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Eigenvalue Problem for a Class of Nonlinear Operators Containing $p(\cdot)$-Laplacian in a Variable Exponent Sobolev Space

ARAMAKI Junichi*

Division of Science, Faculty of Science and Engineering, Tokyo Denki University, Hatoyama-machi, Saitama, 350-0394, Japan.

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Abstract. In this paper, we consider an eigenvalue problem for a class of nonlinear operators containing $p(\cdot)$-Laplacian and mean curvature operator with mixed boundary conditions. More precisely, we are concerned with the problem with the Dirichlet condition on a part of the boundary and the Steklov boundary condition on an another part of the boundary. We show that the eigenvalue problem has infinitely many eigenpairs by using the celebrated Ljusternik-Schnirelmann principle of the calculus of variation. Moreover, in a variable exponent Sobolev space, there are two cases where the infimum of all eigenvalues is equal to zero and is positive.

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1 Introduction

In this paper, we consider the following eigenvalue problem with mixed boundary conditions

$$\begin{cases} -\operatorname{div}[\boldsymbol{a}(x,\nabla u(x))]=0, & \text{in } \Omega, \\ u(x)=0, & \text{on } \Gamma_1, \\ \boldsymbol{n}(x)\cdot\boldsymbol{a}(x,\nabla u(x))=\lambda g(x,u(x)), & \text{on } \Gamma_2. \end{cases} \tag{1.1}$$

*Corresponding author. *Email address:* aramaki@hctv.ne.jp (J. Aramaki)

Here Ω is a bounded domain of $\mathbb{R}^N$ $(N\geq 2)$ with a Lipschitz-continuous $\left(C^{0,1}\text{ for short}\right)$ boundary Γ satisfying that

$$\Gamma_1 \text{ and } \Gamma_2 \text{ are disjoint non-empty open subsets of } \Gamma \text{ such that } \overline{\Gamma_1}\cup\overline{\Gamma_2}=\Gamma, \tag{1.2}$$

and the vector field $\boldsymbol{n}$ denotes the unit, outer, normal vector to Γ. The function $\boldsymbol{a}(x,\boldsymbol{\xi})$ is a Carathéodory function on $\Omega\times\mathbb{R}^N$ satisfying some structure conditions associated with an anisotropic exponent function $p(x)$. Here we say that $\boldsymbol{a}(x,\boldsymbol{\xi})$ is a Carathéodory function on $\Omega\times\mathbb{R}^N$, if for a.e. $x\in\Omega$, the map $\mathbb{R}^N\ni\boldsymbol{\xi}\mapsto\boldsymbol{a}(x,\boldsymbol{\xi})$ is continuous and for every $\boldsymbol{\xi}\in\mathbb{R}^N$, the map $\Omega\ni x\mapsto\boldsymbol{a}(x,\boldsymbol{\xi})$ is measurable on Ω. The operator $u\mapsto \operatorname{div}[\boldsymbol{a}(x,\nabla u(x))]$ is more general than the $p(\cdot)$-Laplacian $\Delta_{p(x)}u(x)=\operatorname{div}\left[|\nabla u(x)|^{p(x)-2}\nabla u(x)\right]$ and the mean curvature operator $\operatorname{div}\left[\left(1+|\nabla u(x)|^2\right)^{(p(x)-2)/2}\nabla u(x)\right]$. This generality brings about difficulties and requires some conditions.

We impose the mixed boundary conditions, that is, the Dirichlet condition on Γ_1 and the Steklov condition on Γ_2. The given data $g:\Gamma_2\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function satisfying some structure conditions and λ is a real number.

The study of differential equations with $p(\cdot)$-growth conditions is a very interesting topic recently. Studying such problem stimulated its application in mathematical physics, in particular, in elastic mechanics (Zhikov [1]), in electrorheological fluids (Diening [2], Halsey [3], Mihăilescu and Rădulescu [4], Růžička [5]).

However, since we find a few papers associate with the problem with the mixed boundary condition in variable exponent Sobolev space as in (1.1) (for example, Aramaki [6,7]). We are convinced of the reason for existence of this paper.

The purpose of this paper is to solve eigenvalue problem (1.1) for a class of operators containing $p(\cdot)$-Laplacian and the mean curvature operator. According to some assumptions on g, we use the Ljusternik-Schnirelmann principle in the constrained variational method. See Ljusternik and Schnirelmann [8] and Szulkin [9].

When $p(x)\equiv p=\text{const.}$, there are many articles for the p-Laplacian. For example, see Lê [10], Anane [11], Friedlander [12]. For the p-Laplacian Dirichlet eigenvalue problem, we can see the following properties hold.

(1) There exists a nondecreasing sequence of non-negative eigenvalues $\{\lambda_n\}$ tending to ∞ as $n\to\infty$.

(2) The first eigenvalue λ_1 is simple and only eigenfunctions associated with λ_1 do not change sign.

(3) The set of eigenvalues is closed.

(4) The first eigenvalue λ_1 is isolated.

On the contrary, recently many authors study the $p(\cdot)$-Laplacian. In particular, Fan [13] has studied the eigenvalue problem for the $p(\cdot)$-Laplacian with zero Neumann boundary condition in a bounded domain, and Fan et. al. [14] has studied the eigenvalue problem for the $p(\cdot)$-Laplacian Dirichlet problem. Mihailescu and Radulescu [15] have studied nonhomogeneous quasilinear eigenvalue problem with variable exponent. In Deng [16], the author treats only the $p(\cdot)$-Laplacian in the case $\Gamma_1 = \emptyset$. This paper is an extension to more general class of operators than [16].

In this paper, we will deal with the mixed boundary value eigenvalue problem (1.1) for a class of operators involving the $p(\cdot)$-Laplacian and the mean curvature operator which is a new topic. We will show that there exist infinitely many eigenvalues $\{\lambda_{(n,\alpha)}\}$ tending to ∞ as $n \to \infty$ for fixed $\alpha > 0$. Moreover, we will derive that under some condition, the infimum λ_* of all eigenvalues of (1.1) is equal to zero, so there does not exist a principal eigenvalue and the set of eigenvalues is not closed. We also show that under some condition on the function g in (1.1), there is a case where λ_* is positive.

The paper is organized as follows. In Section 2, we recall some results on variable exponent Lebesgue-Sobolev spaces. In Section 3, we give the setting of problem (1.1) rigorously and a main theorem (Theorem 3.1) on the eigenvalue problem (1.1). In Section 4, we present some sufficient conditions for $\lambda_* = 0$ and $\lambda_* > 0$, respectively.

2 Preliminaries

Throughout this paper, let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with a $C^{0,1}$-boundary Γ and Ω is locally on the same side of Γ. Moreover, we assume that Γ satisfies (1.2).

In the present paper, we only consider real vector spaces of real valued functions over $\mathbb{R}$. For any space B, we denote B^N by the boldface character $\boldsymbol{B}$. Hereafter, we use this character to denote vectors and vector-valued functions, and we denote the standard inner product of vectors $\boldsymbol{a} = (a_1, \cdots, a_N)$ and $\boldsymbol{b} = (b_1, \cdots, b_N)$ in $\mathbb{R}^N$ by $\boldsymbol{a} \cdot \boldsymbol{b} = \sum_{i=1}^N a_i b_i$ and $|\boldsymbol{a}| = (\boldsymbol{a} \cdot \boldsymbol{a})^{1/2}$. Furthermore, we denote the dual space of B by B^* and the duality bracket by $\langle \cdot, \cdot \rangle_{B^*,B}$.

We recall some well-known results on variable exponent Lebesgue and Sobolev spaces. See Fan and Zhang [17], Kováčik and Rácosník [18], Diening et al. [19] and references therein for more detail. Furthermore, we consider some new properties on variable exponent Lebesgue space. Define $C(\overline{\Omega}) = \{p; p \text{ is a continuous function on } \overline{\Omega}\}$, and for any $p \in C(\overline{\Omega})$, put

$$p^+ = p^+(\Omega) = \sup_{x \in \Omega} p(x) \quad \text{and} \quad p^- = p^-(\Omega) = \inf_{x \in \Omega} p(x).$$

For any $p \in C(\overline{\Omega})$ with $p^- \geq 1$ and for any measurable function u on Ω, a modular (for this notation, see [19, Definition 2.1.1]) $\rho_{p(\cdot)} = \rho_{p(\cdot),\Omega}$ is defined by

$$\rho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)} \mathrm{d}x.$$

The variable exponent Lebesgue space is defined by

$$L^{p(\cdot)}(\Omega) = \{u; u : \Omega \to \mathbb{R} \text{ is a measurable function satisfying } \rho_{p(\cdot)}(u) < \infty\}$$

equipped with the (Luxemburg) norm

$$\|u\|_{L^{p(\cdot)}(\Omega)}=\inf\left\{\tau>0;\, \rho_{p(\cdot)}\left(\frac{u}{\tau}\right)\le 1\right\}.$$

Then $L^{p(\cdot)}(\Omega)$ is a Banach space. We also define, for any integer $m\ge 0$,

$$W^{m,p(\cdot)}(\Omega)=\left\{u\in L^{p(\cdot)}(\Omega);\, \partial^\alpha u\in L^{p(\cdot)}(\Omega) \text{ for } |\alpha|\le m\right\},$$

where $\alpha=(\alpha_1,\cdots,\alpha_N)$ is a multi-index, $|\alpha|=\sum_{i=1}^N\alpha_i$, $\partial^\alpha=\partial_1^{\alpha_1}\cdots\partial_N^{\alpha_N}$ and $\partial_i=\partial/\partial x_i$, endowed with the norm

$$\|u\|_{W^{m,p(\cdot)}(\Omega)}=\sum_{|\alpha|\le m}\|\partial^\alpha u\|_{L^{p(\cdot)}(\Omega)}.$$

Of course, $W^{0,p(\cdot)}(\Omega)=L^{p(\cdot)}(\Omega)$.

The following three propositions are well known ([14], Fan and Zhao [20], Zhao et al. [21]).

Proposition 2.1. *Let $p\in C(\overline{\Omega})$ with $p^-\ge 1$, and let $u,u_n\in L^{p(\cdot)}(\Omega)$ $(n=1,\cdots)$. Then we have the following properties.*

(i) $\|u\|_{L^{p(\cdot)}(\Omega)}<1(=1,>1)\Longleftrightarrow \rho_{p(\cdot)}(u)<1(=1,>1)$.

(ii) $\|u\|_{L^{p(\cdot)}(\Omega)}>1\Longrightarrow \|u\|^{p^-}_{L^{p(\cdot)}(\Omega)}\le\rho_{p(\cdot)}(u)\le\|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}$.

