the universal cover, $\mathbb{H}^s\times\mathbb{C}^s$. Consider the invariant complex structures on $\mathbb{H}^s$ and $\mathbb{C}^t$, $J_{\mathbb{H}^s}$ and $J_{\mathbb{C}^s}$. Let:

$$I=\begin{pmatrix} J_{\mathbb{H}^s} & \\ & J_{\mathbb{C}^s}\end{pmatrix},\qquad J=\begin{pmatrix} J_{\mathbb{H}^s} & \\ & -J_{\mathbb{C}^s}\end{pmatrix}.$$

These are invariant complex structures on $\mathbb{H}^s\times\mathbb{C}^s$, hence pass to the quotient. Moreover, $[I,J]=0$. Defining

$$\Pi=IJ=-\mathrm{Id}_{\mathbb{H}}\oplus\mathrm{Id}_{\mathbb{C}},\quad E_{\mathbb{C}}=T\mathbb{C}^s,\quad E_{\mathbb{H}}=T\mathbb{H}^s.$$

With the associated splitting as above, we see:

$$TM=E_{\mathbb{H}}\oplus E_{\mathbb{C}},\quad T_-M=E_{\mathbb{H}},\quad T_+M=E_{\mathbb{C}}.$$

In our case, since we have a specific split of tangent bundle, we will use the precise component notation ($\mathbb{C}$ or $\mathbb{H}$) instead of the standard notation $(+/-)$. Throughout this paper, we use coordinates $(w,z):=(w_1,\cdots,w_s,z_1,\cdots,z_s)$ on $\mathbb{H}^s\times\mathbb{C}^s$, with respect to the complex structure I. Note that we obtain differential operators

$$d=d_{\mathbb{C}}+d_{\mathbb{H}},\quad d_{\mathbb{C}}=\pi_{\mathbb{C}}\circ d=\partial_z+\partial_{\bar z},\quad d_{\mathbb{H}}=\pi_{\mathbb{H}}\circ d=\partial_w+\partial_{\bar w},$$

where $\pi_{\mathbb{C}}$ and $\pi_{\mathbb{H}}$ are the projections of $E^*_{\mathbb{C}}$ and $E^*_{\mathbb{H}}$, respectively.

Proposition 2.2. *The triple $(h^{a,b},I,J)$ where $h^{a,b}$ is the metric determined by* (2.2)*, is generalized Kähler of commuting type.*

Proof. We will give the proof for the standard metric h in (2.1), with the general case being analogous. First, note the pluriclosed metric h is compatible with both complex structures I and J, and $E_{\mathbb{C}}$ and $E_{\mathbb{H}}$ are h-orthogonal. Second, we let $\omega_I=h(\cdot,I\cdot)$, and $\omega_J=h(\cdot,J\cdot)$. Let (w,z) be the complex coordinate with respect to I. Then:

$$\begin{aligned}\omega_I=\omega_h&=\sum_{i=1}^s\sqrt{-1}\frac{dw_i\wedge d\bar w_i}{(\mathrm{Im}w_i)^2}+\sqrt{-1}\mathrm{Im}w_i dz_i\wedge d\bar z_i,\\ \omega_J&=\sum_{i=1}^s\sqrt{-1}\frac{dw_i\wedge d\bar w_i}{(\mathrm{Im}w_i)^2}-\sqrt{-1}\mathrm{Im}w_i dz_i\wedge d\bar z_i,\end{aligned}$$

and

$$\begin{aligned}d^c_I\omega_I&=\sqrt{-1}(\partial_{\bar w}+\partial_{\bar z}-\partial_w-\partial_z)\omega_I=-(\partial_{\bar w}-\partial_w)(\mathrm{Im}w_i dz_i\wedge d\bar z_i),\\ d^c_J\omega_J&=\sqrt{-1}(\partial_{\bar w}+\partial_z-\partial_w-\partial_{\bar z})\omega_J=(\partial_{\bar w}-\partial_w)(\mathrm{Im}w_i dz_i\wedge d\bar z_i).\end{aligned}$$

Thus, $d^c_I\omega_I=-d^c_J\omega_J$. By direct computation, $dd^c_I\omega_I=-dd^c_J\omega_J=0$. Hence, (h,I,J) is a GK structure. □

We observe that the metric satisfies the leafwise Kähler conditions $d_{\mathbb{C}}\left(\omega_I|_{E_{\mathbb{C}}}\right)=0$ and $d_{\mathbb{H}}\left(\omega_I|_{E_{\mathbb{H}}}\right)=0$, in line with the general theory of commuting-type GK structures developed by Apostolov-Gualtieri [4].

3 Pluriclosed flow on Oeljeklaus-Toma manifolds

3.1 Pluriclosed flow and its normalized flow

The *pluriclosed flow* is the evolution equation [27]:

$$\frac{\partial}{\partial t}\hat{\omega}(t) = -\rho_B^{1,1}(\hat{\omega}(t)), \tag{3.1}$$

where ρ_B is the Bismut-Ricci form. Motivated by the Type III behavior of the Ricci flow, we let the time parameter $s = e^t - 1$ and consider the following blow-down family of metrics:

$$\omega(t) = \frac{\hat{\omega}(s)}{s+1},$$

where $\omega(s)$ satisfies the pluriclosed flow (3.1). It follows that $\widetilde{\omega}(t)$ satisfies

$$\frac{\partial}{\partial t}\omega(t) = -\rho_B^{1,1}(\omega(t)) - \omega(t), \tag{3.2}$$

which we call the *normalized pluriclosed flow*.

