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Global Attractors for the BBM Equation with Fading Memory

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Abstract This paper considers the existence of global attractors for BBM equation with fading memory. Using some new estimate technique to prove the existence of global attractors in topological space $H_1 \times L^2_\mu(\mathbb{R}^+; H_1)$.

Keywords BBM equations, global attractors, fading memory

MSC(2010) 35B40, 35B41.

1. Introduction

In this paper, we study the following BBM equation with fading memory:

$$\begin{cases} u_t - \Delta u_t - \alpha \Delta u - \int_0^\infty k'(s)\Delta u(t-s)ds + (g(u))_x = f, & \text{in } \Omega, t > 0, \\ u(t) = u_0(t), & \text{in } \Omega, t \leqslant 0, \\ u_{|\partial\Omega} = 0, & t \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}$ is a bounded interval of the real line, f is a deterministic time-dependent external forcing term, and memory kernel function $k'(s)$ is assumed to satisfy the posterior adaptation hypothesis, which will be detailed in the subsequent sections. It describes the characteristic that the current state of a material or system is influenced by its past states. $g \in L^2(\Omega)$ is a nonlinear term, and satisfies:

$$g(u) = u + \frac{1}{2}u^2.$$

BBM equation was first proposed by Benjamin, Bona and Mahony, which describes the mathematical model of long waves propagation with nonlinear dispersion and dissipation effects, see [3]. In recent years, the asymptotic behavior of BBM equations has been studied by many authors. In [1, 2, 4, 5, 7], the authors studied the global well-posedness and ill-posedness of BBM equation in H^S and L^P type Sobolev spaces. In [8], the authors proved the existence of global attractors of BBM equations on bounded domains in H^1. Stanislavova et al. proved the existence of global attractors in H^1 for the BBM equation on unbounded domains, see [17, 18]. In [21], Yang et al. proved the upper semi-continuity of the pullback attractor of the three-dimensional non-autonomous BBM equation. Dell 'Oro et al. [11] proved the existence of global attractors of BBM equations with memory. In [19], the authors

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studied the long-term behavior of the generalized BBM equation with damping in a low regularity space.

In recent years, the research on global attractors of equations with fading memory has attracted much attention. For example, in [6, 15] the authors proved the existence of differential equations attractor with fading memory. Zhang et al. [22] showed the existence of strong global attractor for nonclassical diffusion equation with fading memory. Xuan and Munteanu [14,20] respectively proved the attractors of abstract evolution equations and Navier-Stokes Equation with fading memory. The dynamic behavior of the solution of (1.1) in space$H_1 \times L^2_\mu(\mathbb{R}^+; H_1)$ has not yet been considered, and it can be said that it is studied for the first time in this paper as a new problem. In this paper, we will use some new estimate technique to prove the existence of global attractors in topological space $H_1 \times L^2_\mu(\mathbb{R}^+; H_1)$.

The organizational structure of this paper is as follows. In Section 2, we present some notations and proposition of function space. In Section 3, we show the existence of global attractors in $H_1 \times L^2_\mu(\mathbb{R}^+; H_1)$.

2. Preliminaries and abstract results

In this section, we present some notations and proposition of function space.

Set $A = -\Delta$, $H_0 = L^2(\Omega)$, $H_1 = H^1_0(\Omega)$, $H_2 = L^2(\Omega) \cap H^1_0(\Omega)$. Denote as $H_r = D(A^{r/2})$ $(0 \le r \le 2)$, the inner product and norm are as follows:

$$\langle u, v\rangle_{H_r} = \langle A^{r/2}u, A^{r/2}v\rangle, \quad \|u\|_{H_r} = \|A^{r/2}u\|. \tag{2.1}$$

Clearly,

$$H_2 \subset H_1 \subset H_0 = H_0^* \subset H_1^*, \tag{2.2}$$

where H_0^* and H_1^* represent the dual space of H_0, H_1 respectively.

If $\forall m < s$, we have

$$D(A^{\frac{s}{2}}) \subset D(A^{\frac{m}{2}}), \quad D(A^{\frac{s}{2}}) \subset L^{\frac{2n}{n-2s}}(\Omega).$$

Using the Poincaré inequality, we can obtain:

$$\sqrt{\lambda_1}\|v\|_s \le \|v\|_{s+1}, \quad \forall v \in H^1_0. \tag{2.3}$$