(iii) $\|u\|_{L^{p(\cdot)}(\Omega)}<1\Longrightarrow \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}\le\rho_{p(\cdot)}(u)\le\|u\|^{p^-}_{L^{p(\cdot)}(\Omega)}$.

(iv) $\lim_{n\to\infty}\|u_n-u\|_{L^{p(\cdot)}(\Omega)}=0\Longleftrightarrow\lim_{n\to\infty}\rho_{p(\cdot)}(u_n-u)=0$.

(v) $\|u_n\|_{L^{p(\cdot)}(\Omega)}\to\infty$ as $n\to\infty\Longleftrightarrow\rho_{p(\cdot)}(u_n)\to\infty$ as $n\to\infty$.

The following proposition is a generalized Hölder inequality.

Proposition 2.2. *Let $p\in C_+(\overline{\Omega})$, where $C_+(\overline{\Omega}):=\left\{p\in C(\overline{\Omega});p^->1\right\}$. For any $u\in L^{p(\cdot)}(\Omega)$ and $v\in L^{p'(\cdot)}(\Omega)$, we have*

$$\int_\Omega|u(x)v(x)|\mathrm{d}x\le\left(\frac{1}{p^-}+\frac{1}{(p')^-}\right)\|u\|_{L^{p(\cdot)}(\Omega)}\|v\|_{L^{p'(\cdot)}(\Omega)}\le 2\|u\|_{L^{p(\cdot)}(\Omega)}\|v\|_{L^{p'(\cdot)}(\Omega)}.$$

Here and from now on, for any $p\in C_+(\overline{\Omega})$, $p'(\cdot)$ denotes the conjugate exponent of $p(\cdot)$, that is, $p'(x)=p(x)/(p(x)-1)$ for $x\in\overline{\Omega}$.

For $p\in C_+(\overline{\Omega})$, define

$$p^*(x)=\begin{cases}\dfrac{Np(x)}{N-p(x)}, & \text{if } p(x)<N,\\ \infty, & \text{if } p(x)\ge N.\end{cases}$$

Proposition 2.3. *Let Ω be a bounded domain of $\mathbb{R}^N$ with $C^{0,1}$-boundary and let $p\in C_+(\overline{\Omega})$ and $m\geq 0$ be an integer. Then we have the following properties.*

(i) *The spaces $L^{p(\cdot)}(\Omega)$ and $W^{m,p(\cdot)}(\Omega)$ are separable, reflexive and uniformly convex Banach spaces.*

(ii) *If $q(\cdot)\in C(\overline{\Omega})$ with $q^-\geq 1$ satisfies $q(x)\leq p(x)$ for all $x\in\Omega$, then $W^{m,p(\cdot)}(\Omega)\hookrightarrow W^{m,q(\cdot)}(\Omega)$, where $\hookrightarrow$ means that the embedding is continuous.*

(iii) *If $q(x)\in C(\overline{\Omega})$ with $q^-\geq 1$ satisfies that $q(x)<p^*(x)$ for all $x\in\Omega$, then the embedding $W^{1,p(\cdot)}(\Omega)\hookrightarrow L^{q(\cdot)}(\Omega)$ is compact.*

Next we consider the trace (Fan [22]). Let Ω be a bounded domain of $\mathbb{R}^N$ with a $C^{0,1}$-boundary Γ and $p\in C(\overline{\Omega})$ with $p^-\geq 1$. Since $W^{1,p(\cdot)}(\Omega)\subset W^{1,1}(\Omega)$, the trace $\gamma(u)=u\big|_\Gamma$ to Γ of any function u in $W^{1,p(\cdot)}(\Omega)$ is well defined as a function in $L^1(\Gamma)$. We define

$$\mathrm{Tr}\left(W^{1,p(\cdot)}(\Omega)\right)=\left(\mathrm{Tr}W^{1,p(\cdot)}\right)(\Gamma)=\left\{f;f \text{ is the trace to } \Gamma \text{ of a function } F\in W^{1,p(\cdot)}(\Omega)\right\}$$

equipped with the norm

$$\|f\|_{(\mathrm{Tr}W^{1,p(\cdot)})(\Gamma)}=\inf\left\{\|F\|_{W^{1,p(\cdot)}(\Omega)};F\in W^{1,p(\cdot)}(\Omega) \text{ satisfying } F\big|_\Gamma=f\right\}$$

for $f\in\left(\mathrm{Tr}W^{1,p(\cdot)}\right)(\Gamma)$, where the infimum can be achieved. Then we can see that the space $\left(\mathrm{Tr}W^{1,p(\cdot)}\right)(\Gamma)$ is a Banach space. In the later we also write $F\big|_\Gamma=g$ by $F=g$ on Γ. Moreover, for $i=1,2$, we denote

$$\left(\mathrm{Tr}W^{1,p(\cdot)}\right)(\Gamma_i)=\left\{f\big|_{\Gamma_i};f\in\left(\mathrm{Tr}W^{1,p(\cdot)}\right)(\Gamma)\right\}$$

equipped with the norm

$$\|g\|_{(\mathrm{Tr}W^{1,p(\cdot)})(\Gamma_i)}=\inf\left\{\|f\|_{(\mathrm{Tr}W^{1,p(\cdot)})(\Gamma)};f\in\left(\mathrm{Tr}W^{1,p(\cdot)}\right)(\Gamma) \text{ satisfying } f\big|_{\Gamma_i}=g\right\},$$

where the infimum can also be achieved, so for any $g\in\left(\mathrm{Tr}W^{1,p(\cdot)}\right)(\Gamma_i)$, there exists $F\in W^{1,p(\cdot)}(\Omega)$ such that $F\big|_{\Gamma_i}=g$ and $\|F\|_{W^{1,p(\cdot)}(\Omega)}=\|g\|_{(\mathrm{Tr}W^{1,p(\cdot)})(\Gamma_i)}$.

Let $q\in C_+(\Gamma):=\{q\in C(\Gamma);q^->1\}$ and denote the surface measure on Γ induced from the Lebesgue measure $\mathrm{d}x$ on Ω by $\mathrm{d}\sigma_x$. We define

$$L^{q(\cdot)}(\Gamma)=\Big\{u;\, u:\Gamma\to\mathbb{R} \text{ is a measurable function with respect to } d\sigma \\ \text{satisfying } \int_\Gamma |u(x)|^{q(x)}\mathrm{d}\sigma_x<\infty\Big\}$$

and the norm is defined by

$$\|u\|_{L^{q(\cdot)}(\Gamma)}=\inf\left\{\tau>0;\int_\Gamma\left|\frac{u(x)}{\tau}\right|^{q(x)}\mathrm{d}\sigma_x\le 1\right\},$$

and we also define a modular on $L^{q(\cdot)}(\Gamma)$ by

$$\rho_{q(\cdot),\Gamma}(u)=\int_\Gamma|u(x)|^{q(x)}\mathrm{d}\sigma_x.$$

Similarly as Proposition 2.1, we have the following proposition.

Proposition 2.4. *Let $q\in C(\Gamma)$ with $q^-\ge 1$, and let $u,u_n\in L^{q(\cdot)}(\Gamma)$. Then we have the following properties.*

(i) $\|u\|_{L^{q(\cdot)}(\Gamma)}<1(=1,>1)\Longleftrightarrow\rho_{q(\cdot),\Gamma}(u)<1(=1,>1)$.

(ii) $\|u\|_{L^{q(\cdot)}(\Gamma)}>1\Longrightarrow\|u\|^{q^-}_{L^{q(\cdot)}(\Gamma)}\le\rho_{q(\cdot),\Gamma}(u)\le\|u\|^{q^+}_{L^{q(\cdot)}(\Gamma)}$.

(iii) $\|u\|_{L^{q(\cdot)}(\Gamma)}<1\Longrightarrow\|u\|^{q^+}_{L^{q(\cdot)}(\Gamma)}\le\rho_{q(\cdot),\Gamma}(u)\le\|u\|^{q^-}_{L^{q(\cdot)}(\Gamma)}$.

(iv) $\|u_n\|_{L^{q(\cdot)}(\Gamma)}\to 0\Longleftrightarrow\rho_{q(\cdot),\Gamma}(u_n)\to 0$.

(v) $\|u_n\|_{L^{q(\cdot)}(\Gamma)}\to\infty\Longleftrightarrow\rho_{q(\cdot),\Gamma}(u_n)\to\infty$.

The Hölder inequality also holds for functions on Γ.

Proposition 2.5. *Let $q\in C(\Gamma)$ with $q^->1$. Then the following inequality holds.*

$$\int_\Gamma|f(x)g(x)|\mathrm{d}\sigma_x\le 2\|f\|_{L^{q(\cdot)}(\Gamma)}\|g\|_{L^{q'(\cdot)}(\Gamma)}\quad\text{for all }\ f\in L^{q(\cdot)}(\Gamma),\ g\in L^{q'(\cdot)}(\Gamma).$$

Proposition 2.6. *Let Ω be a bounded domain of $\mathbb{R}^N$ with a $C^{0,1}$-boundary Γ and let $p\in C_+(\overline{\Omega})$. If $f\in\left(\mathrm{Tr}W^{1,p(\cdot)}\right)(\Gamma)$, then $f\in L^{p(\cdot)}(\Gamma)$ and there exists a constant $C>0$ such that*

$$\|f\|_{L^{p(\cdot)}(\Gamma)}\le C\|f\|_{(\mathrm{Tr}W^{1,p(\cdot)})(\Gamma)}.$$

In particular, If $f\in\left(\mathrm{Tr}W^{1,p(\cdot)}\right)(\Gamma)$, then $f\in L^{p(\cdot)}(\Gamma_i)$ and

$$\|f\|_{L^{p(\cdot)}(\Gamma_i)}\le C\|f\|_{(\mathrm{Tr}W^{1,p(\cdot)})(\Gamma)}\quad fori=1,2.$$

For $p\in C_+(\overline{\Omega})$, define

$$p^\partial(x)=\begin{cases}\dfrac{(N-1)p(x)}{N-p(x)}, & \text{if } p(x)<N,\\ \infty, & \text{if } p(x)\ge N.\end{cases}$$

The following proposition follows from Yao [23, Proposition 2.6].