3.2 Model solutions

The expected qualitative behavior of the normalized pluriclosed flow on OT manifolds is captured by solutions with initial data the model metrics $\omega_h^{a,b}$. Straightforward computations show that the normalized pluriclosed flow with this initial data is

$$\omega_h^{a,b}(t) = \sum_{i=1}^{s} \sqrt{-1}\left((1-e^{-t})\tfrac{3}{4} + e^{-t}a_i\right)\frac{1}{(\mathrm{Im}w_i)^2} dw_i \wedge d\bar{w}_i + \sqrt{-1}e^{-t} b_i \mathrm{Im}w_i dz_i \wedge d\bar{z}_i. \tag{3.3}$$

Later, we shall denote the time-dependent normalized model metric starting with $\omega_h^{a,b}$ as $\omega_h^{a,b}(t)$. We shall write $\omega_h(t)$ for short if $a_i,\ b_i = 1$, for all $1 \le i \le s$. Observe that for these model solutions the Chern torsion T is uniformly bounded in time, i.e., $|T(t)| \le C$. Moreover, it follows that

$$\omega_h^{a,b}(t) \to \sum_{i=1}^{s} \frac{3}{4(\mathrm{Im}w_i)^2} dw_i \wedge d\bar{w}_i,$$

which can be considered as a degenerate metric on $X(K,U)$. As explained in [11], the blowdown manifolds converge in Gromov-Hausdorff sense to a torus $\mathbb{T}^s$ with a canonical flat metric $d(K,U)$ depending only on the algebraic field K and the rank s subgroup U. We recall the result here, noting that OT manifolds are compact solvmanifolds and the model metrics are left-invariant:

Theorem 3.1 ([11]). *Let ω_0 be a left-invariant pluriclosed metric on an OT manifold M, then the normalized pluriclosed flow starting with ω_0 converges to $(\mathbb{T}^s,d(K,U))$ in the Gromov-Hausdorff sense as $t\to\infty$.*

It is reasonable to conjecture that this behavior holds in the general case:

Conjecture 3.1. *Let $M=X(K,U)$ be an OT manifold of type (s,s), then for any pluriclosed metric ω_0, the normalized pluriclosed flow* (3.2) *with initial metric ω_0 exists on $[0,\infty)$, and converges to $(\mathbb{T}^s,d(K,U))$ in the Gromov-Hausdorff sense.*

We shall prove the first part of the conjecture in Section 3.3. For the second half of the conjecture, we have the following sufficiency condition.

Proposition 3.1. *Let $\omega(t)$ be the normalized pluriclosed flow solution on the OT manifold $X(K,U)$. Suppose that, for all $t\geq 0$, there exists a constant $C>0$ such that*

$$C^{-1}\leq \mathrm{tr}_{\omega(t)}\omega_h(t)\leq C, \qquad \lim_{t\to\infty}\mathrm{tr}_{\omega(t)}\omega_h(t)=1 \quad \text{for all } \ x\in X(K,U).$$

Then, we have

$$(X(K,U),\omega(t))\to(\mathbb{T}^s,d(K,U)),$$

to the flat metric $d(K,U)$ in the Gromov-Hausdorff sense.

Proof. We give a brief sketch as the result is already essentially contained in e.g., [11,33]. Note that for OT manifolds, there is a canonical fibration map:

$$F:X(K,U)\to\mathbb{T}^s$$

where the fiber is diffeomorphic to $\mathbb{T}^{3s}$. In particular, $E_{\mathbb{C}}$ will be in the kernel of dF. Since the quotient of $\{z\}\times\mathbb{C}^s$ is dense in the $\mathbb{T}^{3s}$ fiber ([31, Section 2]), the degenerate metric $\sum_{i=1}^s\frac{3}{4(\mathrm{Im}w_i)^2}dw_i\wedge d\bar{w}_i$ will induce a metric on the base $\mathbb{T}^s$. Let $d(K,U)$ be the metric on $\mathbb{T}^s$ induced from $\sum_{i=1}^s\frac{3}{4(\mathrm{Im}w_i)^2}dw_i\wedge d\bar{w}_i$, which is flat.

Now we choose $G:\mathbb{T}^s\to X(K,U)$ to be any map so that $F\circ G=\mathrm{Id}$, and show that for T large, F and G are $\epsilon(T)$-Gromov-Hausdorff approximations. By our assumption, the limit of $\omega(t)$ exists and is equal to $\sum_{i=1}^s\frac{3}{4(\mathrm{Im}w_i)^2}dw_i\wedge d\bar{w}_i$. Since the quotient of $\{z\}\times\mathbb{C}^s$ is dense in the $\mathbb{T}^{3s}$ fiber, the distance between two points in $F^{-1}(x)$ will converge to 0 and the distance of any two representatives of two fibers will approximate to the distance $d(K,U)$, from which the Gromov-Hausdorff convergence follows. □

Thus, to prove the second half of the conjecture, it is enough to prove that $\mathrm{tr}_{\omega(t)}\omega_h(t)$ is uniformly bounded from above and below and converges to 1. In geometric flow theory, to obtain the long-time behavior, a Type III curvature estimate of the Riemannian curvature is usually needed. Then the collapsing behavior can be shown using the monotonicity formula and the Cheeger-Fukaya-Gromov collapsing theory (e.g., [20,21]).

In our case, using the complex structure, $\mathrm{tr}_{\omega(t)}\omega_h(t)$ will satisfy a strictly parabolic equation (Section 3.3), and thus it is possible to estimate it directly. The Type III estimate for (3.1) starting with an arbitrary initial pluriclosed metric on the OT manifolds remains an interesting open question, even in the Inoue surface case ($s=1$). See a recent study of the long-time behavior of the a related curvature flow following this route in [16].

3.3 Long-time existence of pluriclosed flow

Proof of Theorem 1.1. Let

$$M=X(K,U)=\mathbb{H}^s\times\frac{\mathbb{C}^s}{U\ltimes\mathcal{O}_K}$$

be an OT manifold of type (s,s). Since the universal cover $\widetilde{M}\cong\mathbb{H}^s\times\mathbb{C}^s$ is a product manifold, and the action $U\ltimes\mathcal{O}_K$ acts on $\widetilde{M}$ diagonally, the tangent bundle $TM=E_{\mathbb{H}}\oplus E_{\mathbb{C}}$, splits as a direct sum of two subbundles assocaited with two projection maps, $\mathrm{pr}_{\mathbb{H}}:TM\to E_{\mathbb{H}}$ and $\mathrm{pr}_{\mathbb{C}}:TM\to E_{\mathbb{C}}$. Now, given any Riemannian metric g on M, denote $g_{\mathbb{H}}=g\circ\mathrm{pr}_{\mathbb{H}}$, the induced metric on $E_{\mathbb{H}}$. Similarly, $g_{\mathbb{C}}=g\circ\mathrm{pr}_{\mathbb{C}}$. Note that the model metric (2.1) splits with respect to the splitting of TM, i.e., $h=h_{\mathbb{H}}\oplus h_{\mathbb{C}}$. Thus, the trace $\mathrm{tr}_h g=\mathrm{tr}_{h_{\mathbb{C}}\oplus h_{\mathbb{H}}}g=\mathrm{tr}_{h_{\mathbb{C}}}g_{\mathbb{C}}+\mathrm{tr}_{h_{\mathbb{H}}}g_{\mathbb{H}}$. We can decompose $\mathrm{tr}_g h$ in a similar way. We let $\Upsilon(g,h)=\nabla_g-\nabla_h$, the difference of the Chern connection; Ω^g is the Chern curvature of metric g and Δ is the Chern Laplacian. Denote $\{w,z\}$ as the standard coordinate on $\mathbb{H}^s\times\mathbb{C}^s$. All other letters in the formulas represent general coordinates. Denote:

$$Q_{i\bar{j}}=g^{\bar{l}k}g^{\bar{n}m}T_{ik\bar{n}}T_{\bar{j}\bar{l}m}.$$

Lemma 3.1. *Let $\Delta=\sqrt{-1}\mathrm{tr}_{\omega(t)}\partial\bar{\partial}$ to be the Chern Laplacian. Given the setup above,*

$$\left(\frac{\partial}{\partial t}-\Delta\right)\mathrm{tr}_{g_{\mathbb{H}}}h_{\mathbb{H}}=-g^{r\bar{s}}g^{u\bar{q}}h_{w_i\bar{w}_j}\Upsilon^{w_i}_{ru}\Upsilon^{\bar{w}_j}_{\bar{s}\bar{q}}+g^{r\bar{s}}g^{w_i\bar{w}_j}\Omega^h_{r\bar{s}w_i\bar{w}_j}-g^{w_i\bar{q}}g^{p\bar{w}_j}h_{w_i\bar{w}_j}Q_{p\bar{q}},\tag{3.4}$$

$$\left(\frac{\partial}{\partial t}-\Delta\right)\mathrm{tr}_{h_{\mathbb{H}}}g_{\mathbb{H}}=-g^{r\bar{s}}h^{w_i\bar{w}_j}\Upsilon^{p}_{rw_i}\Upsilon^{\bar{q}}_{\bar{s}\bar{w}_j}g_{p\bar{q}}+h^{w_i\bar{w}_j}Q_{w_i\bar{w}_j}-g^{r\bar{s}}g_{w_i\bar{w}_j}h^{w_i\bar{q}}h^{p\bar{w}_j}\Omega^h_{r\bar{s}p\bar{q}},\tag{3.5}$$

$$\left(\frac{\partial}{\partial t}-\Delta\right)\mathrm{tr}_{g_{\mathbb{C}}}h_{\mathbb{C}}=-g^{r\bar{s}}g^{u\bar{q}}h_{z_i\bar{z}_j}\Upsilon^{z_i}_{ru}\Upsilon^{\bar{z}_j}_{\bar{s}\bar{q}}+g^{r\bar{s}}g^{z_i\bar{z}_j}\Omega^h_{r\bar{s}z_i\bar{z}_j}-g^{z_i\bar{q}}g^{p\bar{z}_j}h_{z_i\bar{z}_j}Q_{p\bar{q}},\tag{3.6}$$

$$\left(\frac{\partial}{\partial t}-\Delta\right)\mathrm{tr}_{h_{\mathbb{C}}}g_{\mathbb{C}}=-g^{r\bar{s}}h^{z_i\bar{z}_j}\Upsilon^{p}_{rz_i}\Upsilon^{\bar{q}}_{\bar{s}\bar{z}_j}g_{p\bar{q}}+h^{z_i\bar{z}_j}Q_{z_i\bar{z}_j}-g^{r\bar{s}}g_{z_i\bar{z}_j}h^{z_i\bar{q}}h^{p\bar{z}_j}\Omega^h_{r\bar{s}p\bar{q}}.\tag{3.7}$$

Proof of Lemma 3.1. The proof is almost identical to [25, Lemma 4.3], except that the metric g does not necessarily split with respect to $E_{\mathbb{H}}$. Here we sketch the computation of $\mathrm{tr}_{g_{\mathbb{H}}}h_{\mathbb{H}}$. Note, $\rho_B^{1,1}=S-Q$, where $S=\mathrm{tr}_\omega\Omega^\omega$. Thus, using the local formula of the Chern curvature:

$$\frac{\partial}{\partial t}\mathrm{tr}_{g_{\mathbb{H}}}h_{\mathbb{H}}=g^{w_i\bar{q}}g^{p\bar{w}_j}h_{w_i\bar{w}_j}(-g_{p\bar{q},r\bar{s}}+g^{u\bar{v}}g_{p\bar{v},r}g_{u\bar{q},\bar{s}})g^{r\bar{s}}-g^{w_i\bar{q}}g^{p\bar{w}_j}h_{w_i\bar{w}_j}Q_{p\bar{q}}.$$

Also, the Chern Laplacian acts on $\mathrm{tr}_{g_{\mathbb{H}}} h_{\mathbb{H}}$ by a standard formula:

$$\begin{aligned}\Delta \mathrm{tr}_{g_{\mathbb{H}}} h_{\mathbb{H}} =& g^{r\bar{s}}\big(g^{w_i\bar{v}}g^{u\bar{q}}g^{p\bar{w}_j}g_{u\bar{v},r}g_{p\bar{q},\bar{s}}h_{w_i\bar{w}_j}+g^{w_i\bar{q}}g^{p\bar{v}}g^{u\bar{w}_j}g_{u\bar{v},r}g_{p\bar{q},\bar{s}}h_{w_i\bar{w}_j}-g^{w_i\bar{q}}g^{p\bar{w}_j}g_{p\bar{q},r\bar{s}}h_{w_i\bar{w}_j}\\ &-g^{w_i\bar{q}}g^{p\bar{w}_j}g_{p\bar{q},\bar{s}}h_{w_i\bar{w}_j,r}-g^{w_i\bar{q}}g^{p\bar{w}_j}g_{p\bar{q},r}h_{w_i\bar{w}_j,\bar{s}}+g^{w_i\bar{w}_j}h_{w_i\bar{w}_j,r\bar{s}}\big).\end{aligned}$$