Let $\mu(s) = -k'(s)$ and $k(\infty) = 0$. Then by (1.1), we have

$$\begin{cases} u_t - \Delta u_t - \alpha\Delta u - \int_0^\infty \mu(s)\Delta\eta(t-s)ds + (g(u))_x = f, \\ \eta^t_t = -\eta^t_s + u, \end{cases} \tag{2.4}$$

with the initial boundary conditions

$$\begin{cases} u(x,t)|_{\partial\Omega} = 0, \quad \eta^t(x,s)|_{\partial\Omega\times\mathbb{R}^+} = 0, \quad t \ge 0, \\ u(x,0) = u_0(x), \quad \eta^0(x,s) = \int_0^s u_0(x,-\tau)d\tau, \quad (x,s) \in \Omega\times\mathbb{R}^+. \end{cases} \tag{2.5}$$

In (1.1), the role of fading memory is reflected in the function $\Delta u(\cdot)$ and the linear convolution term of the memory kernel function $k(\cdot)$. From [14, 20] we make the following assumptions on memory kernel function μ:

$$\mu \in \mathrm{C}^1(\mathrm{R}^+) \cap \mathrm{L}^1(\mathrm{R}^+), \ \mu'(\mathrm{s}) \le 0 \le \mu(\mathrm{s}), \ \forall \mathrm{s} \in \mathbb{R}^+; \tag{2.6}$$

$$\int_0^\infty \mu\ (\mathrm{s})\mathrm{ds} = k_0 > 0,\ \forall \mathrm{s} \in \mathbb{R}^+; \tag{2.7}$$

$$\delta > 0, \mu'(s) + \delta\mu(s) \leq 0. \tag{2.8}$$

Obviously, $k(s)$ and $\mu(s)$ decline exponentially to zero, which is the long-term fading memory we will mention.

Make the following assumptions for the nonlinear term $g(u)$:

$$|g'(u)| \leq c(1 + |u|^{\frac{4}{n-2}}), u \in R. \tag{2.9}$$

$$\lim_{|s|\to\infty} \inf \frac{G(s)}{s^2} \geq 0, \quad G(s) = \int_0^s g\,(s)\,ds. \tag{2.10}$$

$$\lim_{|s|\to\infty} \inf \frac{sg(s) - C_0 G(s)}{s^2} \geq 0. \tag{2.11}$$

$$\lim_{|s|\to\infty} \sup \frac{G(s)}{|s|^\gamma} = 0, \quad 0 \leq \gamma < 2. \tag{2.12}$$

Based on the ideas of literature [16], to define a family of Hilbert Spaces, the inner products and norms are as follows:

$$\langle \beta_1,\beta_2\rangle_{\mu,H_1} = \int_0^\infty \mu\,(s)\,(\nabla\beta_1(s), \nabla\beta_2(s))\mathrm{d}s, \quad \|\beta(s)\|^2_{\mu,H_1} = \int_0^\infty \mu\,(s)\,\|\nabla\beta(s)\|^2 ds,$$

$$\langle \beta_1,\beta_2\rangle_{\mu,D(A)} = \int_0^\infty \mu(s)(\Delta\beta_1(s), \Delta\beta_2(s))\mathrm{d}s, \quad \|\beta(s)\|^2_{\mu,D(A)} = \int_0^\infty \mu(s)\|\Delta\beta(s)\|^2 ds.$$

Let's define $\mathcal{V}_r = \mathrm{H_r} \times \mathrm{L}^2_\mu(\mathrm{R}^+; \mathrm{H_r})$, whose norm is

$$\|z\|_{\mathcal{V}_r} = \|(u, \eta^t)\|_{\mathcal{V}_r} = (\frac{1}{2}(\|u\|^2_{H_r} + \|\eta^t\|^2_{\mu,H_r}))^{\frac{1}{2}}.$$

In order to facilitate the estimation, the following related theories are given

$$\begin{aligned} &\langle u_t, \omega\rangle + \langle u, \omega\rangle_{H_1} + \langle \eta^t(s), \omega\rangle_{\mu,H_1} + (\langle g(u))_x, \omega\rangle = \langle f, \omega\rangle, \\ &\langle \eta^t_t + \eta^t_s, \varphi\rangle_{\mu,H_1} = \langle u, \varphi\rangle_{\mu,H_1}; \end{aligned} \tag{2.13}$$

and for any $\omega \in H_1, \varphi \in L^2_\mu(\mathbb{R}^+; H_1)$ and a.e. $t \in I$, $p > 2$, we have

$$\begin{aligned} &u \in C(I; H_0) \cap L^2([0, T]; H_1) \cap L^p([0, T]; L^p(\Omega)); \\ &\eta^t \in C(I; L^2_\mu(\mathbb{R}^+; H_1)); \\ &\eta^t_t + \eta^t_s \in L^\infty(I; L^2_\mu(\mathbb{R}^+; H^{-1})) \cap L^2(I; L^2_\mu(\mathbb{R}^+; H_0)). \end{aligned}$$