Proposition 2.7. *Let $p\in C_+(\overline{\Omega})$. Then if $q\in C_+(\Gamma)$ satisfies $q(x)<p^\partial(x)$ for all $x\in\Gamma$, then the trace mapping $W^{1,p(\cdot)}(\Omega)\to L^{q(\cdot)}(\Gamma)$ is well-defined and compact. In particular, the trace mapping $W^{1,p(\cdot)}(\Omega)\to L^{p(\cdot)}(\Gamma)$ is compact and there exists a constant $C>0$ such that*

$$\|u\|_{L^{p(\cdot)}(\Gamma)}\le C\|u\|_{W^{1,p(\cdot)}(\Omega)} \quad \text{for } u\in W^{1,p(\cdot)}(\Omega).$$

Now we consider the weighted variable exponent Lebesgue space. Let $p\in C(\overline{\Omega})$ with $p^-\ge 1$ and let $a(x)$ be a measurable function on Ω with $a(x)>0$ a.e. $x\in\Omega$. We define a modular

$$\rho_{(p(\cdot),a(\cdot))}(u)=\int_\Omega a(x)|u(x)|^{p(x)}\mathrm{d}x$$

for any measurable function u in Ω . Then the weighted Lebesgue space is defined by

$$L^{p(\cdot)}_{a(\cdot)}(\Omega)=\{u;u \text{ is a measurable function on } \Omega \text{ satisfying } \rho_{(p(\cdot),a(\cdot))}(u)<\infty\}$$

equipped with the norm

$$\|u\|_{L^{p(\cdot)}_{a(\cdot)}(\Omega)}=\inf\left\{\tau>0;\int_\Omega a(x)\left|\frac{u(x)}{\tau}\right|^{p(x)}\mathrm{d}x\le 1\right\}.$$

Then $L^{p(\cdot)}_{a(\cdot)}(\Omega)$ is a Banach space.

We have the following proposition (Fan [24, Proposition 2.5]).

Proposition 2.8. *Let $p\in C(\overline{\Omega})$ with $p^-\ge 1$. For u, $u_n\in L^{p(\cdot)}_{a(\cdot)}(\Omega)$, we have the following.*

(i) *For $u\neq 0$, $\|u\|_{L^{p(\cdot)}_{a(\cdot)}(\Omega)}=\tau\Longleftrightarrow\rho_{(p(\cdot),a(\cdot))}\left(\frac{u}{\tau}\right)=1$.*

(ii) $\|u\|_{L^{p(\cdot)}_{a(\cdot)}(\Omega)}<1(=1,>1)\Longleftrightarrow\rho_{(p(\cdot),a(\cdot))}(u)<1(=1,>1)$.

(iii) $\|u\|_{L^{p(\cdot)}_{a(\cdot)}(\Omega)}>1\Longrightarrow\|u\|^{p^-}_{L^{p(\cdot)}_{a(\cdot)}(\Omega)}\le\rho_{(p(\cdot),a(\cdot))}(u)\le\|u\|^{p^+}_{L^{p(\cdot)}_{a(\cdot)}(\Omega)}$.

(iv) $\|u\|_{L^{p(\cdot)}_{a(\cdot)}(\Omega)}<1\Longrightarrow\|u\|^{p^+}_{L^{p(\cdot)}_{a(\cdot)}(\Omega)}\le\rho_{(p(\cdot),a(\cdot))}(u)\le\|u\|^{p^-}_{L^{p(\cdot)}_{a(\cdot)}(\Omega)}$.

(v) $\lim_{n\to\infty}\|u_n-u\|_{L^{p(\cdot)}_{a(\cdot)}(\Omega)}=0\Longleftrightarrow\lim_{n\to\infty}\rho_{(p(\cdot),a(\cdot))}(u_n-u)=0$.

(vi) $\|u_n\|_{L^{p(\cdot)}_{a(\cdot)}(\Omega)}\to\infty$ as $n\to\infty\Longleftrightarrow\rho_{(p(\cdot),a(\cdot))}(u_n)\to\infty$ as $n\to\infty$.

The author of [24] also derived the following proposition ([24, Theorem 2.1]).

Proposition 2.9. *Let Ω be a bounded domain of $\mathbb{R}^N$ with a $C^{0,1}$-boundary and $p\in C_+(\overline{\Omega})$. Moreover, let $a\in L^{\alpha(\cdot)}(\Omega)$ satisfy $a(x)>0$ a.e. $x\in\Omega$ and $\alpha\in C_+(\overline{\Omega})$. If $q\in C(\overline{\Omega})$ satisfies*

$$1\le q(x)<\frac{\alpha(x)-1}{\alpha(x)}p^*(x) \quad \text{for all } x\in\overline{\Omega},$$

then the embedding $W^{1,p(\cdot)}(\Omega)\hookrightarrow L^{q(\cdot)}_{a(\cdot)}(\Omega)$ is compact.

Similarly, let $q\in C(\Gamma)$ with $q^-\geq 1$ and let $b(x)$ be a measurable function with respect to σ on Γ with $b(x)>0$ σ-a.e. $x\in\Gamma$. We define a modular

$$\rho_{(q(\cdot),b(\cdot)),\Gamma}(u)=\int_\Gamma b(x)|u(x)|^{q(x)}\mathrm{d}\sigma_x.$$

Then the weighted Lebesgue space on Γ is defined by

$$L^{q(\cdot)}_{b(\cdot)}(\Gamma)=\{u;u \text{ is a } \sigma\text{-measurable function on } \Gamma \text{ satisfying } \rho_{(q(\cdot),b(\cdot)),\Gamma}(u)<\infty\}$$

equipped with the norm

$$\|u\|_{L^{q(\cdot)}_{b(\cdot)}(\Gamma)}=\inf\left\{\tau>0;\int_\Gamma b(x)\left|\frac{u(x)}{\tau}\right|^{q(x)}\mathrm{d}\sigma_x\leq 1\right\}.$$

Then $L^{q(\cdot)}_{b(\cdot)}(\Gamma)$ is a Banach space.

Then we have the following proposition.

Proposition 2.10. *Let $q\in C(\Gamma)$ with $q^-\geq 1$. For u, $u_n\in L^{q(\cdot)}_{b(\cdot)}(\Gamma)$, we have the following.*

(i) $\|u\|_{L^{q(\cdot)}_{b(\cdot)}(\Gamma)}<1(=1,>1)\Longleftrightarrow\rho_{(q(\cdot),b(\cdot)),\Gamma}(u)<1(=1,>1)$.

(ii) $\|u\|_{L^{q(\cdot)}_{b(\cdot)}(\Gamma)}>1\Longrightarrow\|u\|^{q^-}_{L^{q(\cdot)}_{b(\cdot)}(\Gamma)}\leq\rho_{(q(\cdot),b(\cdot)),\Gamma}(u)\leq\|u\|^{q^+}_{L^{q(\cdot)}_{b(\cdot)}(\Gamma)}$.

(iii) $\|u\|_{L^{q(\cdot)}_{b(\cdot)}(\Gamma)}<1\Longrightarrow\|u\|^{q^+}_{L^{q(\cdot)}_{b(\cdot)}(\Gamma)}\leq\rho_{(q(\cdot),b(\cdot)),\Gamma}(u)\leq\|u\|^{q^-}_{L^{q(\cdot)}_{b(\cdot)}(\Gamma)}$.

(iv) $\lim_{n\to\infty}\|u_n-u\|_{L^{q(\cdot)}_{b(\cdot)}(\Gamma)}=0\Longleftrightarrow\lim_{n\to\infty}\rho_{(q(\cdot),b(\cdot)),\Gamma}(u_n-u)=0$.

(v) $\|u_n\|_{L^{q(\cdot)}_{b(\cdot)}(\Gamma)}\to\infty$ as $n\to\infty\Longleftrightarrow\rho_{(q(\cdot),b(\cdot)),\Gamma}(u_n)\to\infty$ as $n\to\infty$.

The following proposition plays an important role in the present paper.

Proposition 2.11. *Let Ω be a bounded domain of $\mathbb{R}^N$ with a $C^{0,1}$-boundary Γ and let $p\in C_+(\overline{\Omega})$. Assume that $0<b\in L^{\beta(\cdot)}(\Gamma)$, $\beta\in C_+(\Gamma)$. If $r\in C(\Gamma)$ satisfies*

$$1\leq r(x)<\frac{\beta(x)-1}{\beta(x)}p^\partial(x)\quad\text{for all }\ x\in\Gamma,$$

then the embedding $W^{1,p(\cdot)}(\Omega)\hookrightarrow L^{r(\cdot)}_{b(\cdot)}(\Gamma)$ is compact.

Proof. Let $u\in W^{1,p(\cdot)}(\Omega)$. Set $h(x)=\beta'(x)r(x)$. From the hypothesis, we have $h(x)<p^\partial(x)$ for all $x\in\Gamma$. By Proposition 2.7, the embedding $W^{1,p(\cdot)}(\Omega)\hookrightarrow L^{h(\cdot)}(\Gamma)$ is compact. Since $|u(x)|^{r(x)}\in L^{\beta'(\cdot)}(\Gamma)$, it follows from the Hölder inequality (Proposition 2.5) that

$$\int_\Gamma b(x)|u(x)|^{r(x)}\mathrm{d}\sigma_x\leq 2\|b\|_{L^{\beta(\cdot)}(\Gamma)}\||u|^{r(\cdot)}\|_{L^{\beta'(\cdot)}(\Gamma)}<\infty.$$

Hence $W^{1,p(\cdot)}(\Omega) \subset L^{r(\cdot)}_{b(\cdot)}(\Gamma)$. We show that the embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}_{b(\cdot)}(\Gamma)$ is compact. Let $u_n \to 0$ weakly in $W^{1,p(\cdot)}(\Omega)$. Then $u_n \to 0$ strongly in $L^{h(\cdot)}(\Gamma)$. Since

$$\rho_{\beta'(\cdot),\Gamma}\left(|u_n|^{r(\cdot)}\right) = \int_\Gamma |u_n(x)|^{r(x)\beta'(x)} d\sigma_x = \int_\Gamma |u_n(x)|^{h(x)} d\sigma_x \to 0,$$

we have $\||u_n|^{r(\cdot)}\|_{L^{\beta'(\cdot)}(\Gamma)} \to 0$ from Proposition 2.10 (iv). Therefore, we have

$$\int_\Gamma b(x)|u_n(x)|^{r(x)} d\sigma_x \le 2\|b\|_{L^{r(\cdot)}(\Gamma)} \||u_n|^{r(\cdot)}\|_{L^{\beta'(\cdot)}(\Gamma)} \to 0.$$

Thus it also follows from Proposition 2.10 (iv) that $\|u_n\|_{L^{r(\cdot)}_{b(\cdot)}(\Gamma)} \to 0$, so $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}_{b(\cdot)}(\Gamma)$ is compact. □

Now we consider the Nemytskii operator.