Combining these two formulas, with the cancellation of the first order terms and the completing of squares, the result follows. The other results are analogous. □

To show Theorem 1.1, we need to show that, for any time interval $[0,T]$, with $T>0$, the metrics $g(t)$ is $C(T)$-equivalent to h, i.e.,

$$C(T)^{-1}h\le g(t)\le C(T)h$$

for any $t\in[0,T]$. In other words, we need to show time dependent estimates of $\mathrm{tr}_g h$ and $\mathrm{tr}_h g$. To achieve this, we need to use the Chern curvature of the model metric, more precisely, the fact that the Chern curvature on $E_{\mathbb{H}}$ is negative. In particular, using Proposition 2.1 and (3.4) we have

$$\left(\frac{\partial}{\partial t}-\Delta\right)\mathrm{tr}_{g_{\mathbb{H}}} h_{\mathbb{H}}\le -\frac{1}{2}(\mathrm{tr}_{g_{\mathbb{H}}} h_{\mathbb{H}})^2.$$

By the maximum principle: $\mathrm{tr}_{g_{\mathbb{H}}} h_{\mathbb{H}}\le \frac{1}{C_0+\frac{1}{2}t}$, where C_0 only depends on the initial metric $g(0)$. Now, for the full trace, using Lemma 3.1 and the above estimate we have

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\Delta\right)\mathrm{tr}_g h &\le \sum_{i=1}^{s} g^{z_i\bar{z}_i}g^{w_i\bar{w}_i}\Omega^h_{w_i\bar{w}_i z_i\bar{z}_i}\\ &\le \frac{1}{4}(\mathrm{tr}_{g_{\mathbb{C}}} h_{\mathbb{C}})(\mathrm{tr}_{g_{\mathbb{H}}} h_{\mathbb{H}})\\ &\le \frac{1}{4C_0+2t}\mathrm{tr}_g h.\end{aligned}$$

By the maximum principle again,

$$\mathrm{tr}_g h\le C_0'(4C_0+2t)^{\frac{1}{2}},$$

where C_0' again only depends on $g(0)$. Thus, we have a time-dependent lower bound:

$$g(t)\ge C_1(T)h.$$

Now, for every fixed $T>0$, we can apply [24, Theorem 1.8], we have a time dependent upper bound, $C_2(T)$. Hence:

$$C_1(T)h\le g(t)\le C_2(T)h.$$

Using the higher regularity of uniformly parabolic solutions of pluriclosed flow [12, 19, 24], the long-time existence follows. □

4 Generalized Kähler-Ricci flow on Oeljeklaus-Toma manifolds

In this section we consider the more restricted setting of GKRF on OT manifolds. As explained in Section 2, OT manifolds come equipped with generalized Kähler metrics of commuting type, and the pluriclosed flow starting with such a metric will preserve this structure [28], leading to new estimates.

4.1 Cohomology constraint and scalar reduction

The key point established in [25] is that after accounting for a certain cohomological constraint, the flow (3.2) can be reduced to a scalar twisted Monge-Ampère type equation. We recall some aspects of this cohomology group ([24] Definition 7.3 and 7.4, [13]).

Definition 4.1 ([24]). *Let (M^{2n},g,I,J) be a generalized Kähler manifold of commuting type. Let*

$$\mathcal{H}(M)=\frac{\{\omega\in\Lambda^{1,1}_{I,\mathbb{R}}|\sqrt{-1}\partial\bar{\partial}\omega=0\}}{\{\sqrt{-1}(\partial_+\bar{\partial}_+-\partial_-\bar{\partial}_-)u|u\in C^\infty(M)\}}.$$

Note $\omega\in[\omega_0]$, if and only if there exists a smooth function $f\in C^\infty(M)$ such that

$$\omega=\omega_0+\sqrt{-1}(\partial_+\bar{\partial}_+-\partial_-\bar{\partial}_-)f. \tag{4.1}$$

Firstly, the dimension of the cohomology group $\mathcal{H}(M)$ (Definition 4.1) on the Inoue surface S_M is known.

Theorem 4.1 ([8]). *For a complex surface with split tangent bundle, M, then*

$$\mathcal{H}(S_M)=\Big\{[\omega_h^{a,b}]\mid a,b\in\mathbb{R}\Big\}.$$

In particular, $\dim\mathcal{H}(S_M)=2$.

For more general OT manifolds we prove a conjecturally sharp lower bound on the dimension of this space by exhibiting a space spanned by the model metrics.

Proposition 4.1. *For positive constants, a_i, b_i, and a_i', b_i', such that there exists i with either $a_i\neq a_i'$ or $b_i\neq b_i'$, then*

$$\big[\omega_h^{a,b}\big]\neq\big[\omega_h^{a',b'}\big].$$

Proof. It suffices to show that for constants a_i and b_i such that a_i, $b_i\in\mathbb{R}$, if there exists $u\in C^\infty(X(K,U))$ such that on the universal cover $\mathbb{H}^s\times\mathbb{C}^s$,

$$(\partial_z\partial_{\bar{z}}-\partial_w\partial_{\bar{w}})u=\sum_{i=1}^s a_i\frac{dw_i\wedge d\bar{w}_i}{(\mathrm{Im}w_i)^2}+b_i\mathrm{Im}w_i dz_i\wedge d\bar{z}_i,$$

then $a_i = b_i = 0$. Tracing the above equation with $\omega_J^{a,b}$ yields

$$\Delta^{Ch}_{\omega_J^{a,b}} u = \text{const}$$

on $X(K,U)$. Since the Chern Laplacian differs from the Riemannian Laplacian by a strictly first-order term (the action of the Lee vector field), it follows from the maximum principle that $u \equiv C$, hence $a_i = b_0 = 0$ as required. □

The relevance of the above cohomology group stems from its relationship to the Bismut Ricci form for commuting-type generalized Kähler metrics. In particular from [13, Proposition 8.20], and specializing to the case of OT manifolds here, for such metrics $\rho_B^{1,1}$ can be decomposed:

$$\rho_B^{1,1}(\omega) = \rho_{\mathbb{C}}(\omega_{\mathbb{C}}) - \rho_{\mathbb{H}}(\omega_{\mathbb{C}}) - \rho_{\mathbb{C}}(\omega_{\mathbb{H}}) + \rho_{\mathbb{H}}(\omega_{\mathbb{H}}). \tag{4.2}$$

