Journal of Nonlinear Modeling and Analysis
Volume 8, Number 2, April 2026, 494–519
https://global-sci.com/jnma
DOI:10.12150/jnma.2026.494

Numerical Solutions of Fuzzy Fractional Variable-Order Differential Equations

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Received 21 February 2025; Accepted 7 July 2025

Abstract This paper addresses a class of fuzzy fractional differential equations (FFDEs) with variable-order (VO) derivatives, where the variable-order derivative is defined in the Caputo sense for fuzzy-valued functions. Using the $\Upsilon-$cut representation of fuzzy-valued functions, the original problem is reformulated into a new problem. To solve it, we apply operational matrices (OMs) derived from shifted Chebyshev polynomials of the third kind (SCP3). By approximating the unknown function and its derivative with SCP3, the problem is reduced to a system of nonlinear algebraic equations. A theoretical error analysis of the numerical solution is presented, along with an example to validate the method's accuracy.

Keywords Fuzzy fractional differential equations, variable-order, shifted Chebyshev polynomials of the third kind, operational matrix

MSC(2010) 26A33, 42C05, 34A07.

1. Introduction

Fractional calculus (FC) has emerged as a powerful mathematical framework for modeling complex and memory-dependent phenomena in diverse fields such as damping, viscoelasticity, wave propagation, diffusion, control systems, and signal processing [1–3]. The widespread applicability of fractional differential equations (FDEs) has led to the development of a broad range of analytical and numerical methods to solve them efficiently [4–6]. These methods not only enhance our understanding of such systems but also pave the way for developing more advanced computational techniques.

To further improve modeling accuracy, FDEs have been generalized through the introduction of VO operators, in which the order of differentiation or integration varies as a function of the independent variable. Several VO formulations have been proposed, including the Riemann-Liouville (RL) [7], Coimbra [9], Caputo–Fabrizio (CF) [8], and Atangana-Baleanu (AB) [10] operators. These provide greater flexibility for modeling processes with evolving memory and hereditary behavior.

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Despite their potential, VO-FDEs pose significant computational challenges due to the complexity of their integral kernels. In response, various numerical strategies have been proposed. Ganji et al. [10, 11] developed orthogonal polynomial-based techniques, while Bernstein polynomials have been employed in solving variable-order diffusion-wave equations [12]. More recently, Tingting et al. [13] used operational matrices with collocation to solve VO partial differential equations (PDEs), Jafari et al. [14] applied Hosoya polynomials to stochastic VO problems, Saha et al. [15] extended these approaches to VO-FDEs, and Mohd et al. [16] proposed a Bernstein-based method for nonlinear coupled reaction-diffusion systems.

In parallel, uncertainty modeling has progressed through the development of FFDEs, which integrate fuzzy logic into fractional calculus. FFDEs account for imprecision in system parameters, initial conditions, or boundary values using fuzzy numbers characterized by normality, convexity, upper semi-continuity, and compact support [17]. Numerous techniques have been proposed to solve FFDEs, including fuzzy Laplace transforms [18], Mittag-Leffler functions [19], fractional Euler methods [20], spline collocation [21], Chebyshev-based approaches [22], differential transforms [23], spectral methods [24]. Notable examples also include power series methods for fuzzy logistic equations [25] and fuzzy modeling in drug administration systems [26].

Combining the flexibility of VO operators with the uncertainty-handling capability of fuzzy sets leads to fuzzy variable-order fractional differential equations (VO-FFDEs). These hybrid models are well suited to capture both time-dependent memory effects and data uncertainty, making them applicable in engineering, physics, biology, finance, and other fields. Despite their promise, VO-FFDEs remain relatively underexplored, and challenges persist in their numerical formulation, stability, and implementation.

To contribute to this evolving area, Jafari et al. [27] proposed an approach using operational matrices based on shifted Legendre polynomials to solve fuzzy VO-FDEs involving Mittag-Leffler kernels.

In this study, we propose a spectral collocation method for solving VO-FFDEs using operational matrices based on shifted Chebyshev polynomials of the third kind. SCP3 polynomials are chosen for their orthogonality, computational efficiency, and strong endpoint clustering on $[0, 1]$, which improves accuracy near boundaries particularly important for VO-FDEs with non-uniform kernel behavior. Their structure also enables well-conditioned and sparse operational matrices, enhancing numerical stability.

The proposed method reformulates the fuzzy VO-FDE into a system of algebraic equations using the Υ-cut representation and SCP3 based operational matrices. The resulting scheme is general, computationally efficient, and capable of handling linear fuzzy systems.

We consider the following form of a fuzzy variable-order fractional differential equation:

$$\begin{cases} {}_0^C D_{\mathtt{t}_1}^{\varsigma(\mathtt{t}_1)} \mathsf{U}(\mathtt{t}_1) = \mathfrak{f}(\mathtt{t}_1, \mathsf{U}(\mathtt{t}_1)), \\ \\ \qquad \mathsf{U}(0) = \mathsf{U}_0 \in \mathbb{E}_{\mathrm{F}}. \end{cases} \tag{1.1}$$

Here, $0 < \varsigma(\mathtt{t}_1) < 1$, and $\mathbb{E}_{\mathrm{F}}$ represents the space of fuzzy numbers. The function $\mathfrak{f} : [0, 1] \times \mathbb{E}_{\mathrm{F}} \to \mathbb{E}_{\mathrm{F}}$ is a continuous fuzzy function, and the unknown