Proposition 2.12. *Let $q \in C(\overline{\Omega})$ with $q^- \ge 1$ and a be a measurable function with $a(x) > 0$ for a.e. $x \in \Omega$. Assume that*

(F.1) *A function $F(x,t)$ is a Carathéodory function on $\Omega \times \mathbb{R}$.*

(F.2) *The growth condition holds: there exist $c \in L^{q_1(\cdot)}(\Omega)$ with $c(x) \ge 0$ a.e. $x \in \Omega$, $q_1 \in C(\overline{\Omega})$ with $q_1^- \ge 1$, and a constant $c_1 > 0$ such that*

$$|F(x,t)| \le c(x) + c_1 a(x)^{\frac{1}{q_1(x)}} |t|^{\frac{q(x)}{q_1(x)}} \quad \text{for a.e. } x \in \Omega \text{ and all } t \in \mathbb{R}.$$

Then the Nemytskii operator $N_F : L^{q(\cdot)}_{a(\cdot)}(\Omega) \ni u \mapsto F(x,u(x)) \in L^{q_1(\cdot)}(\Omega)$ is continuous and there exists a constant $C > 0$ such that

$$\rho_{q_1(\cdot)}(N_F(u)) \le C\left(\rho_{q_1(\cdot)}(u) + \rho_{(q(\cdot),a(\cdot))}(u)\right) \quad \text{for all } u \in L^{q(\cdot)}_{a(\cdot)}(\Omega).$$

In particular, if $q_1(x) \equiv 1$, then $N_F : L^{q(\cdot)}_{a(\cdot)}(\Omega) \to L^1(\Omega)$ is continuous.

For the proof, see Aramaki [25, Proposition 7].

Remark 2.1. This proposition is an extension of [7, Proposition 2.12].

Similarly we have the following proposition.

Proposition 2.13. *Let $r \in C(\overline{\Gamma_2})$ with $r^- \ge 1$ and b be a σ-measurable function with $b(x) > 0$ σ-a.e. $x \in \Gamma_2$. Assume that*

(G.1) *A function $G(x,t)$ is a Carathéodory function on $\Gamma_2 \times \mathbb{R}$.*

(G.2) *The growth condition holds: there exist $d \in L^{r_1(\cdot)}(\Gamma_2)$ with $d(x) \ge 0$ σ-a.e. $x \in \Gamma_2$, $r_1 \in C(\overline{\Gamma_2})$ with $r_1 \ge 1$, and a constant $d_1 > 0$ such that*

$$|G(x,t)| \le d(x) + d_1 b(x)^{\frac{1}{r_1(x)}} |t|^{\frac{r(x)}{r_1(x)}} \quad \text{for } \sigma\text{-a.e. } x \in \Gamma_2 \text{ and all } t \in \mathbb{R}.$$

Then the Nemytskii operator $N_G : L^{r(\cdot)}_{b(\cdot)}(\Gamma_2) \ni v \mapsto G(x,v(x)) \in L^{r_1(\cdot)}(\Gamma_2)$ *is continuous and there exists a constant* $C>0$ *such that*

$$\rho_{r_1(\cdot),\Gamma_2}(N_G(v)) \le C\left(\rho_{r_1(\cdot),\Gamma_2}(d) + d_1\rho_{(r(\cdot),b(\cdot)),\Gamma_2}(v)\right) \quad \text{for all } v \in L^{r(\cdot)}_{b(\cdot)}(\Gamma_2).$$

In particular, if $r_1(x)\equiv 1$*, then* $N_G : L^{r(\cdot)}_{b(\cdot)}(\Gamma_2) \to L^1(\Gamma_2)$ *is continuous.*

Now define a space by

$$X = \left\{ v \in W^{1,p(\cdot)}(\Omega);\ v=0 \text{ on } \Gamma_1 \right\}. \tag{2.1}$$

Then it is clear to see that X is a closed subspace of $W^{1,p(\cdot)}(\Omega)$, so X is a reflexive and separable Banach space. We can see the following Poincaré-type inequality (Ciarlet and Dinca [26]).

Proposition 2.14. *Let* Ω *be a bounded domain of* $\mathbb{R}^N$ *with a* $C^{0,1}$*-boundary and let* $p \in C_+(\overline{\Omega})$*. Then there exists a constant* $C = C(\Omega,N,p) > 0$ *such that*

$$\|u\|_{L^{p(\cdot)}(\Omega)} \le C \|\boldsymbol{\nabla} u\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)} \quad \text{for all } u \in X.$$

In particular, $\|\boldsymbol{\nabla} u\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)}$ *is equivalent to* $\|u\|_{W^{1,p(\cdot)}(\Omega)}$ *for* $u \in X$*.*

For the direct proof, see [6, Lemma 2.5].

Thus we can define the norm on X so that

$$\|v\|_X = \|\boldsymbol{\nabla} v\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)} \quad \text{for } v \in X, \tag{2.2}$$

which is equivalent to $\|v\|_{W^{1,p(\cdot)}(\Omega)}$ from Proposition 2.14.

3 Assumptions and the main theorem

In this section, we state some assumptions and a main theorem.

Let $p \in C_+(\overline{\Omega})$ be fixed. Assume that the following (A.0)-(A.5) hold.

(A.0) $A : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a function satisfying that for a.e. $x \in \Omega$, the function $A(x,\cdot) : \mathbb{R}^N \ni \boldsymbol{\xi} \mapsto A(x,\boldsymbol{\xi})$ is of C^1-class, and for all $\boldsymbol{\xi} \in \mathbb{R}^N$, the function $A(\cdot,\boldsymbol{\xi}) : \Omega \ni x \mapsto A(x,\boldsymbol{\xi})$ is measurable. Moreover, suppose that $A(x,\boldsymbol{0}) = 0$ and put $\boldsymbol{a}(x,\boldsymbol{\xi}) = \boldsymbol{\nabla}_{\boldsymbol{\xi}} A(x,\boldsymbol{\xi})$. Then $\boldsymbol{a}(x,\boldsymbol{\xi})$ is a Carathéodory function.

In the following (A.1)-(A.3), $c,k_0,k_1 > 0$ denote some constants, a function $h_0 \in L^{p'(\cdot)}(\Omega)$ is non-negative and $h_1 \in L^1_{\text{loc}}(\Omega)$ with $h_1(x) \ge 1$ for a.e. $x \in \Omega$.

(A.1)$|\boldsymbol{a}(x,\boldsymbol{\xi})| \le c(h_0(x) + h_1(x)|\boldsymbol{\xi}|^{p(x)-1})$ for all $\boldsymbol{\xi} \in \mathbb{R}^N$ and a.e. $x \in \Omega$.

(A.2) A is $p(\cdot)$-uniformly convex, that is,

$$A\left(x, \frac{\boldsymbol{\xi}+\boldsymbol{\eta}}{2}\right) + k_1 h_1(x) |\boldsymbol{\xi}-\boldsymbol{\eta}|^{p(x)} \le \frac{1}{2}A(x,\boldsymbol{\xi}) + \frac{1}{2}A(x,\boldsymbol{\eta})$$

for all $\boldsymbol{\xi},\boldsymbol{\eta}\in\mathbb{R}^N$ and a.e. $x\in\Omega$.

(A.3) $k_0h_1(x)|\boldsymbol{\xi}|^{p(x)}\le \boldsymbol{a}(x,\boldsymbol{\xi})\cdot\boldsymbol{\xi}\le p(x)A(x,\boldsymbol{\xi})$ for all $\boldsymbol{\xi}\in\mathbb{R}^N$ and a.e. $x\in\Omega$.

(A.4) $\big(\boldsymbol{a}(x,\boldsymbol{\xi})-\boldsymbol{a}(x,\boldsymbol{\eta})\big)\cdot(\boldsymbol{\xi}-\boldsymbol{\eta})>0$ for all $\boldsymbol{\xi},\boldsymbol{\eta}\in\mathbb{R}^N$ with $\boldsymbol{\xi}\neq\boldsymbol{\eta}$ and a.e. $x\in\Omega$.

(A.5) $A(x,-\boldsymbol{\xi})=A(x,\boldsymbol{\xi})$ for all $\boldsymbol{\xi}\in\mathbb{R}^N$ and a.e. $x\in\Omega$.

Remark 3.1. (i) The condition (A.1) is more general than that of Mashiyev et al. [27] who considered the case $h_1(x)\equiv 1$. In our case, to overcome this we have to consider the space Y defined by (3.1) later as a basic space rather than the space X defined by (2.1).

(ii) (A.3) implies that A is $p(\cdot)$-sub-homogeneous, that is,

$$A(x,s\boldsymbol{\xi})\le A(x,\boldsymbol{\xi})s^{p(x)}\quad \text{for any } \boldsymbol{\xi}\in\mathbb{R}^N,\text{ a.e. } x\in\Omega \text{ and } s>1. \tag{3.1}$$

For the proof, see Aramaki [28, (4.14)].

Example 3.1. (i) $A(x,\boldsymbol{\xi})=\frac{h(x)}{p(x)}|\boldsymbol{\xi}|^{p(x)}$ with $p^-\ge 2$, $h\in L^1_{\mathrm{loc}}(\Omega)$ satisfying $h(x)\ge 1$ a.e. $x\in\Omega$.

(ii) $A(x,\boldsymbol{\xi})=\frac{h(x)}{p(x)}\Big(\big(1+|\boldsymbol{\xi}|^2\big)^{p(x)/2}-1\Big)$ with $p^-\ge 2$, $h\in L^{p'(\cdot)}(\Omega)$ satisfying $h(x)\ge 1$ a.e. $x\in\Omega$.

Then $A(x,\boldsymbol{\xi})$ and $\boldsymbol{a}(x,\boldsymbol{\xi})=\boldsymbol{\nabla}_{\boldsymbol{\xi}}A(x,\boldsymbol{\xi})$ of (i) and (ii) satisfy (A.0)-(A.5).

Remark 3.2. In Example 3.1, when $h(x)\equiv 1$, (i) corresponds to the $p(\cdot)$-Laplacian and (ii) corresponds to the prescribed mean curvature operator for nonparametric surface.

For the function $h_1\in L^1_{\mathrm{loc}}(\Omega)$ with $h_1(x)\ge 1$ for a.e. $x\in\Omega$, we define a modular on X by

$$\widetilde{\rho}_{(p(\cdot),h_1(\cdot))}(v)=\int_\Omega h_1(x)|\boldsymbol{\nabla} v(x)|^{p(x)}\mathrm{d}x\quad \text{for } v\in X,$$

where the space X is defined by (2.1). Define our basic space

$$Y=Y(\Omega)=\Big\{v\in X;\ \widetilde{\rho}_{(p(\cdot),h_1(\cdot))}(v)<\infty\Big\} \tag{3.2}$$

equipped with the norm

$$\|v\|_Y=\inf\Big\{\tau>0;\ \widetilde{\rho}_{(p(\cdot),h_1(\cdot))}\Big(\frac{v}{\tau}\Big)\le 1\Big\}.$$

Proposition 3.1. *The space $(Y,\|\cdot\|_Y)$ is a separable and reflexive Banach space.*

For the proof, see Aramaki [29, Proposition 3.4].

