Journal of Nonlinear Modeling and Analysis
Volume 8, Number 2, April 2026, 431–452
https://global-sci.com/jnma
DOI:10.12150/jnma.2026.431

Solvability for a Class of Logarithmic Type Initial Conditions of Hadamard Fractional Differential System on an Infinite Interval*

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Received 25 January 2025; Accepted 5 June 2025

Abstract This paper investigates a class of Hadamard fractional differential system involving logarithmic type initial conditions on an infinite interval. Based on some obtained properties of the Green's functions, Schauder fixed point theorem and Banach contraction mapping principle, we establish the existence and uniqueness results for the system. Finally, we illustrate examples which show our main results.

Keywords Solvability, Hadamard fractional differential system, logarithmic type initial conditions, infinite interval

MSC(2010) 26A33, 34A37, 34B15, 34B18.

1. Introduction

We examine a system comprising nonlinear Hadamard fractional differential equations (HFDEs for short)

$$\begin{cases} D^p_{1^+}x(t) + f(t, x(t), D^{q-1}_{1^+}y(t)) = 0, t \in (1, \infty), \\ D^q_{1^+}y(t) + g(t, x(t), D^{p-1}_{1^+}y(t)) = 0, t \in (1, \infty), \end{cases} \tag{1.1}$$

supplemented with some logarithmic type initial conditions

$$\begin{cases} \lim_{t\to 1}(\log t)^{2-p}x(t) = \lim_{t\to\infty} D^{p-1}_{1^+}x(t) = \sum_{i=1}^m a_i x(\eta_i), \\ \lim_{t\to 1}(\log t)^{2-q}y(t) = \lim_{t\to\infty} D^{q-1}_{1^+}y(t) = \sum_{j=1}^n b_j y(\zeta_j), \end{cases} \tag{1.2}$$

where $D^v_{1^+}$ is the Hadamard type fractional derivative of order $v \in \{p, q\}, p, q \in (1, 2], f, g \in C([1, \infty) \times \mathbf{R} \times \mathbf{R}, \mathbf{R}), 1 < \eta_1 < \eta_2 < \cdots < \eta_m < \infty, 1 < \zeta_1 < \zeta_2 < \cdots < \zeta_n < \infty, a_i(i = 1, 2, \ldots, m), b_j(j = 1, 2, \ldots, n)$ denote positive real constants with

$$1 - \sum_{i=1}^m a_i\left(\frac{(\log \eta_i)^{p-1}}{\Gamma(p)} + (\log \eta_i)^{p-2}\right) > 0,$$

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*This paper is supported by Fundamental Research Program of Shanxi Province(202303021221068).

$$1-\sum_{j=1}^{n} b_j\Big(\frac{(\log \zeta_j)^{q-1}}{\Gamma(q)}+(\log \zeta_j)^{q-2}\Big)>0.$$

For variables in infinite intervals, we have to consider different function spaces and infinite integrals. Therefore, we need the following assumptions:
(A_1) $f:[1,\infty)\times \mathbf{R}\times \mathbf{R}\to \mathbf{R}$, satisfies the a-Caratheodory conditions, that is
(i) $t\to f(t,\frac{1+(\log t)^{\sigma+2}}{(\log t)^{2-p}}x,\frac{1+(\log t)^{\sigma+2}}{\log t}y)$ is measurable on $[1,\infty)$ for every $(x,y)\in \mathbf{R}^2$;
(ii) $(x,y)\to f(t,\frac{1+(\log t)^{\sigma+2}}{(\log t)^{2-p}}x,\frac{1+(\log t)^{\sigma+2}}{\log t}y)$ is continuous on $\mathbf{R}^2$ for all $t\in[1,\infty)$;
(iii) for any $r>0$, there exists a function $\varphi_r(t)\geq 0$ with $\int_1^\infty \varphi_r(t)\frac{dt}{t}<\infty$, such that

$$f(t,\frac{1+(\log t)^{\sigma+2}}{(\log t)^{2-p}}x,\frac{1+(\log t)^{\sigma+2}}{\log t}y)\leq \varphi_r(t) \text{ for } t\in[1,\infty),|x|,|y|\leq r.$$

(A_2) $g:[1,\infty)\times \mathbf{R}\times \mathbf{R}\to \mathbf{R}$, satisfies the a-Caratheodory conditions, that is
(i) $t\to g(t,\frac{1+(\log t)^{\sigma+2}}{(\log t)^{2-q}}x,\frac{1+(\log t)^{\sigma+2}}{\log t}y)$ is measurable on $[1,\infty)$ for every $(x,y)\in \mathbf{R}^2$;
(ii) $(x,y)\to g(t,\frac{1+(\log t)^{\sigma+2}}{(\log t)^{2-q}}x,\frac{1+(\log t)^{\sigma+2}}{\log t}y)$ is continuous on $\mathbf{R}^2$ for all $t\in[1,\infty)$;
(iii) for any $r>0$, there exists a function $\phi_r(t)\geq 0$ with $\int_1^\infty \phi_r(t)\frac{dt}{t}<\infty$, such that

$$g(t,\frac{1+(\log t)^{\sigma+2}}{(\log t)^{2-q}}x,\frac{1+(\log t)^{\sigma+2}}{\log t}y)\leq \phi_r(t) \text{ for } t\in[1,\infty),|x|,|y|\leq r.$$

(A_3) $f(t,0,0),g(t,0,0)\not\equiv 0$ on any subinterval of $[1,\infty)$;

Fractional calculus is an important branch of mathematics, which is used to study the differentiation and integration of any real order or complex order, and can be used to solve many problems which are difficult to be solved by integral calculus. In the past few decades, fractional differential equations (FDEs for short) have been widely used in physics, mechanics, chemistry, economics, biomedicine and other fields, which are mainly used to describe the behavior of complex systems, describe the properties of complex media mechanics, describe molecular diffusion behavior, establish nonlinear economic models, describe biomedical signals and so on, see [1–4]. So far, several different fractional derivatives have been proposed, for example, the Riemann-Liouville, Caputo, Hadamard, and Caputo-Fabrizio fractional derivatives, see [5–13, 24]. Different derivative operators are often used according to their different structures. In this paper, we mainly study FDEs with Hadamard type derivatives. In order to gain insight into the many applications of fractional calculus in different fields, we refer the reader to [14–17, 25–41].

In [22], Nyamoradi and Ahmad applied the Leggett-Williams fixed point theorem and the concept of iterative positive solutions to prove the existence of at least two or three positive solutions for the following HFDE:

$$\begin{cases} D_{1^+}^{\varsigma}x(t)+\upsilon(t)f(t,x(t))=0, t\in(1,\infty),\\ \lim_{t\to1}(\log t)^{2-\varsigma}x(t)=\lim_{t\to\infty}D_{1^+}^{\varsigma-1}x(t)=\int_1^\infty \tau(s)x(s)\frac{ds}{s},\end{cases} \tag{1.3}$$

where $D_{1^+}^{\varsigma}$ is the Hadamard fractional derivative of order $\varsigma\in(1,2]$, $f:(1,\infty)\times\mathbf{R}\to\mathbf{R}$, $\upsilon:(1,\infty)\to(1,\infty)$ are continuous functions, $0<\int_1^\infty \upsilon(s)\frac{ds}{s}<\infty$, $\tau\in L^1(1,\infty)$ and

$$\Psi_\varsigma=\int_1^\infty\left(\frac{(\log s)^{\varsigma-1}}{\Gamma(\varsigma)}+(\log s)^{\varsigma-2}\right)\tau(s)\frac{ds}{s}<1.$$