Lemma 2.1. *Let* $I = [0, T]$, *for any* $T > 0$. *Let* μ *satisfy* (2.6)-(2.8), $\forall \eta^t \in C^1(\mathrm{I};\ \mathrm{L}^2_\mu(\mathrm{R}^+; \mathrm{H_r}))$. *Then*

$$\langle \eta^t,\ \eta^t_s\rangle_{\mu,\mathrm{H_r}} \geq \frac{\delta}{2}\|\eta^t\|^2_{\mu,\mathrm{H_r}}.$$

The relevant theory of the attractor is given below.

Definition 2.2. *Let X be the Banach space and B be the bounded set in X. $X \times X$ in a function defined on $\phi(\cdot, \cdot)$ is called a contraction function on $B \times B$, if for any sequence $\{x_n\}_{n=1}^{\infty} \subset B$, there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty} \subset \{x_n\}_{n=1}^{\infty}$ such that*

$$\lim_{k \to \infty} \lim_{l \to \infty} \phi(x_{n_k}, x_{n_l}) = 0.$$

Denote $\mathfrak{C}(B)$ for the set of contraction functions defined on $B \times B$.

Lemma 2.2. *Let $\{S(t)\}_{t \geqslant 0}$ be a semigroup on the Banach space $(X, \|\cdot\|)$ and the existence of the bounded absorption set B_0. Further, for any $\varepsilon > 0$, there exists $T = T(B_0, \varepsilon)$ and $\phi_T(\cdot, \cdot) \in \mathfrak{C}(B_0)$, such that*

$$\|S(T)x - S(T)y\| \leqslant \varepsilon + \phi_T(x, y), \qquad \forall x, \ y \in B_0, \tag{2.14}$$

where ϕ_T depends on T. Then $\{S(t)\}_t \geqslant 0$ is asymptotically compact in X, i.e. for any bounded sequence $\{y_n\}_{n=1}^{\infty} \subset X$ and $\{t_n\}$, $t_n \to \infty, \{S(t_n)y_n\}_{n=1}^{\infty}$ is relatively compact in X when $n \to \infty$.

3. Global attractor

In this section, we show that (2.4) generates a global attractor in $H_1 \times L^2_{\mu}(R^+; H_1)$.

Theorem 3.1. *(existence and uniqueness of solutions) Let (2.6)-(2.12) hold. Then for any given initial value $z_0 \in \mathcal{V}_1$ and any $T > 0$, (2.4) exists a unique solution $z = (u, \eta^t)$ in $\mathcal{V}_1$, and satisfies*

$$\begin{aligned} &u \in L^2([0,T]; H_1) \cap L^p([0,T]; L^p(\Omega)), \\ &z \in L^2([0,T]; \mathcal{V}_1) \cap L^{\infty}([0,\infty); \mathcal{V}_1), \end{aligned} \tag{3.1}$$

and the mapping $z_0 \to z(t)$ is strongly and weakly continuous in $\mathcal{V}_1$.

According to Theorem 3.1, it is possible to define a solution semigroup on the space $\mathcal{V}_1$, i.e.

$$S(t): \ \mathcal{V}_1 \to \mathcal{V}_1, z_0 = (u_0, \eta^0) \to (u(t), \eta^t) = S(t)z_0.$$

Next, the solution semigroup of (2.4)-(2.5) on space $\mathcal{V}_1$ is represented by $\{S(t)\}_{t \geqslant 0}$.

With the help of the ideas and estimation techniques in literature [13, 19], the following results can be obtained:

Theorem 3.2. *(bounded absorbing sets) Let $z(t) = (u, \eta^t), (\alpha \geq 0)$ be the problem (2.4) solution, and $\{S_{\alpha}(t)\}_{t \geq 0}$ be the corresponding semigroup. If (2.9)-(2.12) was established, then $\{S_{\alpha}(t)\}_{t \geq 0}$ has a bounded absorbing set $B_1 = B_{\mathcal{V}_1}(0, \mu_2)$ in $\mathcal{V}_1$. That is, for any bounded set B, there exists $t_1 > 0$, such that for $t \geq t_1$, we have $S_{\alpha}(t)B \subset B_1$.*

According to the fundamental theorem of the existence of global attractors for infinite-dimensional dynamical systems, it is necessary to prove the asymptotic compactness of solution semigroups $\{S(t)\}_t \geqslant 0$ in $\mathcal{V}_1$.