Moreover this leads to the following transgression formula for commuting-type GK metrics:

$$\rho_B^{1,1}(\omega_1) - \rho_B^{1,1}(\omega_2) = -\sqrt{-1}(\partial_z \partial_{\bar{z}} - \partial_w \partial_{\bar{w}}) \log \frac{\det g^1_{\mathbb{C}} \cdot \det g^2_{\mathbb{H}}}{\det g^1_{\mathbb{H}} \cdot \det g^2_{\mathbb{C}}}. \tag{4.3}$$

Definition 4.2 (Model flow)**.** *To simplify notation we set $P = \rho_B^{1,1}(h)$, where h is the model metric* (2.1)*. Let $\widetilde{\omega}(t) = e^{-t}\widetilde{\omega}_0 - (1-e^{-t})P$, where $\widetilde{\omega}_0$ is a representative in $[\omega(0)]$, and $P := \rho_B^{1,1}(\omega_h) = \sum_{i=1}^{s} -\frac{3\sqrt{-1}}{4} h_{w_i \bar{w}_j} dw_i \wedge d\bar{w}_j$, the Bismut curvature of the model metric* (2.1)*.*

Analogous to the Kähler Ricci flow, we can reduce the GKRF to a parabolic fully nonlinear twisted Monge-Ampère equation.

Lemma 4.1. *Suppose $\omega(t) = \widetilde{\omega}(t) + \sqrt{-1}(\partial_z \partial_{\bar{z}} - \partial_w \partial_{\bar{w}})\phi$, where ϕ satisfies*

$$\frac{\partial}{\partial t}\phi(t) + \phi(t) = \log \frac{\det g_{\mathbb{C}}}{\det h_{\mathbb{C}}} - \log \frac{\det g_{\mathbb{H}}}{\det h_{\mathbb{H}}} + c(t), \tag{4.4}$$

where $c(t)$ is some time-dependent constant. Then, ω_t solves (3.2)*.*

Proof. Use the transgression formula (4.3), we have:

$$\begin{aligned}
\frac{\partial}{\partial t}\omega(t) &= -e^{-t}\omega_0 - e^{-t}P + P - P + \sqrt{-1}(\partial_z \partial_{\bar{z}} - \partial_w \partial_{\bar{w}}) \frac{\partial}{\partial t}\phi(t) \\
&= -\omega(t) + \sqrt{-1}(\partial_z \partial_{\bar{z}} - \partial_w \partial_{\bar{w}})(\frac{\partial}{\partial t}\phi + \phi) - \rho_B^{1,1}(\omega_h) \\
&= -\omega(t) - \rho_B^{1,1}(\omega(t)).
\end{aligned}$$

□

4.2 A priori estimates

Using the scalar reduction of Lemma 4.1 we prove new a priori estimates. First, by a specific choice of $c(t)$ we obtain a C^0 estimate for the potential.

Lemma 4.2. *If we choose* $c(t)=st+s\log\frac{3}{4}-\log c_1$, *where* $c_1 \geq \sup\frac{\det g^0_{\mathbb{C}}}{\det h_{\mathbb{C}}}$, *then:*

$$-C_0 \leq \phi(t) \leq C_0 e^{-t}(1+Ct), \quad \textit{for some } \; C_0, C>0.$$

Proof. Recall:

$$\omega(t)=\widetilde{\omega}(t)+\sqrt{-1}(\partial_z\partial_{\bar{z}}-\partial_w\partial_{\bar{w}})\phi$$

From our choice, we have $\omega(0)=\omega_0$ and $\phi(0)=0$. At the maximum point, with the choice of $c(t)$, we have:

$$\begin{aligned}\frac{\partial}{\partial t}(\max\phi(t)) &\leq \log\frac{\det(\widetilde{g}_{\mathbb{C}})}{c_1(\det e^{-t}h_{\mathbb{C}})}-\log\frac{\det\widetilde{g}_{\mathbb{H}}}{\det\frac{3}{4}h_{\mathbb{H}}}-\max\phi(t)\\ &\leq \log\frac{\det e^{-t}g^0_{\mathbb{C}}}{c_1(\det e^{-t}h_{\mathbb{C}})}-\log\frac{\det(e^{-t}g^0_{\mathbb{H}}+\frac{3}{4}(1-e^{-t})h_{\mathbb{H}})}{\det\frac{3}{4}h_{\mathbb{H}}}-\max\phi(t).\end{aligned}$$

To estimate the first two terms, observe

$$e^t\log\frac{\det\frac{3}{4}h_{\mathbb{C}}}{\det((C_1e^{-t}+\frac{3}{4}(1-e^{-t}))h_{\mathbb{C}}}\leq e^t s\log\frac{1}{1+C_1e^{-t}}\leq C_2.$$

By the maximum principle, since we choose $c_1\geq\sup\frac{\det g^0_{\mathbb{C}}}{\det h_{\mathbb{C}}}$, then $\phi(x,t)\leq f(t)$, where f satisfies:

$$\frac{\partial}{\partial t}(e^t f(t))=C_2.$$

Thus, we have $\phi(t)\leq C_0e^{-t}(1+Ct)$, for some C_0, $C>0$. At a minimum point, note that $\det g_{\mathbb{C}^0}/c_1\det h_{\mathbb{C}}\geq c_0$, for some $c_0>0$, we have:

$$\frac{\partial}{\partial t}(e^t\min\phi(t))\geq e^t s(\log\frac{C_3}{1+C_1e^{-t}})\geq -C_4e^t.$$

Thus, the lower bound is obtained similarly. □

Remark 4.1. We note that for the Chern Ricci flow on Inoue surfaces [9] it is possible to get decaying upper and lower bounds directly via the maximum principle. It is unclear how to directly achieve this for the twisted parabolic Monge-Ampère equation here, a manifestation of the mixed convex/concave structure.

If we assume that the initial metric is in $[\omega_h^{a,b}]$ (ref: (2.2)), we have a refined C^0 estimate.

