JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **39**, No. 1, pp. 1-30

doi: 10.4208/jpde.v39.n1.1
March 2026

# Eigenvalue Problem for a Class of Nonlinear Operators Containing $p(\cdot)$-Laplacian in a Variable Exponent Sobolev Space

ARAMAKI Junichi*

*Division of Science, Faculty of Science and Engineering, Tokyo Denki University, Hatoyama-machi, Saitama, 350-0394, Japan.*

Received 8 October 2023; Accepted 6 July 2024

**Abstract.** In this paper, we consider an eigenvalue problem for a class of nonlinear operators containing $p(\cdot)$-Laplacian and mean curvature operator with mixed boundary conditions. More precisely, we are concerned with the problem with the Dirichlet condition on a part of the boundary and the Steklov boundary condition on an another part of the boundary. We show that the eigenvalue problem has infinitely many eigenpairs by using the celebrated Ljusternik-Schnirelmann principle of the calculus of variation. Moreover, in a variable exponent Sobolev space, there are two cases where the infimum of all eigenvalues is equal to zero and is positive.

**AMS Subject Classifications**: 49R50, 35A01, 35J62, 35J57

**Chinese Library Classifications**: O175.27

**Key Words**: Eigenvalue problem; $p(\cdot)$-Laplacian; mean curvature operator; mixed boundary value problem; variable exponent Sobolev space.

## 1 Introduction

In this paper, we consider the following eigenvalue problem with mixed boundary conditions

$$\begin{cases} -\mathrm{div}[\boldsymbol{a}(x,\nabla u(x))]=0, & \text{in } \Omega, \\ u(x)=0, & \text{on } \Gamma_1, \\ \boldsymbol{n}(x)\cdot\boldsymbol{a}(x,\nabla u(x))=\lambda g(x,u(x)), & \text{on } \Gamma_2. \end{cases} \tag{1.1}$$

*Corresponding author. *Email address:* `aramaki@hctv.ne.jp` (J. Aramaki)

Here $\Omega$ is a bounded domain of $\mathbb{R}^N$ $(N\geq 2)$ with a Lipschitz-continuous $\left(C^{0,1}\text{ for short}\right)$ boundary $\Gamma$ satisfying that

$$\Gamma_1 \text{ and } \Gamma_2 \text{ are disjoint non-empty open subsets of } \Gamma \text{ such that } \overline{\Gamma_1}\cup\overline{\Gamma_2}=\Gamma, \tag{1.2}$$

and the vector field $\boldsymbol{n}$ denotes the unit, outer, normal vector to $\Gamma$. The function $\boldsymbol{a}(x,\boldsymbol{\xi})$ is a Carathéodory function on $\Omega\times\mathbb{R}^N$ satisfying some structure conditions associated with an anisotropic exponent function $p(x)$. Here we say that $\boldsymbol{a}(x,\boldsymbol{\xi})$ is a Carathéodory function on $\Omega\times\mathbb{R}^N$, if for a.e. $x\in\Omega$, the map $\mathbb{R}^N\ni\boldsymbol{\xi}\mapsto\boldsymbol{a}(x,\boldsymbol{\xi})$ is continuous and for every $\boldsymbol{\xi}\in\mathbb{R}^N$, the map $\Omega\ni x\mapsto\boldsymbol{a}(x,\boldsymbol{\xi})$ is measurable on $\Omega$. The operator $u\mapsto\operatorname{div}[\boldsymbol{a}(x,\nabla u(x))]$ is more general than the $p(\cdot)$-Laplacian $\Delta_{p(x)}u(x)=\operatorname{div}\left[|\nabla u(x)|^{p(x)-2}\nabla u(x)\right]$ and the mean curvature operator $\operatorname{div}\left[\left(1+|\nabla u(x)|^2\right)^{(p(x)-2)/2}\nabla u(x)\right]$. This generality brings about difficulties and requires some conditions.

We impose the mixed boundary conditions, that is, the Dirichlet condition on $\Gamma_1$ and the Steklov condition on $\Gamma_2$. The given data $g:\Gamma_2\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function satisfying some structure conditions and $\lambda$ is a real number.

The study of differential equations with $p(\cdot)$-growth conditions is a very interesting topic recently. Studying such problem stimulated its application in mathematical physics, in particular, in elastic mechanics (Zhikov [1]), in electrorheological fluids (Diening [2], Halsey [3], Mihăilescu and Rădulescu [4], Růžička [5]).

However, since we find a few papers associate with the problem with the mixed boundary condition in variable exponent Sobolev space as in (1.1) (for example, Aramaki [6,7]). We are convinced of the reason for existence of this paper.

The purpose of this paper is to solve eigenvalue problem (1.1) for a class of operators containing $p(\cdot)$-Laplacian and the mean curvature operator. According to some assumptions on $g$, we use the Ljusternik-Schnirelmann principle in the constrained variational method. See Ljusternik and Schnirelmann [8] and Szulkin [9].

When $p(x)\equiv p=\text{const.}$, there are many articles for the $p$-Laplacian. For example, see Lê [10], Anane [11], Friedlander [12]. For the $p$-Laplacian Dirichlet eigenvalue problem, we can see the following properties hold.

(1) There exists a nondecreasing sequence of non-negative eigenvalues $\{\lambda_n\}$ tending to $\infty$ as $n\to\infty$.

(2) The first eigenvalue $\lambda_1$ is simple and only eigenfunctions associated with $\lambda_1$ do not change sign.

(3) The set of eigenvalues is closed.

(4) The first eigenvalue $\lambda_1$ is isolated.

On the contrary, recently many authors study the $p(\cdot)$-Laplacian. In particular, Fan [13] has studied the eigenvalue problem for the $p(\cdot)$-Laplacian with zero Neumann boundary condition in a bounded domain, and Fan et. al. [14] has studied the eigenvalue