We note that $C_0^\infty(\Omega)\subset Y$. Since $h_1(x)\ge 1$ a.e. $x\in\Omega$, it follows that

$$\widetilde{\rho}_{(p(\cdot),h_1(\cdot))}(v)=\rho_{p(\cdot)}\Big(h_1^{\frac{1}{p(\cdot)}}\boldsymbol{\nabla} v\Big)\ge\rho_{p(\cdot)}(\boldsymbol{\nabla} v)\quad \text{for } v\in Y$$

and

$$\|v\|_Y=\Big\|h_1^{\frac{1}{p(\cdot)}}\boldsymbol{\nabla} v\Big\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)}\ge\|\boldsymbol{\nabla} v\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)}=\|v\|_X\quad \text{for } v\in Y. \tag{3.3}$$

From (3.3) and Proposition 2.1, we have the following proposition.

Proposition 3.2. *Let $p\in C_+(\overline{\Omega})$ and let $u,u_n\in Y$ $(n=1,\cdots)$. Then the following properties hold.*

(i) $Y\hookrightarrow X$ *and* $\|u\|_X\le\|u\|_Y$.

(ii) $\|u\|_Y>1(=1,<1)\Longleftrightarrow\widetilde{\rho}_{(p(\cdot),h_1(\cdot))}(u)>1(=1,<1)$.

(iii) $\|u\|_Y>1\Longrightarrow\|u\|_Y^{p^-}\le\widetilde{\rho}_{(p(\cdot),h_1(\cdot))}(u)\le\|u\|_Y^{p^+}$.

(iv) $\|u\|_Y<1\Longrightarrow\|u\|_Y^{p^+}\le\widetilde{\rho}_{(p(\cdot),h_1(\cdot))}(u)\le\|u\|_Y^{p^-}$.

(v) $\lim_{n\to\infty}\|u_n-u\|_Y=0\Longleftrightarrow\lim_{n\to\infty}\widetilde{\rho}_{(p(\cdot),h_1(\cdot))}(u_n-u)=0$.

(vi) $\|u_n\|_Y\to\infty$ *as* $n\to\infty\Longleftrightarrow\widetilde{\rho}_{(p(\cdot),h_1(\cdot))}(u_n)\to\infty$ as $n\to\infty$.

We assume the following (g.0)-(g.2) on the function g in (1.1).

(g.0) $g(x,t)$ is a Carathéodory function on $\Gamma_2\times\mathbb{R}$ and there exist $1\le b\in L^{\beta(\cdot)}(\Gamma_2)$ with $\beta\in C_+(\overline{\Gamma_2})$ and $r\in C_+(\overline{\Gamma_2})$ satisfying

$$r(x)<\frac{\beta(x)-1}{\beta(x)}p^{\partial}(x)\qquad\text{for all } x\in\overline{\Gamma_2}$$

such that

$$|g(x,t)|\le d\left(1+b(x)|t|^{r(x)-1}\right)\qquad\text{for }\sigma\text{-a.e. } x\in\Gamma_2\text{ and } t\in\mathbb{R}$$

for some constant $d>0$.

(g.1) For σ-a.e. $x\in\Gamma_2$, $g(x,t)$ is an odd function with respect to $t\in\mathbb{R}$.

(g.2) $0<g(x,t)t=r(x)G(x,t)$ for σ-a.e. $x\in\Gamma_2$ and $t>0$, where

$$G(x,t)=\int_0^t g(x,s)\mathrm{d}s\qquad\text{for }\sigma\text{-a.e. } x\in\Gamma_2\text{ and } t\in\mathbb{R}.$$

We note that from (g.2), we obtain the homogeneity of G with respect to t, that is,

$$G(x,t)=G(x,1)|t|^{r(x)}.\tag{3.4}$$

Indeed, from (g.2), we have $\frac{g(x,s)}{G(x,s)}=\frac{r(x)}{s}$ for σ-a.e. $x\in\Gamma_2$ and $s>0$. Taking that $G(x,t)$ is an even function with respect to t into consideration, for $t\ge0$, if we integrate this equality from 1 to t, we easily get (3.4).

For example, a function $g(x,t)=b(x)|t|^{r(x)-2}t$, where b is a function as in (g.0), satisfies (g.0)-(g.2).

Here we introduce the notions of a weak solution and an eigenfunction for the problem (1.1).

Definition 3.1. (i) *We say that a pair $(u,\lambda)\in Y\times\mathbb{R}$ is a weak solution of* (1.1), *if*

$$\int_\Omega \boldsymbol{a}(x,\nabla u(x))\cdot\nabla v(x)\mathrm{d}x=\lambda\int_{\Gamma_2}g(x,u(x))v(x)\mathrm{d}\sigma_x\qquad\text{for all } v\in Y.\tag{3.5}$$

(ii) *Such a pair $(u,\lambda)\in Y\times\mathbb{R}$ with $u\neq0$ is called an eigenpair, λ is called an eigenvalue and u is called an associated eigenfunction.*

Define functionals on Y by

$$\Phi(u)=\int_{\Omega} A(x,\boldsymbol{\nabla} u(x))\mathrm{d}x, \quad K(u)=\int_{\Gamma_2} G(x,u(x))\mathrm{d}\sigma_x \quad \text{for } u\in Y. \tag{3.6}$$

It follows from (A.5) and (g.1) that Φ and K are even functionals, that is, $\Phi(-u)=\Phi(u)$ and $K(-u)=K(u)$ for all $u\in Y$.

Lemma 3.1. (i) *We have*

$$\frac{k_0}{p^+}\widetilde{\rho}_{(p(\cdot),h_1(\cdot))}(u)\le\Phi(u)\le c\left(2\|h_0\|_{L^{p'(\cdot)}(\Omega)}\|\boldsymbol{\nabla} u\|_{L^{p(\cdot)}(\Omega)}+\widetilde{\rho}_{(p(\cdot),h_1(\cdot))}(u)\right)$$

for $u\in Y$, where c and k_0 are the constants in (A.1) *and* (A.3).

(ii) *We have*

$$\Phi\left(\frac{u+v}{2}\right)+k_1\widetilde{\rho}_{(p(\cdot),h_1(\cdot))}(u-v)\le\frac{1}{2}\Phi(u)+\frac{1}{2}\Phi(v)$$

for all $u,v\in Y$, where k_1 is the constant in (A.2)*, in particular, Φ is convex, that is, $\Phi((1-\tau)u+\tau v)\le(1-\tau)\Phi(u)+\tau\Phi(v)$ for all $u,v\in Y$ and $\tau\in[0,1]$.*

Proof. (i) easily follows from (A.3) and the Hölder inequality (Proposition 2.2). (ii) easily follows from (A.2) and the continuity of $A(x,\boldsymbol{\xi})$ with respect to $\boldsymbol{\xi}$. □

Proposition 3.3. (i) *Φ is coercive, that is, $\Phi(u)\to\infty$ as $\|u\|_Y\to\infty$.*

(ii) *Φ is sequentially weakly lower-semicontinuous on Y.*

(iii) *$\Phi\in C^1(Y,\mathbb{R})$ and we have*

$$\langle\Phi'(u),v\rangle_{Y^*,Y}=\int_{\Omega}\boldsymbol{a}(x,\boldsymbol{\nabla} u(x))\cdot\boldsymbol{\nabla} v(x)\mathrm{d}x \quad \text{for } u,v\in Y. \tag{3.7}$$

Proof. (i) follows from Lemma 3.1 (i) and Proposition 3.2 (iii). (ii) follows from [28, Proposition 4.4 (iii)]. (iii) follows from [28, Proposition 4.1]. □

Proposition 3.4. *If $u_n\to u$ weakly in Y and $\Phi(u_n)\to\Phi(u)$ as $n\to\infty$, then we have $\Phi\left(\frac{u_n-u}{2}\right)\to 0$ as $n\to\infty$, so $u_n\to u$ strongly in Y.*

Proof. Assume that the conclusion is false. Then there exist $\varepsilon_0>0$ and a subsequence $\{u_{n'}\}$ of $\{u_n\}$ such that $\Phi\left(\frac{u_{n'}-u}{2}\right)\ge\varepsilon_0$ for all n'. From Lemma 3.1 (i), there exists a constant $C(\varepsilon_0)=C(\varepsilon_0,p)>0$ such that

$$\begin{aligned}\varepsilon_0&\le\Phi\left(\frac{u_{n'}-u}{2}\right)\le c\left(2\|h_0\|_{L^{p'(\cdot)}(\Omega)}\left\|\frac{u_{n'}-u}{2}\right\|_Y+\left\|\frac{u_{n'}-u}{2}\right\|_Y^{p^+}\vee\left\|\frac{u_{n'}-u}{2}\right\|_Y^{p^-}\right)\\&\le\frac{\varepsilon_0}{2}+C(\varepsilon_0)\left(\|u_{n'}-u\|_Y^{p^+}\vee\|u_{n'}-u\|_Y^{p^-}\right).\end{aligned}$$

Here and from now on, $a\vee b=\max\{a,b\}$ and $a\wedge b=\min\{a,b\}$ for real numbers a and b. Hence

$$C(\varepsilon_0)\|u_{n'}-u\|_Y^{p^+}\vee\|u_{n'}-u\|_Y^{p^-}\geq\frac{\varepsilon_0}{2},$$

so by Proposition 3.2 (v), there exists a subsequence of $\{u_{n'}\}$ (still denoted by $\{u_{n'}\}$) and $c_1>0$ such that $k_1\widetilde{\rho}_{(p(\cdot),h_1(\cdot))}(u_{n'}-u)\geq c_1$. Hence it follows from Lemma 3.1 (ii) that

$$\Phi\left(\frac{u_{n'}+u}{2}\right)+c_1\leq\frac{1}{2}\Phi(u_{n'})+\frac{1}{2}\Phi(u).$$

Since $(u_{n'}+u)/2\to u$ weakly in Y and Φ is sequencially weakly lower semi-continuous, we have

$$\Phi(u)+c_1\leq\liminf_{n'\to\infty}\Phi\left(\frac{u_{n'}+u}{2}\right)+c_1\leq\frac{1}{2}\liminf_{n'\to\infty}\Phi(u_{n'})+\frac{1}{2}\Phi(u)=\Phi(u).$$

This is a contradiction. Thus $\Phi((u_n-u)/2)\to 0$ as $n\to\infty$. By Lemma 3.1 (i), we can see that $\widetilde{\rho}_{(p(\cdot),h_1(\cdot))}\left(\frac{u_n-u}{2}\right)\to 0$ as $n\to\infty$. Thus from Proposition 3.2 (v), $u_n\to u$ strongly in Y. □

Proposition 3.5. (i) $\Phi\in\mathcal{W}_Y$, *that is, if* $u_n\to u$ *weakly in* Y *as* $n\to\infty$ *and* $\liminf_{n\to\infty}\Phi(u_n)\leq\Phi(u)$, *then there exists a subsequence* $\{u_{n'}\}$ *of* $\{u_n\}$ *such that* $u_{n'}\to u$ *strongly in* Y *as* $n'\to\infty$.

