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Geometric Inequality for CR-Slant Warped Product Submanifolds in Nearly Lorentzian Para-Sasakian Manifold

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Abstract This paper examines CR-slant warped product submanifolds of the form $B \times_f N_\theta$ within a nearly Lorentzian para-Sasakian manifold. Here, B represents a CR-product submanifold, N_θ is a slant submanifold and f denotes the warping function. We establish an inequality relating the squared norm of the second fundamental form to the warping function, considering two cases based on the behavior of the structure vector field. Additionally, the conditions under which equality holds are investigated.

Keywords Warped product, CR-slant submanifold, nearly Lorentzian para-Sasakian manifold

MSC(2010) 53B25, 53C15, 53C40, 53C42, 53D10.

1. Introduction

The concept of warped products is well-established in both differential geometry and physics. In 1969, Bishop and O'Neill [6] introduced warped products as a tool for studying manifolds with negative curvature, extending the idea of Riemannian product manifolds. They defined these manifolds as follows: Let (B, g_1) and (F, g_2) be two Riemannian manifolds, and let f be a differentiable function on B. For the product manifold $B \times F$, with projections $\gamma_1 : B \times F \to B$ and $\gamma_2 : B \times F \to F$, the warped product of B and F, denoted by $N = B \times_f F$, is endowed with a Riemannian structure defined as follows [6]

$$g(X_1, X_2) = g_1(\gamma_{1*}X_1, \gamma_{1*}X_2) + (f \circ \gamma_1)^2 g_2(\gamma_{2*}X_1, \gamma_{2*}X_2),$$

for any vector field $X_1, X_2 \in \Gamma(TN)$, where $*$ denotes the tangent map. A warped product manifold is considered trivial or simply a Riemannian product manifold, if the warping function f is constant. It is well known that, for a vector field X_1 on B and X_3 on F, the following holds: [6]

$$\nabla_{X_3} X_1 = \nabla_{X_1} X_3 = X_1(lnf)X_3, \tag{1.1}$$

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where ∇ is the Levi-Civita connection on N. Additionally, it is well established that B is totally geodesic and F is totally umbilical in the warped product $B \times_f F$ ([6], [10]).

In [10], Chen introduced the concept of CR-warped products in Kaehler manifolds, demonstrating results on the existence of warped products and proving general sharp inequalities for the second fundamental form in relation to the warping function. Subsequently, numerous articles have appeared addressing similar inequalities in almost Hermitian and almost contact metric manifolds ([7], [4], [20], [21], [17], [14]).

Sahin introduced the concept of CR-slant warped products, which are referred to as skew CR-warped products, in Kaehler manifolds in [22]. Following this, Chen et al. explored the pointwise CR-slant warped products in Kaehler manifolds in [11]. More recently, Alqahtani and Almudawi [1] examined CR-slant warped product submanifolds in nearly trans-Sasakian manifolds. They established an inequality for the second fundamental form in two cases, depending on the behavior of the structure vector field. Several geometers have also studied CR-slant warped product submanifolds, including works in ([23], [24] [3], [2], [25]).

On the other hand, a class of almost paracontact metric manifolds known as Lorentzian para-Sasakian manifolds was introduced by Matsumoto [15]. Subsequently, Mihai et al. [16] independently introduced the same concept and derived several results on these manifolds. Lorentzian para-Sasakian manifolds have also been studied by De et al. [12], Rahman et al. [19], and others. In [20], Rahman established an inequality for contact CR-warped product submanifolds of nearly Lorentzian para-Sasakian manifolds.

Building on previous studies, we investigate CR-slant warped product submanifold of the form $B \times_f N_\theta$ within a nearly Lorentzian para-Sasakian manifold. Here, $B = N_T \times N_\perp$ is the CR-product of invariant and anti-invairant submanifolds of $\tilde{M}$, N_θ is a slant submanifold and f represents the warping function. We derive a sharp estimate for the squared norm of the second fundamental form in relation to the warping function, considering two cases based on whether the structure vector field is tangent to the invariant or anti-invariant submanifold. We also investigate the equality cases of these inequalities.

2. Preliminaries

Let $\tilde{M}$ be an m-dimensional Lorentzian almost paracontact manifold equipped with an almost paracontact structure (ϕ, ξ, η, g), where ϕ is a (1, 1)-tensor field, ξ is a characteristic vector field, η is a 1-form and g is a Riemannian metric satisfying the following conditions [15]:

$$\phi^2 X_1 = X_1 + \eta(X_1)\xi, \quad \eta(\xi) = -1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \tag{2.1}$$

$$g(\phi X_1, \phi X_2) = g(X_1, X_2) + \eta(X_1)\eta(X_2), \tag{2.2}$$

$$g(\phi X_1, X_2) = g(X_1, \phi X_2), \quad \eta(X_1) = g(X_1, \xi), \tag{2.3}$$

for all vector fields X_1 and X_2 on $\tilde{M}$. Then the structure (ϕ, ξ, η, g) is said to be Lorentzian para-contact structure.

A Lorentzian paracontact manifold $\tilde{M}$ is called a Lorentzian para-Sasakian(LP-Sasakian) manifold if [15]

$$(\tilde{\nabla}_{X_1}\phi)X_2 = g(X_1, X_2)\xi + \eta(X_2)X_1) + 2\eta(X_1)\eta(X_2)\xi, \tag{2.4}$$