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A Robust Fixed Point Method to Solve System of Nonlinear Equations

M.R. Amattouch[1,†]

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Abstract Solving nonlinear partial differential equations requires addressing systems of nonlinear algebraic equations. In this article, we propose a new fixed-point method for solving these systems. We assume that the method is both fast and globally convergent. Additionally, this method can be accelerated using Aitken or Anderson acceleration techniques. Several numerical test cases are presented to illustrate the efficiency of the proposed method.

Keywords Brouwer fixed point theorem, system of equations, global convergence

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1. Introduction

Nonlinear partial differential equations are widely used to simulate various industrial and biological phenomena. Discretizing these equations results in a system of nonlinear equations, as seen in [1–3]. Thus, we focus on solving the system of equations

$$F(x) = 0, \tag{1.1}$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear, continuously differentiable function, particularly when n is large. We consider cases where $n \geq 3$.

The most popular method for solving (1.1) is the Newton method [4, 5]. This method, known for its physical interpretation and strong convergence properties, poses a major challenge: it requires calculating the Jacobian matrix of F: F', and evaluating its inverse at each iteration, which is computationally expensive, especially for large dimensions n. Another limitation is that convergence is not guaranteed in the presence of singularities, and the method converges slowly for flat functions.

Alternative methods to Newton's method include quasi-Newton and inexact Newton methods, which approximate F'^{-1} to reduce computational costs. Various quasi-Newton methods are discussed in [6–10]. However, these methods still entail high computational costs for many problems. A solution is to accelerate the method's convergence, such as with the BFGS method [11, 12]. Another accelerator is the Krylov method [13, 14]. Yet, these methods also face difficulties with large-scale problems, particularly for flat functions and problems with singularities.

[†] the corresponding author.
Email address:amattouch36@gmail.com.
[1] Department of Mathematics and Informatics, University AbdelMalik Essaadi,
High national school of engineering of Alhoceima, BP 146, Alhoceima, 28806,
Morocco

As an alternative to Newton-based methods, accelerated fixed-point methods are often employed. Two common accelerators are Anderson acceleration ([15,16]) and line search methods ([17,18]). These methods offer the advantage of global convergence and are more practical for large-scale problems arising from partial differential equations.

In this article, we introduce a new fixed-point method and provide proofs of both its local and global convergence, as well as an explanation of its hyper-convergence properties. Specifically, we propose a convergence theorem under certain assumptions, modify the general equations to satisfy these assumptions, and achieve fast global convergence through these modifications. This method is highly efficient and applicable to linear systems of equations, infinite-dimensional problems, and optimization tasks. Its computational complexity is favorable, as it only requires basic operations such as matrix multiplication and addition, without necessitating the solution of minimization problems or matrix inversion. Through various test cases, we demonstrate the method's efficiency, particularly for problems involving singularities and large-scale systems. Additionally, the method's performance can be further enhanced using acceleration techniques like Anderson or Aitken methods.

2. The proposed method

Notice that the function F of problem 1.1 can be written as:

$$F = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}$$

where $f_1, f_2, ..., f_n$ are functions defined on $\mathbb{R}$.

The Jacobian of the function F is defined by the matrix

$$F'(x) = (f_{i,j}(x))_{1 \leq i,j \leq n},$$

where $f_{i,j}(x) = \dfrac{\partial f_i}{\partial x_j}$.

We also define the function sign defined on $\mathbb{R}$ as

$$sign(x) = \begin{cases} 1 & \text{if } x > 1. \\ -1 & \text{if } x < 1. \\ 0 & \text{if } x = 0. \end{cases}$$

Consider $x_0 \in \mathbb{R}^n$ as an initial guess. The sequence of our method is constructed as:

$$x_{k+1} = x_k - \frac{1}{d(x_k)} H(x_k) F(x_k), \tag{2.1}$$

where

$$d(x) = \max_{k \in [\![1;n]\!]} \sum_{i=1}^{n} |\frac{\partial f_i(x)}{\partial x_k}| + MN|\frac{\partial f_k}{\partial x_k}|$$