

On Constacyclic Codes of Length $4p^s$ over $F_{p^m}[u, v]/\langle u^2, v^2, uv - vu \rangle$

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Abstract In this paper, we determine the algebraic structure of all λ -constacyclic codes of length $4p^s$ over the ring $R_{u^2, v^2, p^m} = F_{p^m}[u, v]/\langle u^2, v^2, uv - vu \rangle$ and $u^2 = 0, v^2 = 0$ where $\lambda = (\alpha + \beta u + \gamma v + \delta uv)$ with $\beta, \gamma, \delta \in F_{p^m}, \alpha \in F_{p^m}^*$ and β, γ are not both zero. If λ is a square, each λ -constacyclic codes of length $4p^s$ is expressed as a direct sum of an $-\alpha$ -constacyclic code and α -constacyclic code of length $2p^s$. In the primary case where the unit λ is not square, it is shown that any non-zero polynomial of degree ≤ 4 over F_{p^m} is invertible in the ring $R_{\alpha, \beta, \gamma, \delta} = R_{u^2, v^2, p^m}[x]/\langle x^{4p^s} - \lambda \rangle$ when $\lambda = (\alpha + \beta u + \gamma v + \delta uv)$ for non-zero elements $\alpha, \beta, \gamma \in F_{p^m}^*, \delta \in F_{p^m}$, it follows that the ring $R_{\alpha, \beta, \gamma, \delta}$ is a chain ring with maximal ideal $\langle x^4 - \alpha_0 \rangle$ and $(\alpha + \beta u + \gamma v + \delta uv)$ -constacyclic codes are $\langle (x^4 - \alpha_0)^i \rangle$, for $0 \leq i \leq 2p^s$. In the second case where λ is not square and $\lambda = \gamma$ for $\gamma \in F_{p^m}$, it is obtain that $R_{u^2, v^2, p^m}[x]/\langle x^{4p^s} - \gamma \rangle$ is a local ring with maximal ideal $\langle x^4 - \gamma_0, u \rangle$ but not a chain ring, such λ -constacyclic codes are classified into four distinct types. We provide the number of codewords for each type, and give the details of their dual codes.

Keywords Constacyclic codes, codes over rings, dual codes

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1. Introduction

In the realm of error-correcting code theory, constacyclic codes are significant due to their efficient encoding capabilities using simple shift registers. These codes possess robust algebraic structures that facilitate effective error correction and detection, making them essential for both practical applications and theoretical explorations. However, the classification of these codes, particularly for lengths such as $4p^s$, presents a complex challenge. Historically, only specific classes of constacyclic codes of certain lengths over particular finite rings have been successfully classified. Let R denote a finite commutative ring with unity, and let λ be a unit of R . A λ -constacyclic code of length n over R can be represented as an ideal of the quotient ring $R[x]/\langle x^n - \lambda \rangle$. When the code length n is coprime to the characteristic of the residue field of R , these codes are categorized as simple root codes, otherwise, they are referred to as repeated root codes.

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Recent years have seen a surge of interest in codes over finite rings, leading to the discovery of several effective codes as Gray images of codes over rings. Notably, special classes of simple and repeated root constacyclic codes over various finite rings have been extensively studied. For instance, see [1–15]. Yildiz et al. [16] studied the ring $F_2[u, v]/\langle u^2, v^2, uv - vu \rangle$ which is not a chain ring and examined cyclic codes of odd length over it, and identified effective binary codes as Gray images of cyclic codes. A generalization of these codes was explored in [17], where the authors investigated the cyclic codes over $F_2[u_1, u_2, \dots, u_k]/\langle u_i^2, u_j^2, u_i u_j - u_j u_i \rangle$. Subsequently, Sobhani and Molakarimi [18] extended these studies to cyclic codes over the ring $F_{2^m}[u, v]/\langle u^2, v^2, uv - vu \rangle$. In addition, in 2015, Kewat et al. [19] investigated cyclic codes over the ring $Z_p[u, v]/\langle u^2, v^2, uv - vu \rangle$.

Rings of these forms have been used as alphabets of certain cyclic codes by numerous authors [20, 21]. Dinh et al. [22] determined the algebraic structures of all constacyclic codes of length p^s over $R_{u,v}$, except for the $(\alpha + \delta uv)$ -constacyclic codes where $\delta \neq 0$ which are studied in [23]. In the case $p = 2$, Dinh et al. classified all self-dual λ -constacyclic codes of length 2^s over $R_{u,v}$. Y. Ahendouz and I. Akharraa [26] studied some cases of constacyclic codes of length $2p^s$ over $R_{u,v}$.

Motivated by the above-mentioned works and to our knowledge, no notable work yet that conducts that research has been published on extending the classification of constacyclic codes to the length $4p^s$ over the ring $R_{u,v} = F_{p^m}[u, v]/\langle u^2, v^2, uv - vu \rangle$, where p is an odd prime number. This ring serves as a crucial example of a finite local ring that is not a chain ring. Its unit has the form $\lambda = (\alpha + \beta u + \gamma v + \delta uv)$ with $\alpha, \beta, \gamma, \delta \in F_{p^m}, \alpha \in F_{p^m}^*$. Our investigation will build upon previous works, particularly focusing on the algebraic structures of λ -constacyclic codes of this new length, thereby contributing to the ongoing discourse in the field of error-correcting codes.

The rest of this paper is organized as follows: Section 2 provides fundamental background on constacyclic codes. Section 3 investigates λ -constacyclic codes of length $4p^s$ over $R_{u,v}$, focusing on the number of elements in each code and their duals when λ is a square. Section 4 extends this analysis to λ -constacyclic codes in scenarios where λ is not a square. Lastly, section 5 examines the duals of the codes discussed in Section 4.

2. Preliminaries

A ring R is a principal ideal ring if its ideals are principal. An ideal I of R is called principal if it is generated by one element. Let R be a finite commutative ring. If the ring R has a unique maximal ideal then R is called local ring with unity. Also if this unique maximal ideal is principal then R is a chain ring.

Definition 2.1. [25] The socle of a ring R is denoted by $Soc(R)$, defined as the sum of all minimal left ideals of R . A minimal left ideal is an ideal that is not the zero ideal and does not contain any other non-zero left ideals.

Definition 2.2. [29] The Jacobson radical of R is defined to be $J(R)$, which is the intersection of all maximal left ideals of R . If R satisfies the isomorphism

$$\frac{R}{J(R)} \cong Soc(R),$$

then R is called a Frobenius ring.