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A NOTE ON THE VALUATION OF AMERICAN OPTION*

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Dedicated to the 80th birthday of Professor Zhou Yulin

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Abstract American options give holder a right to exercise it at any time at will, the holder should to make the exercise policy in such a way that the expected payoff from the option will be maximized. In this note we prove that it is equivalent to a fact which makes the option value and option delta continuous.

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1. Introduction

An American put option is a contract which give the holder a right to sell the underlying assert at any time before or at the expiry date. The holder should choose best strategy to exercise the contract. It means that the holder should make his exercise policy in such a way that the expected payoff from the option will be maximized. It has been showed that the maximize principle is equivalent to the fact which makes the option value and option delta continuous in the case of the perpetual American put option in which the option pricing can be found in a close form [1].

In this note we prove that the equivalence is still true for American options with a finite time to expiry. We claim that for American put option the option value is maximized if and only if the owner of option exercises such that $\Delta = \frac{\partial V}{\partial s}$ is continuous.

2. Assumptions and Basic Facts

In the risk neutral world, the underlying assert price S_t is assumed to follow the lognormal diffusion process

$$\frac{dS_t}{S_t} = (r - q)dt + \sigma dW_t$$

where r and q are the constant riskless interest rate and dividend yield, respectively, σ is the constant volatility and dW_t is the standard Wiener process (See [1] and [2])

$$E(dW_t) = 0,$$

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$$Var(dW_t) = dt.$$

Now let us consider an American option on the underlying asset S_t. American option can be exercised at any time up to expiration date. It gives holder a right to do something, but holder does not have to exercise this right. So threr is a cost to entering into an option contract(the holder has to pay premium for the contract initially). This cost is the price of American option.

How to define the price of an American option? It is well known that the value of European option is governed by Black-Scholes equation ([1] and [2]). In view of probability theory, the pricing problem of American optionis can be reduced to a optimal stopping problem and then a free boundary problem of a parabolic partial differential equation [3]. In this paper, we prove that the pricing problem of American optionis is equivalent to free boundary problem by using the PDE approach. We consider the American put option only.

Let K be the striking price of option, s be the price of stock and $V^*(s,t)$ be the value of American put option with payoff $(K-s)^+$ until time expiry T. According to arbitrage-free principle, we conclude that the price of American option $V^*(s,t)$ is a nonnegative continuous function. For American option with same strike price, the owner of long-life option has all the exercise opportunities open to the owner of short-life option and more, so the value of option is monotone increase with respect to the life time of the option $T-t$ (Ch.8 in [4]). Hence $V^*(s,t)$ is monotone decreasing function of t. Mathematically, we have

$$\begin{aligned}
&V^*(s,T) = (K-s)^+\\
&V^*(0,t) = K\\
&V^*(\infty,t) = 0.
\end{aligned}$$

And arbitrage-free principle implies

$$(K-s)^+ \le V^*(s,t) \le K.$$

For American put option, based on Δ-hedging argument and Ito's lemma, we deduce that $V^*(s,t)$ satisfies the following Black-Scholes equation [1]

$$\mathcal{L}V = \frac{\partial V}{\partial t} + \frac{\sigma^2}{2}s^2\frac{\partial^2 V}{\partial s^2} + (r-q)s\frac{\partial V}{\partial s} - rV = 0, \qquad (1)$$

in the continuation region

$$\mathcal{D}^* = \{(s,t) \mid V^*(s,t) > (K-s)^+\}.$$

Let

$$s^*(t) = \sup\{s \in [0,+\infty) | V^*(s,t) = (K-s)^+\}, \quad t \in [0,T).$$