(ii) Φ *is bounded on every bounded subset of* Y.

Proof. (i) Let $u_n\to u$ weakly in Y as $n\to\infty$ and $\liminf_{n\to\infty}\Phi(u_n)\leq\Phi(u)$. Since Φ is sequentially weakly lower semi-continuous from Proposition 3.3 (ii), we have $\Phi(u)\leq\liminf_{n\to\infty}\Phi(u_n)$, so $\liminf_{n\to\infty}\Phi(u_n)=\Phi(u)$. Hence there exists a subsequence $\{u_{n'}\}$ of $\{u_n\}$ such that $\lim_{n'\to\infty}\Phi(u_{n'})=\Phi(u)$. By Proposition 3.4, $u_{n'}\to u$ strongly in Y.

(ii) easily follows from Lemma 3.1 (i). □

Proposition 3.6. (i) Φ' *is strictly monotone in* Y, *that is,*

$$\langle\Phi'(u)-\Phi'(v),u-v\rangle_{Y^*,Y}>0\quad\text{for all } u,v\in Y \text{ with } u\neq v.$$

Moreover, Φ' *is bounded on every bounded subset of* Y *and coercive in the sense that*

$$\lim_{\|u\|_Y\to\infty}\frac{\langle\Phi'(u),u\rangle_{Y^*,Y}}{\|u\|_Y}=\infty.$$

(ii) Φ' *is of* (S_+)*-type, that is, if* $u_n\to u$ *weakly in* Y *and*

$$\limsup_{n\to\infty}\langle\Phi'(u_n),u_n-u\rangle_{Y^*,Y}\leq 0,$$

then $u_n\to u$ *strongly in* Y.

(iii) *The mapping* $\Phi':Y\to Y^*$ *is a homeomorphism.*

For the proof, see [25, Proposition 21].

For the functional K defined by (3.6), we have the following proposition.

Proposition 3.7. *Under the hypotheses* (g.0), *we have the following.*

(i) $K \in C^1(Y,\mathbb{R})$ *and*

$$\langle K'(u),v\rangle_{Y^*,Y} = \int_{\Gamma_2} g(x,u(x))v(x)\mathrm{d}\sigma_x \quad \text{for } u,v \in Y. \tag{3.8}$$

(ii) *K is sequentially weakly continuous in Y.*

(iii) *$K': Y \to Y^*$ is weakly-strongly continuous, that is, if $u_n \to u$ weakly in Y as $n \to \infty$, then $K'(u_n) \to K'(u)$ strongly in Y^* as $n \to \infty$.*

Proof. Since (i) and (ii) follow from [28, Propositions 4.2 and 4.4], we only derive (iii). Let $u_n \to u$ weakly in Y. Then

$$\langle K'(u_n) - K'(u),v\rangle_{Y^*,Y} = \int_{\Gamma_2} (g(x,u_n(x)) - g(x,u(x)))v(x)\mathrm{d}\sigma_x \quad \text{for } v \in Y.$$

From Proposition 2.11 and $(g.0)$, the embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}_{b(\cdot)}(\Gamma_2)$ is compact. Since $Y \hookrightarrow X \hookrightarrow W^{1,p(\cdot)}(\Omega)$, there exists a constant $C > 0$ such that

$$\|v\|_{L^{r(\cdot)}_{b(\cdot)}(\Gamma_2)} \le C\|v\|_Y \quad \text{for all } v \in Y.$$

By the Hölder inequality (Proposition 2.5), for any $v \in Y$,

$$\begin{aligned}
&|\langle K'(u_n) - K'(u),v\rangle_{Y^*,Y}| \\
\le& \int_{\Gamma_2} b(x)^{-1/r(x)}|g(x,u_n(x)) - g(x,u(x))|b(x)^{\frac{1}{r(x)}}|v(x)|\mathrm{d}\sigma_x \\
\le& 2\|b(\cdot)^{-1/r(\cdot)}|g(\cdot,u_n(\cdot)) - g(\cdot,u(\cdot))|\|_{L^{r'(\cdot)}(\Gamma_2)}\|b(\cdot)^{\frac{1}{r(\cdot)}}|v(\cdot)|\|_{L^{r(\cdot)}(\Gamma_2)}.
\end{aligned}$$

Since

$$\|b(\cdot)^{\frac{1}{r(\cdot)}}v(\cdot)\|_{L^{r(\cdot)}(\Gamma_2)} = \|v\|_{L^{r(\cdot)}_{b(\cdot)}(\Gamma_2)} \le C\|v\|_Y,$$

we have

$$\|K'(u_n) - K'(u)\|_{Y^*} \le 2C\|b(\cdot)^{\frac{-1}{r(\cdot)}}|g(\cdot,u_n(\cdot)) - g(\cdot,u(\cdot))|\|_{L^{r'(\cdot)}(\Gamma_2)}.$$

We want to show that $\|K'(u_n) - K'(u)\|_{Y^*} \to 0$ as $n \to \infty$. By Proposition 2.4 (iv), it suffices to show that

$$\rho_{r'(\cdot),\Gamma_2}\left(b(\cdot)^{\frac{-1}{r(\cdot)}}g(\cdot,u_n(\cdot)) - b(\cdot)^{\frac{-1}{r(\cdot)}}g(\cdot,u(\cdot))\right) \to 0 \quad \text{as } n \to \infty. \tag{3.9}$$

We can see that

$$\rho_{r'(\cdot),\Gamma_2}\left(b(\cdot)^{\frac{-1}{r(\cdot)}}g(\cdot,u_n(\cdot)) - b(\cdot)^{\frac{-1}{r(\cdot)}}g(\cdot,u(\cdot))\right)$$

$$=\int_{\Gamma_2} b(x)^{\frac{-r'(x)}{r(x)}} |g(x,u_n(x))-g(x,u(x))|^{r'(x)} \mathrm{d}\sigma_x.$$

Since $u_n \to u$ weakly in Y and the embedding $Y \hookrightarrow L^{r(\cdot)}_{b(\cdot)}(\Gamma_2)$ is compact, we can see that $u_n \to u$ strongly in $L^{r(\cdot)}_{b(\cdot)}(\Gamma_2)$. From [7, Theorem A.1], there exist a subsequence $\{u_{n'}\}$ of $\{u_n\}$ and $f \in L^{r(\cdot)}(\Gamma_2)$ such that $b(x)^{1/r(x)}u_{n'}(x) \to b(x)^{1/r(x)}u(x)$ σ-a.e. $x \in \Gamma_2$ and $|b(x)^{1/r(x)}u_{n'}(x)| \le f(x)$ for σ-a.e. $x \in \Gamma_2$. Since $b(x)>0$ and g is a Carathéodory function, $g(x,u_{n'}(x)) \to g(x,u(x))$ σ-a.e. $x \in \Gamma_2$. From $(g.0)$ and $b(x) \ge 1$, we have

$$\begin{aligned}
&b(x)^{\frac{-r'(x)}{r(x)}} |g(x,u_{n'}(x)-g(x,u(x))|^{r'(x)} \\
&\le C_1 b(x)^{\frac{-r'(x)}{r(x)}} \left(1+b(x)|u_{n'}(x)|^{r(x)-1}+b(x)|u(x)|^{r(x)-1}\right)^{r'(x)} \\
&\le C_1 \left(b(x)^{\frac{-r'(x)}{r(x)}} + b(x)^{r'(x)-\frac{r'(x)}{r(x)}} \left(|u_{n'}(x)|^{r(x)}+|u(x)|^{r(x)}\right)\right) \\
&\le C_1 \left(1+b(x)\left(|u_{n'}(x)|^{r(x)}+|u(x)|^{r(x)}\right)\right) \\
&\le 2C_1 \left(1+f(x)^{r(x)}\right) \qquad \text{for some positive constant } C_1.
\end{aligned}$$

The last term is an integrable function in Ω independent of n'. Thus by the Lebesgue dominated convergence theorem, we have

$$\rho_{r'(\cdot),\Gamma_2}\left(b(\cdot)^{\frac{-1}{r(\cdot)}} g(\cdot,u_{n'}(\cdot)) - b(\cdot)^{\frac{-1}{r(\cdot)}} g(\cdot,u(\cdot))\right) \to 0 \text{ as } n' \to \infty.$$

From the convergent principle (Zeidler [30, Proposition 10.13]), we see that (3.9) holds, so $\|K'(u_n)-K'(u)\|_{Y^*} \to 0$ as $n \to \infty$. □

Remark 3.3. From (3.7)-(3.8) and Definition 3.1, we can see that $(u,\lambda) \in Y \times \mathbb{R}$ is a weak solution of the problem (1.1) if and only if

$$\Phi'(u) = \lambda K'(u). \tag{3.10}$$

In particular, we have $\langle \Phi'(u),u\rangle_{Y^*,Y} = \lambda \langle K'(u),u\rangle_{Y^*,Y}$. If (u,λ) is an eigenpair of (1.1), then it follows from (A.3) and (g.2) that

$$\begin{aligned}
\langle \Phi'(u),u\rangle_{Y^*,Y} &= \int_\Omega \boldsymbol{a}(x,\boldsymbol{\nabla} u(x)) \cdot \boldsymbol{\nabla} u(x) \mathrm{d}x \\
&\ge k_0 \int_\Omega h_1(x) |\boldsymbol{\nabla} u(x)|^{p(x)} \mathrm{d}x \ge k_0 \|u\|_Y^{p^+} \wedge \|u\|_Y^{p^-} > 0
\end{aligned}$$

and so

$$\langle K'(u),u\rangle_{Y^*,Y} = \int_{\Gamma_2} g(x,u(x))u(x) \mathrm{d}\sigma_x > 0,$$

so we have

$$\lambda=\frac{\langle\Phi'(u),u\rangle_{Y^*,Y}}{\langle K'(u),u\rangle_{Y^*,Y}}>0. \tag{3.11}$$

This means that any eigenvalue of the problem (1.1) is positive.