Corollary 4.1. *If there exist a^i, b^i such that $\omega_0 \in [\omega_h^{a,b}]$, then we can choose $c(t) = st + s\log\frac{3}{4}$ so that:*

$$-C_0 e^{-t}(1+Ct) \le \phi(t) \le C_0 e^{-t}(1+Ct), \quad \text{for some } C_0, C > 0.$$

Proof. We can choose the initial representative $\omega_0 = \omega_h^{a,b}$. In this case, $\phi(0)$ is a function satisfies $\omega(0) = \omega_0 + \sqrt{-1}(\partial_z\partial_{\bar z} - \partial_w\partial_{\bar w})\phi(0)$. Also, note that $\rho^{1,1}(\omega_h^{a,b}) = P$. Thus, we can choose the representative cohomological background metric h to be $h^{a,b}$ as well. The rest of the proof is the same as Lemma 4.2. Note that in this case, we do not need to choose the normalization constant c_1 since the initial background metric is exactly the model metric $\omega_h^{a,b}$. □

Remark 4.2. The choice of $c(t)$ can be seen in a more geometric way. If we denote $h(t)$ to be the normalized model flow of the model flow in Section 3.2, then we see that $P = \rho^{1,1}(\omega_h(t))$, which is a constant. With the choice of $c(t) = st + s\log\frac{3}{4}$. Thus, we can express:

$$\frac{\partial}{\partial t}\phi(t) + \phi(t) = \log\frac{\det g_{\mathbb{C}}(t)}{\det h_{\mathbb{C}}(t)} - \log\frac{\det g_{\mathbb{H}}(t)}{\det h_{\mathbb{H}}(t)},$$

which is measured with respect to the moving model metric $h(t)$. And the choice of $\log c_1$ is a normalizing constant to match the initial condition. Since the Gromov-Hausdorff limit, at least from all the existing results, is independent of the initial metric. Then the potential with respect to the moving model metric should converge to 0 uniformly. Thus, the choice of $c(t)$ is not surprising, and if ω_0 is in the cohomology class of ω_h, then we will achieve better decaying C^0 estimate.

Also, we need the estimate for $\dot\phi$, the time derivative of the potential. To derive the equation, several lemmas are needed.

Lemma 4.3. *For the metrics $g_{\mathbb{H}}$ and $g_{\mathbb{C}}$ on bundles $E_{\mathbb{H}}$ and $E_{\mathbb{C}}$ respectively, we have the following evolution equation:*

$$(\partial_t - \Delta)\log\frac{\det g_{\mathbb{H}}}{\det h_{\mathbb{H}}} = \frac{1}{2}|T|^2 + \frac{1}{2}\mathrm{tr}_{g_{\mathbb{H}}} h_{\mathbb{H}} - s, \tag{4.5}$$

$$(\partial_t - \Delta)\log\frac{\det g_{\mathbb{C}}}{\det h_{\mathbb{C}}} = \frac{1}{2}|T|^2 - \frac{1}{4}\mathrm{tr}_{g_{\mathbb{H}}} h_{\mathbb{H}} - s. \tag{4.6}$$

Proof. By the computation of Proposition 2.1, for metrics on ($E_{\mathbb{H}}$ and $E_{\mathbb{C}}$), we have:

$$\rho(h_{\mathbb{H}}) = \sum_{i=1}^{s} \frac{1}{4}\frac{1}{(\mathrm{Im}\, w_i)^2} dw_i \wedge d\bar w_i$$

$$\rho(h_{\mathbb{C}}) = \sum_{i=1}^{s} -\frac{1}{2}\frac{1}{(\mathrm{Im}\, w_i)^2} dw_i \wedge d\bar w_i$$

Thus, by computing the trace, we have $\mathrm{tr}_g\rho_{\mathbb{H}} = \frac{1}{4}\mathrm{tr}_{g_{\mathbb{H}}} h_{\mathbb{H}}$ and $\mathrm{tr}_g\rho_{\mathbb{C}} = -\frac{1}{2}\mathrm{tr}_{g_{\mathbb{H}}} h_{\mathbb{H}}$. By using [25, Lemma 4.1], it is done. □

A direct corollary of the above evolution equations is the a priori lower bound for the trace of the metric on $E_{\mathbb{H}}$:

Corollary 4.2. *There exists a constant C, depends on the initial metric, g_0, such that, on $E_{\mathbb{H}}$:*

$$\mathrm{tr}_{h_{\mathbb{H}}} g_{\mathbb{H}} \geq C.$$

Proof. On $E_{\mathbb{H}}$, by the AM-GM inequality:

$$\frac{\mathrm{tr}_{g_{\mathbb{H}}} h_{\mathbb{H}}}{s} \geq \left(\frac{\det h_{\mathbb{H}}}{\det g_{\mathbb{H}}}\right)^{\frac{1}{s}}.$$

Thus, for (4.5), we have:

$$(\partial_t - \Delta)\log\frac{\det g_{\mathbb{H}}}{\det h_{\mathbb{H}}} \geq \frac{1}{2}\left(\frac{\det h_{\mathbb{H}}}{\det g_{\mathbb{H}}}\right)^{\frac{1}{s}} s - s.$$

By the ODE comparison, we have:

$$\mathrm{tr}_{h_{\mathbb{H}}} g_{\mathbb{H}} \geq s\left(\frac{\det g_{\mathbb{H}}}{\det h_{\mathbb{H}}}\right)^{\frac{1}{s}} \geq \left(\frac{1}{2} + C_0 e^{-t}\right)^{s} \geq C.$$ □

Now, we look at equation of $\dot{\phi}$:

Lemma 4.4. *For the time derivative of the potential, $\dot{\phi}$, we have:*

$$\left(\frac{\partial}{\partial t} - \Delta\right)(\dot{\phi} + \phi) = -\frac{3}{4}\mathrm{tr}_{g_{\mathbb{H}}} h_{\mathbb{H}} + s.$$