Theorem 3.3. *Let $\{S(t)\}_{t\geqslant 0}$ be the semigroup generated in $\mathcal{V}_1$ by the solutions of* (2.4)*. If the nonlinear term $g(u)$ satisfies the conditions* (2.9)-(2.12)*, then $\{S(t)\}_t \geqslant 0$ is asymptotically compact in $\mathcal{V}_1$.*

Proof. Let $z_m = (u_m, \eta_m^t)$ and $z_n = (u_n, \eta_n^t)$ be two solutions of (2.4)-(2.5), and that the initial values $z_m^0 = (u_m^0, \eta_m^0)$ and $z_n^0 = (u_n^0, \eta_n^0)$ belong to the bounded absorbing set B_0 of the semigroup $\{S(t)\}_{t\geqslant 0}$ in $\mathcal{V}_1$. Let $\omega = u_m - u_n$, $\xi^t = \eta_m^t - \eta_n^t$. Then by (2.4), it is obtained

$$\begin{cases}\omega_t - \Delta\omega_t - \alpha\Delta\omega - \int_0^\infty \mu(s)\Delta\xi'(s)ds + (g(u_m))_x - (g(u_n))_x = 0, \\ \omega(t) = \xi_t + \xi_s.\end{cases} \qquad (3.2)$$

Multiplying (3.2) by ω, we deduce that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\omega\|^2 + \|\nabla\omega\|^2 + \left\|\xi^t\right\|_{\mu,2}^2\right) + \alpha\|\nabla\omega\|^2 + \frac{\delta}{2}\left\|\xi^t\right\|_{\mu,2}^2 \\ &\leq -\langle g(u_m)_x - g(u_n)_x, \omega\rangle.\end{aligned} \qquad (3.3)$$

Then by (2.9)-(2.12), we obtain

$$-\big\langle g(u_m)_x - g(u_n)_x, \omega\big\rangle \leq c\|\omega\|_2^2. \qquad (3.4)$$

Therefore, we get from (3.3) and (3.4) that

$$\frac{1}{2}\frac{d}{dt}\left(\|\omega\|_2^2 + \left\|\xi^t\right\|_{\mu,2}^2\right) + \alpha\|\omega\|_2^2 + \frac{\delta}{2}\left\|\xi^t\right\|_{\mu,2}^2 \leq c\|\omega\|_2^2. \qquad (3.5)$$

Taking $\epsilon = \min\{\alpha, \frac{\delta}{2}\}$, we get that

$$\frac{d}{dt}\left(\|\omega\|_2^2 + \left\|\xi^t\right\|_{\mu,2}^2\right) + 2\epsilon\left(\|\omega\|_2^2 + \left\|\xi^t\right\|_{\mu,2}^2\right) \leq 2c\|\omega\|_2^2. \qquad (3.6)$$

We get from the Gronwall inequality that

$$\begin{aligned}&\|\omega(T)\|_2^2 + \left\|\xi^T\right\|_{\mu,2}^2 \\ &\leq e^{-2\epsilon T}\left(\|\omega_0\|_2^2 + \left\|\xi^0\right\|_{\mu,2}^2\right) + 2c\int_0^T \|\omega\|_2^2 dr.\end{aligned} \qquad (3.7)$$

Let $\phi(z_n, z_m) = 2c\int_0^T \|\omega\|_2^2$, $\phi(z_n, z_m) \in \mathfrak{C}(B_0)$. The asymptotic compactness of the solution semigroup $\{S(t)\}_{t\geqslant 0}$ is easily obtained by Lemma 2.2, so it is necessary to verify that $\phi(z_n, z_m)$ is a contraction function.

Since $H_2 \hookrightarrow H_1$, u_n is bounded on H_2 and $u_n \in C(I; H_2)$,

$$\lim_{k\to\infty}\lim_{l\to\infty}\int_0^T \|u_{n_k}(r) - u_{n_l}(r)\|_1^2\, dr = 0. \qquad (3.8)$$

So the solution semigroup $\{S(t)\}_t \geqslant 0$ obtained by (3.7)-(3.8) is asymptotically compact. □

Theorem 3.4. *Let $\{S(t)\}_{t\geq 0}$ be the semigroup of* (2.4)-(2.5)*. If the nonlinear term $g(u)$ satisfies the conditions* (2.9)-(2.12)*, and the conditions of Theorem 3.2 and Theorem 3.3 are true, then $\{S(t)\}_t \geqslant 0$ has the global attractor A_1, which attracts any bounded set of H_1 with the norm of $\|\cdot\|_{H_1}$.*

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