In order to solve the eigenvalue problem (3.10), we apply the constrained variational method. We take Φ as an objective functional and K as a constraint functional. For any fixed $\alpha>0$, put

$$M_\alpha=\{u\in Y;K(u)=\alpha\}. \tag{3.12}$$

If $u\in M_\alpha$, then from (g.2),

$$\langle K'(u),u\rangle_{Y^*,Y}=\int_{\Gamma_2}g(x,u(x))u(x)\mathrm{d}\sigma_x\geq r^-\int_{\Gamma_2}G(x,u(x))\mathrm{d}\sigma_x=r^-K(u)=r^-\alpha>0, \tag{3.13}$$

so $K'(u)\neq 0$. Hence M_α is a C^1-submanifold of Y with codimension one. Moreover, M_α is a weakly closed subset of Y. Indeed, let $u_j\in M_\alpha$ and $u_j\to u$ weakly in Y as $j\to\infty$. Since K is sequentially weakly continuous from Proposition 3.7 (ii), $\alpha=K(u_j)\to K(u)$, so $u\in M_\alpha$.

It is well known that $(u,\lambda)\in Y\times\mathbb{R}$ with $K(u)=\alpha>0$ solves (3.10) if and only if u is a critical point of Φ with respect to M_α, that is,

$$\langle\Phi'(u),h\rangle_{Y^*,Y}=0 \quad\text{for all } h\in T_uM_\alpha,$$

(for example, [30, Proposition 43.21]). Here T_uM_α is the tangent space of M_α at $u\in M_\alpha$ and we can see that

$$T_uM_\alpha=\mathrm{Ker}(K'(u))=\{v\in Y;\langle K'(u),v\rangle_{Y^*,Y}=0\}.$$

Let $P:Y\to T_uM_\alpha$ be the natural projection. Note that the bounded linear map $K'(u):Y\to\mathbb{R}$ is surjective. We denote the restriction of Φ to M_α by $\widetilde{\Phi}=\Phi|_{M_\alpha}$ and the derivative $d\widetilde{\Phi}(u)\in Y^*$ of $\widetilde{\Phi}$ at $u\in M_\alpha$ can be defined by $\langle d\widetilde{\Phi}(u),v\rangle_{Y^*,Y}=\langle\Phi'(u),Pv\rangle_{Y^*,Y}$ for $v\in Y$.

For $u\in M_\alpha$, put $w=(\Phi')^{-1}(K'(u))$. Then since we have (3.13), we see that $K'(u)\neq 0$. From (A.5), the functional Φ is even, so Φ' is odd and so $\Phi'(0)=0$. Since $(\Phi')^{-1}$ is injective, we have $w\neq 0$. From strict monotonicity of Φ' (Proposition 3.6 (i)),

$$\langle K'(u),w\rangle_{Y^*,Y}=\big\langle K'(u),(\Phi')^{-1}\big(K'(u)\big)\big\rangle_{Y^*,Y}=\langle\Phi'(w),w\rangle_{Y^*,Y}>0. \tag{3.14}$$

Hence since $w=(\Phi')^{-1}\big(K'(u)\big)\notin T_uM_\alpha$, we can see that

$$Y=T_uM_\alpha\oplus\Big\{\beta(\Phi')^{-1}\big(K'(u)\big);\beta\in\mathbb{R}\Big\}.$$

For every $v\in Y$, there exists a unique $\beta\in\mathbb{R}$ such that $v=Pv+\beta(\Phi')^{-1}\big(K'(u)\big)$. Since $Pv\in T_uM_\alpha=\mathrm{Ker}\big(K'(u)\big)$, we have

$$\big\langle K'(u),v\big\rangle_{Y^*,Y}=\beta\big\langle K'(u),(\Phi')^{-1}\big(K'(u)\big)\big\rangle_{Y^*,Y}.$$

Thus from (3.14), we can write

$$\beta=\frac{\langle K'(u),v\rangle_{Y^*,Y}}{\langle K'(u),(\Phi')^{-1}(K'(u))\rangle_{Y^*,Y}}.$$

Now we have

$$\begin{aligned}&\big\langle d\widetilde{\Phi}(u),v\big\rangle_{Y^*,Y}=\big\langle \Phi'(u),Pv\big\rangle_{Y^*,Y}\\ =&\big\langle \Phi'(u),v\big\rangle_{Y^*,Y}-\left\langle \Phi'(u),\frac{\langle K'(u),v\rangle_{Y^*,Y}}{\langle K'(u),(\Phi')^{-1}(K'(u))\rangle_{Y^*,Y}}(\Phi')^{-1}(K'(u))\right\rangle_{Y^*,Y}\\ =&\left\langle \Phi'(u)-\frac{\langle \Phi'(u),(\Phi')^{-1}(K'(u))\rangle_{Y^*,Y}}{\langle K'(u),(\Phi')^{-1}(K'(u))\rangle_{Y^*,Y}}K'(u),v\right\rangle_{Y^*,Y}.\end{aligned}$$

Thus we have

$$d\widetilde{\Phi}(u)=\Phi'(u)-\lambda(u)K'(u),$$

where

$$\lambda(u)=\frac{\big\langle \Phi'(u),(\Phi')^{-1}(K'(u))\big\rangle_{Y^*,Y}}{\big\langle K'(u),(\Phi')^{-1}(K'(u))\big\rangle_{Y^*,Y}}.$$

Proposition 3.8. *For any $\alpha>0$, the functional $\widetilde{\Phi}:M_\alpha\to\mathbb{R}$ verifies* (PS)*-condition, that is, if any sequence $\{u_n\}\subset M_\alpha$ such that $\widetilde{\Phi}(u_n)\to c\in\mathbb{R}$ and $\|d\widetilde{\Phi}(u_n)\|_{Y^*}\to 0$ as $n\to\infty$, then $\{u_n\}$ contains a convergent subsequence.*

Proof. Let $\{u_n\}\subset M_\alpha$ satisfy that $\widetilde{\Phi}(u_n)\to c$ and $d\widetilde{\Phi}(u_n)\to 0$ in Y^* as $n\to\infty$. Then, since (A.3) produces

$$\widetilde{\Phi}(u_n)=\Phi(u_n)\geq\frac{k_0}{p^+}\int_\Omega h_1(x)|\boldsymbol{\nabla} u_n(x)|^{p(x)}\mathrm{d}x\geq\frac{k_0}{p^+}\|u_n\|_Y^{p^+}\wedge\|u_n\|_Y^{p^-},$$

$\{u_n\}$ is bounded in Y. Since Y is a reflexive Banach space from Proposition 3.1, there exist a subsequence $\{u_{n'}\}$ of $\{u_n\}$ and $u_0\in Y$ such that $u_{n'}\to u_0$ weakly in Y. By Proposition 3.7 (ii) and (iii), $K'(u_{n'})\to K'(u_0)$ in Y^* and $K(u_{n'})\to K(u_0)$ as $n\to\infty$. Thereby, $u_0\in M_\alpha$. Put $w_{n'}=(\Phi')^{-1}(K'(u_{n'}))$. Since $K'(u_{n'})\to K'(u_0)\neq 0$ in Y^* from (3.13), we see that $w_{n'}\to w_0\neq 0$ in Y, where $w_0=(\Phi')^{-1}(K'(u_0))$. Thus

$$\big\langle K'(u_{n'}),(\Phi')^{-1}(K'(u_{n'}))\big\rangle_{Y^*,Y}=\big\langle \Phi'(w_{n'}),w_{n'}\big\rangle_{Y^*,Y}\to\big\langle \Phi'(w_0),w_0\big\rangle_{Y^*,Y}>0. \tag{3.15}$$

On the other hand, we have

$$\big|\big\langle \Phi'(u_{n'}),(\Phi')^{-1}(K'(u_{n'}))\big\rangle_{Y^*,Y}\big|=\big|\big\langle \Phi'(u_{n'}),w_{n'}\big\rangle_{Y^*,Y}\big|\leq\big\|\Phi'(u_{n'})\big\|_{Y^*}\|w_{n'}\|_Y.$$

Since $u_{n'}\to u_0$ weakly in Y, $\{u_{n'}\}$ is bounded in Y, so by Proposition 3.6 (i), $\|\Phi'(u_{n'})\|_{Y^*}$ is bounded. Hence, there exists a constant $c_2>0$ such that

$$\big|\big\langle \Phi'(u_{n'}),(\Phi')^{-1}(K'(u_{n'}))\big\rangle_{Y^*,Y}\big|\leq c_2. \tag{3.16}$$

From (3.15) and (3.16), $\{\lambda(u_{n'})\}$ is bounded in $\mathbb{R}$. Passing to a subsequence, we may assume that $\lambda(u_{n'})\to\lambda_0$ for some $\lambda_0\in\mathbb{R}$. Since $d\widetilde{\Phi}(u_{n'})\to 0$ in Y^*, we see that $\Phi'(u_{n'})-\lambda(u_{n'})K'(u_{n'})\to 0$ as $n'\to\infty$. Hence, since $K'(u_{n'})\to K'(u_0)$ in Y^*,

$$\Phi'(u_{n'})=\big(\Phi'(u_{n'})-\lambda(u_{n'})K'(u_{n'})\big)+\lambda(u_{n'})K'(u_{n'})\to\lambda_0K'(u_0)\quad \text{in } Y^* \text{ as } n'\to\infty.$$

Therefore, we see that $u_{n'}\to(\Phi')^{-1}\big(\lambda_0K'(u_0)\big)$ strongly in Y as $n'\to\infty$. $\square$

Here we recall the notion of *genus* which is introduced in Rabinowitz [31, Chapter 7] or [30, Section 44.3]. Let E be a real Banach space and let $\mathcal{E}$ denote the family of subsets $A\subset E\setminus\{0\}$ such that A is closed in E and symmetric with respect to 0, that is, $x\in A$ implies $-x\in A$. For $\emptyset\neq A\in\mathcal{E}$, define the genus of A to be $n\geq 1$ (denoted by $\gamma(A)=n$) if there is a map $\varphi\in C(A,\mathbb{R}^n\setminus\{\mathbf{0}\})$ with φ odd and n is the smallest integer with this property. When there does not exist a finite such n, set $\gamma(A)=\infty$. Finally set $\gamma(\emptyset)=0$.

The main properties of genus will be listed in the next proposition.