Proof. Using (4.4) and Lemma 4.3, we have:

$$\left(\frac{\partial}{\partial t} - \Delta\right)(\dot{\phi} + \phi) = -\frac{1}{4}\mathrm{tr}_{g_{\mathbb{H}}} h_{\mathbb{H}} - \frac{1}{2}\mathrm{tr}_{g_{\mathbb{H}}} h_{\mathbb{H}} + \frac{\partial}{\partial t}c(t) = -\frac{3}{4}\mathrm{tr}_{g_{\mathbb{H}}} h_{\mathbb{H}} + s.$$ □

Now, we derive the upper bound estimate for $\dot{\phi}$:

Lemma 4.5. *There exists a constant $C > 0$, such that:*

$$\dot{\phi} \leq C.$$

Proof. Note that, from (4.1), the Chern Laplacian on potential ϕ can be purely expressed as the geometric quantity:

$$\begin{aligned}\Delta\phi &= \mathrm{tr}_{g_{\mathbb{C}}}\partial_z\partial_{\bar{z}}\phi + \mathrm{tr}_{g_{\mathbb{H}}}\partial_w\partial_{\bar{w}}\phi = \mathrm{tr}_{g_{\mathbb{C}}}(g_{\mathbb{C}} - \widetilde{g}_{\mathbb{C}}) + \mathrm{tr}_{g_{\mathbb{H}}}(\widetilde{g}_{\mathbb{H}} - g_{\mathbb{H}}) \\ &= -e^{-t}\mathrm{tr}_{g_{\mathbb{C}}} g^0_{\mathbb{C}} + e^{-t}\mathrm{tr}_{g_{\mathbb{H}}} g^0_{\mathbb{H}} + (1 - e^{-t})\frac{3}{4}\mathrm{tr}_{g_{\mathbb{H}}} h_{\mathbb{H}}.\end{aligned}$$

Consider the quantity

$$\dot{\phi}-(C_1-1)\phi=\dot{\phi}+\phi-C_1\phi,$$

for some constant $C_1>1$, which will be chosen later. Then:

$$\begin{aligned}&\Big(\frac{\partial}{\partial t}-\Delta\Big)(\dot{\phi}-(C_1-1)\phi)\\ =&-\frac{3}{4}\mathrm{tr}_{g_{\mathbb{H}}}h_{\mathbb{H}}+s-C_1\phi-C_1e^{-t}\mathrm{tr}_{g_{\mathbb{C}}}g^0_{\mathbb{C}}+C_1e^{-t}\mathrm{tr}_{g_{\mathbb{H}}}g^0_{\mathbb{H}}+C_1(1-e^{-t})\frac{3}{4}\mathrm{tr}_{g_{\mathbb{H}}}h_{\mathbb{H}}\\ =&s-C_1\dot{\phi}-C_1e^{-t}\mathrm{tr}_{g_{\mathbb{C}}}g^0_{\mathbb{C}}+C_1e^{-t}\mathrm{tr}_{g_{\mathbb{H}}}g^0_{\mathbb{H}}+(C_1(1-e^{-t})-1)\frac{3}{4}\mathrm{tr}_{g_{\mathbb{H}}}h_{\mathbb{H}}.\end{aligned}$$

If for $A>0$, a very large constant, such that $\dot{\phi}-(C_1-1)\phi>A$. Then, by Lemma 4.2,

$$\dot{\phi}\geq(C_1-1)C_0+A=A'$$

for some large constant $A'>0$. By the lower bound of the metric on $E_{\mathbb{H}}$ (Corollary 4.2), we have:

$$\Big(\frac{\partial}{\partial t}-\Delta\Big)(\dot{\phi}-(C_1-1)\phi)\leq s+C_1C_0-C_1\dot{\phi},$$

for some constant $C_0=C_0(s,g(0))>0$. Thus, for A large, by the maximum principle:

$$\dot{\phi}-(C_1-1)\phi\leq A.$$

By Lemma 4.2 again, done. □

4.3 Results on the Inoue surface

When s=1, then the OT manifolds $X(K,U)$ become the well-known Inoue surfaces of type S_M. On these surfaces, several estimates can be strengthened. First note that by Theorem 4.1, for any GK initial metric ω_0, there will be constants a and b such that $\omega_0\in[\omega_h^{a,b}]$. Thus, we always have the improved C^0 estimate for ϕ from Corollary 4.1.

Corollary 4.3. *Let ω_0 be any GK metrics on the Inoue surface S_M. Then, if we choose $c(t)=t+\log(\frac{3}{4})$, then:*

$$-C_0e^{-t}(1+Ct)\leq\phi(t)\leq C_0e^{-t}(1+Ct),\quad \textit{for some}\ \ C_0,C>0.$$

Secondly, when $s=1$, then $E_{\mathbb{C}}$ and $E_{\mathbb{H}}$ are holomorphic line bundles. Thus, the AM-GM inequality becomes equality in this case, and we can have the following lower bound of $\dot{\phi}$.

Lemma 4.6. *On Inoue surface S_M, for the potential ϕ, there exists a constant $C>0$, such that:*

$$\dot{\phi}\geq -C.$$

Proof. Note $s=1$. Choose a large constant $\Lambda>0$, consider the quantity $\dot{\phi}+\phi+\frac{1}{\Lambda}\phi$. Then

$$\begin{aligned}&\Big(\frac{\partial}{\partial t}-\Delta\Big)\Big(\dot{\phi}+\phi+\frac{1}{\Lambda}\phi\Big)\\ =&1+\frac{1}{\Lambda}\dot{\phi}+\frac{1}{\Lambda}e^{-t}\mathrm{tr}_{g_{\mathbb{C}}}g^0_{\mathbb{C}}-\frac{1}{\Lambda}e^{-t}\mathrm{tr}_{g_{\mathbb{H}}}g^0_{\mathbb{H}}-\Big(\frac{1}{\Lambda}(1-e^{-t})+1\Big)\frac{3}{4}\mathrm{tr}_{g_{\mathbb{H}}}h_{\mathbb{H}}.\end{aligned}$$

Now, for $A>0$, a very large constant, such that $\dot{\phi}+\phi+\frac{1}{\Lambda}\phi<-A$. By (4.4), notice that in this case $s=1$, we know:

$$\dot{\phi}+\phi=\log\frac{\det g_{\mathbb{C}}}{c_1(\det e^{-t}h_{\mathbb{C}})}-\log\frac{\det g_{\mathbb{H}}}{\det\frac{3}{4}h_{\mathbb{H}}}=-\log(c_1e^{-t}\mathrm{tr}_{g_{\mathbb{C}}}h_{\mathbb{C}})+\log\Big(\frac{3}{4}\mathrm{tr}_{g_{\mathbb{H}}}h_{\mathbb{H}}\Big).$$

