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The Analysis of Willmore Surfaces and Its Generalizations in Higher Dimensions

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Abstract. We review recent progress concerning the analysis of Lagrangians on immersions into $\mathbb{R}^d$ depending on the first and second fundamental forms and their covariant derivatives.

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1 Introduction

The study of topological and geometrical properties of manifolds through the search of special metrics "equipping" these manifolds and solving some special Partial Differential Equations has a long and rich history which takes its origin maybe in the construction of special curves (like Brachistochrone curves in the XVII$^{\text{th}}$ century, or Euler's elastica in the XVIII$^{\text{th}}$ century) and constant Gauss curvature metrics on closed surfaces in relation with the uniformization theorem for Riemann Surfaces in the XX$^{\text{th}}$ century. The development of what could be called "intrinsic geometric analysis" has been the source of spectacular results in differential topology, differential geometry and complex geometry with the search of constant scalar curvature metric, Einstein metrics, Kähler-Einstein metrics, solutions to the Ricci flow... Another branch of geometric analysis is dealing with the study of "special submanifolds" within a given Riemannian manifold and its interaction with the geometry of the manifold itself. This branch of geometric analysis is maybe rooted originally both in the calculus of variations with the variational constructions of closed

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geodesics, the resolution of the Lagrange-Plateau problem in Euclidean space as well in the "explicit differential geometry" of special submanifolds such as plane algebraic curves, algebraic surfaces, etc. The central objects of what can be described as "extrinsic geometric analysis" are minimal surfaces and their generalizations (constant and prescribed mean curvature surfaces). The underlying PDEs are related to the area variations under various constraints. In the present work, we consider a branch of extrinsic variational geometric analysis that has seen substantial development in recent decades, and concerns immersions that arise as critical points of Lagrangians depending on the first and second fundamental form, and their covariant derivatives. The simplest and maybe most studied model for such a Lagrangian is the Willmore functional of immersed surfaces into $\mathbb{R}^m$ with $m\geq 3$. Let Σ be a closed oriented two-dimensional manifold and let $\vec{\Phi}$ be a C^2 immersion of this surface into $\mathbb{R}^m$. Denoting by $g_{\vec{\Phi}}$ the first fundamental form of this immersion and by $\vec{\mathbb{I}}_{\vec{\Phi}}$ its second fundamental form, the mean curvature of the immersion is given by

$$\vec{H}_{\vec{\Phi}} := \frac{1}{2}\mathrm{tr}_{g_{\vec{\Phi}}}\vec{\mathbb{I}}_{\vec{\Phi}}.$$

The Willmore energy is

$$W(\vec{\Phi}) := \int_\Sigma |\vec{H}_{\vec{\Phi}}|^2 \, d\mathrm{vol}_{g_{\vec{\Phi}}}.$$

This energy has been initially introduced by Sophie Germain and Siméon Denis Poisson in an attempt to generalize to two-dimensional elastic membranes the famous Euler Elastica modelizing the free energy of a beam [64, 163]. The very first derivation of the Euler–Lagrange equation of W was made by Poisson himself around 1814 in the case of a graph (see page 60 of [163] and chapter 10 of [141]) in $\mathbb{R}^3$. Using concepts which were not completely clarified at the time (such as the Gauss curvature and the Laplace–Beltrami operator), we can rewrite Poisson's Euler–Lagrange equation for the Lagrangian W in the following form

$$\Delta_g H_{\vec{\Phi}} + 2H_{\vec{\Phi}}(H_{\vec{\Phi}}^2 - K_{\vec{\Phi}}) = 0, \tag{1.1}$$

where $H_{\vec{\Phi}} = \vec{n}_{\vec{\Phi}} \cdot \vec{H}_{\vec{\Phi}}$ is the mean curvature and $\vec{n}_{\vec{\Phi}}$ the unit Gauss map to the immersion, Δ_g is the negative Laplace–Beltrami operator which reads in local coordinates (x^1, x^2)

$$\Delta_g \cdot := (\det(g))^{-\frac{1}{2}} \partial_{x_i}\left(g^{ij}(\det(g))^{\frac{1}{2}}\partial_{x_j}\cdot\right)$$

and $K_{\vec{\Phi}} := \det_{g_{\vec{\Phi}}}(\vec{n}_{\vec{\Phi}} \cdot \vec{\mathbb{I}}_{\vec{\Phi}})$ is the Gauss curvature.[†] Throughout this paper, we shall often write $g = g_{\vec{\Phi}}$ when there is no ambiguity. With these notations we have also

$$\vec{H}_{\vec{\Phi}} = H_{\vec{\Phi}}\, \vec{n}_{\vec{\Phi}} = \frac{1}{2}\left(\vec{n}_{\vec{\Phi}} \cdot \Delta_g \vec{\Phi}\right) \vec{n}_{\vec{\Phi}} = \frac{1}{2}\Delta_g \vec{\Phi}. \tag{1.2}$$

[†]The Euler-Lagrange equation in higher codimension has the same cubic structure in the second fundamental form, see [190, Theorem 2.1].