By Corollary 4.3, in particular $|\phi|\le C_0$. Then, there is a large constant $A'=A-C_0>0$ such that $\dot{\phi}\le -A'$, and

$$\mathrm{tr}_{g_{\mathbb{H}}}h_{\mathbb{H}}\le\frac{4}{3}e^{\frac{C_0}{\Lambda}-A}e^{-t}c_1\mathrm{tr}_{g_{\mathbb{C}}}h_{\mathbb{C}}\le\frac{4}{3}e^{\frac{C_0}{\Lambda}-A}c_0e^{-t}\mathrm{tr}_{g_{\mathbb{C}}}g^0_{\mathbb{C}},$$

for some constant c_0, only depends on the initial metric. By choosing Λ large, such that $1-\frac{C_0}{\Lambda}>\frac{2}{3}$, fixed, when A is large enough, $c_0e^{\frac{C_0}{\Lambda}-A}\le\frac{1}{\Lambda}$. Thus, choose an $A=\frac{\Lambda}{2}$ large enough, such that $\dot{\phi}(t)\ge-\frac{1}{2}A$, for $t\in[0,T]$, where T is large enough. Then for the first time $\dot{\phi}+\phi+\frac{1}{\Lambda}\phi=-A$, we have:

$$\Big(\frac{\partial}{\partial t}-\Delta\Big)\Big(\dot{\phi}+\phi+\frac{1}{\Lambda}\phi\Big)\ge\Big(\frac{1}{\Lambda}-\frac{C_1e^{-t}}{\Lambda}-c_0e^{\frac{C_0}{\Lambda}-A}\Big)e^{-t}\mathrm{tr}_{g_{\mathbb{C}}}h_{\mathbb{C}}+1+\frac{1}{\Lambda}\dot{\phi}>0.$$

By the maximum principle, the result follows. □

Combine with the potential derivative upper bound (Lemma 4.5), we have the following metric lower bound.

Theorem 4.2. *On an Inoue surface S_M, for generalized Kähler metric ω_0, there is a constant C, such that the normalized pluriclosed flow solution $\omega(t)$ with initial data ω_0 will satisfy*

$$\omega(t)\ge C\omega_h(t),$$

where $\omega_h(t)$ is the model flow.

Proof. By Lemmas 4.5 and 4.6, we have that:

$$-C\le\dot{\phi}+\phi\le C$$

for some constant C. Now, from the equation (4.4) and $s=1$, we have:

$$e^{-C}\le\frac{\mathrm{tr}_{g_{\mathbb{H}}}h_{\mathbb{H}}}{\mathrm{tr}_{g_{\mathbb{C}}}e^{-t}h_{\mathbb{C}}}\le e^{C}.$$

Note that, in the normalized pluriclosed flow model case, (3.3), the $E_{\mathbb{C}}$ part will shrink at the rate of e^{-t}, while the $E_{\mathbb{H}}$ part will be equivalent to $h_{\mathbb{H}}$. Thus, by Corollary 4.2, applied here in the case that $E_{\mathbb{H}}$ is a line bundle, we have the desired lower bound for $E_{\mathbb{C}}$. □

Proof of Theorem 1.2. Combine Lemmas 4.2, 4.5 and 4.6, Corollaries 4.1 and 4.3, and Theorem 4.2, the results follow. □

5 Closing remarks

As pluriclosed flow is in particular a solution to generalized Ricci flow [29], there are scalar curvature monotonicity formulas as explained in [26]. The key extra input is a solution of the *dilaton flow*

$$\frac{\partial}{\partial t}\psi = \Delta\psi + \tfrac{1}{6}|H|^2.$$

To state the monotonicity, we recall the notation of weighted Ricci and scalar curvatures:

$$\begin{aligned} \mathrm{Ric}^{H,\psi} &= \mathrm{Ric} - \frac{1}{4}H^2 + \mathrm{Hess}\,\psi - \frac{1}{2}(d_g^* H + i_{\nabla\psi}H) \\ R^{H,\psi} &= R - \frac{1}{12}|H|^2 + 2\Delta\psi - |\nabla\psi|^2, \end{aligned}$$

Along the *normalized* generalized Ricci flow we then obtain

Corollary 5.1 ([26] Prop 1.1)**.** *If (g_t, H_t) is a solution of normalized generalized Ricci flow and ψ is a solution of the dilaton flow, we have*

$$\left(\frac{\partial}{\partial t} - \Delta\right) R^{H,\psi} = 2|\mathrm{Ric}^{H,\psi}|^2 + R^{H,\psi},$$

and the estimate:

$$R^{H,\psi} \geq \inf_{M\times\{0\}} R^{H,\psi} e^t.$$

For a solution to pluriclosed flow there is a natural class of solutions to the dilaton flow, namely the induced metrics on flat line bundles ([24] Lemma 6.1). On Inoue surfaces for instance there is a natural such bundle $\det E_{\mathbb{H}} \otimes (\det E_{\mathbb{C}})^{\otimes 2}$. In particular by direct computations it can be shown that

$$\psi = \frac{1}{9}\left(\log\frac{\det g_{\mathbb{H}}}{\det h_{\mathbb{H}}} + 2\log\frac{\det g_{\mathbb{C}}}{\det e^{-t}h_{\mathbb{C}}}\right)$$

is a solution of the dilaton flow after appropriate normalization and gauge-fixing. In particular, nonnegativity of $R^{H,\psi}$ is preserved. Noting that in context now all the data defining $R^{H,\psi}$ is canonically determined by a pluriclosed metric alone, it will be interesting to ask whether $X(K,U)$ admits a metric such that $R^{H,\psi} \geq 0$, which becomes an elliptic problem.

Conjecture 5.1. *On an OT manifold $X(K,U)$ there exists no pluriclosed metric with $R^{H,\psi} \geq 0$.*

This question bears a resemblence to a result of Albanese [1] showing that the Inoue surface S_M does not have any positive scalar curvature metric.

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