Proposition 3.9. *Let $A,B\in\mathcal{E}$. Then the following properties hold.*

(i) *If there exits an odd map $f\in C(A,B)$, then $\gamma(A)\leq\gamma(B)$.*

(ii) *If $A\subset B$, then $\gamma(A)\leq\gamma(B)$.*

(iii) $\gamma(A\cup B)\leq\gamma(A)+\gamma(B)$.

(iv) *If A is compact, then $\gamma(A)<\infty$ and there exists $\delta>0$ such that if we put*

$$N_\delta(A)=\{x\in E;\|x-A\|:=\inf\{\|x-y\|;y\in A\}\leq\delta\},$$

then $N_\delta(A)\in\mathcal{E}$ and $\gamma(N_\delta(A))=\gamma(A)$.

(v) *If Ω is a bounded neighborhood of $\mathbf{0}$ in $\mathbb{R}^n$, and there exists a mapping $h:A\to\partial\Omega$ with h an odd homeomorphism, then $\gamma(A)=n$.*

For the proof, see [31, Lemma 7.5 and Proposition 7.7] or [9, Proposition 2.3]. We note that it can be easily seen that when $A\in\mathcal{E}$, $A\neq\emptyset$ if and only if $\gamma(A)\geq 1$.

Let $\Sigma_\alpha=\{H\subset M_\alpha;H$ is compact and symmetric$\}$, $\gamma(H)$ be the genus of $H\in\Sigma_\alpha$, and define

$$c_{(n,\alpha)}=\inf_{H\in\Sigma_\alpha,\gamma(H)\geq n}\sup_{u\in H}\widetilde{\Phi}(u)\quad(n=1,\cdots).\tag{3.17}$$

The following proposition is due to [9, Corollary 4.3].

Proposition 3.10 (Ljusternik-Schnirelmann principle)**.** *Assume that M is a closed symmetric C^1-submanifold of a real Banach space B and $0\notin M$. Let $f\in C^1(M,\mathbb{R})$ be an even functional and bounded from below. Define*

$$c_j=\inf_{H\in\Gamma_j}\sup_{u\in H}f(u)\ \text{ for } j=1,2,\cdots,$$

where

$$\Gamma_j=\{H\subset M;H \text{ is compact, symmetric and } \gamma(H)\geq j\}.$$

If $\Gamma_k\neq\emptyset$ for some $k\geq 1$ and f satisfies $(\mathrm{PS})_c$-condition for $c:=c_m=c_{m+1}=\cdots=c_k$ with $1\leq m\leq k$, then f has at least $k-m+1$ distinct pairs of critical points.

Since Y is a separable reflexive Banach space, it is well known that there exist $\{e_n\}_{n=1}^{\infty}\subset Y$ and $\{f_n\}_{n=1}^{\infty}\subset Y^*$ such that $\langle f_n,e_m\rangle_{Y^*,Y}=\delta_{nm}$, where δ_{nm} is the Kronecker delta and

$$Y=\overline{\mathrm{span}\{e_1,e_2,\cdots\}}\quad \text{and}\quad Y^*=\overline{\mathrm{span}\{f_1,f_2,\cdots\}}.$$

Define spaces

$$Y_j=\mathrm{span}\{e_j\},\quad Z_n=\bigoplus_{j=1}^{n}Y_j,\quad W_n=\overline{\bigoplus_{j=n}^{\infty}Y_j}.$$

If we apply Proposition 3.10 with $B=Y$, $M=M_\alpha$ and $f=\widetilde{\Phi}$, then we obtain the following lemma. We note that $\widetilde{\Phi}$ is bounded from below on M_α and satisfies $(\mathrm{PS})_c$-condition with respect ot M_α for any $c\in\mathbb{R}$ by Proposition 3.8.

Lemma 3.2. *For any $m\in\mathbb{N}$, we have $\Gamma_m\neq\emptyset$. Thus we see that all $c_{(n,\alpha)}$ defined by* (3.17) *are critical values of $\widetilde{\Phi}$ with respect to M_α and*

$$-\infty<c_{(n,\alpha)}\le c_{(n+1,\alpha)}<\infty\quad \text{for every }\ n\in\mathbb{N}.$$

Proof. For any fixed $m\in\mathbb{N}$, we claim that

$$c(m):=\inf\{K(u);u\in Z_m,\|u\|_Y=1\}>0. \tag{3.18}$$

Indeed, assume that $c(m)=0$. Then there exists a sequence $\{u_j\}\subset Z_m$ such that $\|u_j\|_Y=1$ and

$$0\le K(u_j)\le\frac{1}{j}. \tag{3.19}$$

Since $\{u_j\}$ is bounded in Y, there exist a subsequence $\{u_{j'}\}$ of $\{u_j\}$ and $u_0\in Y$ such that $u_{j'}\to u_0$ weakly in Y as $j'\to\infty$. Since $\langle f_k,u_{j'}\rangle_{Y^*,Y}=0$ for any $k>m$, we have $\langle f_k,u_0\rangle_{Y^*,Y}=0$, so we see that $u_0\in Z_m$. Since $\dim Z_m=m<\infty$, $u_{j'}\to u_0$ strongly in Z_m, so in Y. Thereby $\|u_0\|_Y=1$, so we can see that $K(u_0)>0$. On the other hand, letting $j\to\infty$ in (3.19), we see that $K(u_0)=0$. This is a contradiction.

For $0\neq u\in Z_m$, since $\|u/\|u\|_Y\|_Y=1$, it follows from (3.18) and (3.4) that

$$\begin{aligned}c(m)\le K\left(\frac{u}{\|u\|_Y}\right)&=\int_{\Gamma_2}G(x,1)\left|\frac{u(x)}{\|u\|_Y}\right|^{r(x)}\mathrm{d}\sigma_x\\&=\int_{\Gamma_2}\frac{1}{\|u\|_Y^{r(x)}}G(x,u(x))\mathrm{d}\sigma_x\le\frac{1}{\|u\|_Y^{r^+}\wedge\|u\|_Y^{r^-}}K(u).\end{aligned}$$

Thus we have $K(u)\ge c(m)\|u\|_Y^{r^+}\wedge\|u\|_Y^{r^-}$ for all $u\in Z_m$. In particular, $\alpha=K(u)\ge c(m)\|u\|_Y^{r^+}\wedge\|u\|_Y^{q^-}$ for $u\in Z_m\cap M_\alpha$. Therefore, $Z_m\cap M_\alpha$ is a bounded and closed subset of Z_m, so is compact by $\dim Z_m<\infty$. Let $G=\{u=u_1e_1+\cdots+u_me_m\in Z_m;K(u)<\alpha\}$. Then G can be identified with an open neighborhood of $\mathbf{0}$ in $\mathbb{R}^m$ by a trivial odd homeomorphism. Since the identity map: $Z_m\cap M_\alpha\to\partial G$ is an odd homeomorphism, using Proposition 3.9 (v), we have $\gamma(Z_m\cap M_\alpha)=m$, so $\Gamma_m\neq\emptyset$. $\square$

It follows that the following lemma holds.

Lemma 3.3. *Assume that a functional* $\Psi : Y \to \mathbb{R}$ *is sequentially weakly continuous and satisfies* $\Psi(0)=0$. *Then for any fixed* $r>0$,

$$\lim_{n\to\infty} \sup_{u\in W_n,\|u\|_Y\le r} |\Psi(u)|=0. \tag{3.20}$$

Proof. Put $d_n=\sup_{u\in W_n,\|u\|_Y\le r}|\Psi(u)|$. Then there exists $u_j\in W_n$ with $\|u_j\|_Y\le r$ such that $\lim_{j\to\infty}|\Psi(u_j)|=d_n$. Since Y is a reflexive Banach space, there exist a subsequence $\{u_{j'}\}$ of $\{u_j\}$ and $u^{(n)}\in Y$ such that $u_{j'}\to u^{(n)}$ weakly in Y. Hence $\|u^{(n)}\|_Y\le \liminf_{j'\to\infty}\|u_{j'}\|_Y\le r$. Since W_n is a closed subspace of Y, W_n is weakly closed, so $u^{(n)}\in W_n$. Since Ψ is sequentially weakly continuous, $|\Psi(u_{j'})|\to|\Psi(u^{(n)})|$ as $j'\to\infty$. Thereby $|\Psi(u^{(n)})|=d_n$. Since clearly $d_{n+1}\le d_n$ for all $n\in\mathbb{N}$, $\lim_{n\to\infty}d_n=d_0\ge 0$ exists. Since $\{u^{(n)}\}$ satisfies $\|u^{(n)}\|_Y\le r$, there exists a subsequence $\{u^{(n')}\}$ of $\{u^{(n)}\}$ and $u_0\in Y$ such that $u^{(n')}\to u_0$ weakly in Y, so $\|u_0\|_Y\le r$. Since again Ψ is sequentially weakly continuous, $|\Psi(u^{(n')})|=d_{n'}\to|\Psi(u_0)|=d_0$. Since Y is reflexive, we can look upon $u_0\in Y^{**}=Y$. Therefore, for any $f_j\in Y^*$, since $u^{(n')}\in W_{n'}$, we have

$$\langle u_0,f_j\rangle_{Y^{**},Y^*}=\langle f_j,u_0\rangle_{Y^*,Y}=\lim_{n'\to\infty}\langle f_j,u^{(n')}\rangle_{Y^*,Y}=0.$$

Thus we have $u_0=0$, so $d_0=\Psi(0)=0$, that is, (3.20) holds. □

Proposition 3.11. *We have* $\lim_{n\to\infty}\inf_{u\in W_n\cap M_\alpha}\|u\|_Y=\infty$.

Proof. Suppose that the conclusion is false. Then there exist $c_1>0$ and $u_n\subset W_n\cap M_\alpha$ such that $\|u_n\|_Y\le c_1$ for large $n\in\mathbb{N}$. Then

$$\sup_{u\in W_n,\|u\|_Y\le c_1}|K(u)|\ge|K(u_n)|=\alpha.$$

Therefore,

$$\lim_{n\to\infty}\sup_{u\in W_n,\|u\|_Y\le c_1}|K(u)|\ge\lim_{n\to\infty}|K(u_n)|=\alpha>0.$$

If we apply Lemma 3.3 with $\Psi=K$, this is a contradiction. □

Proposition 3.12. *We have*

$$\lim_{n\to\infty}c_{(n,\alpha)}=\infty. \tag{3.21}$$

Proof. By Proposition 3.11, for any $c>1$, there exists $n_0\in\mathbb{N}$ such that for any $n\ge n_0$ and $u\in W_n\cap M_\alpha$, $\|u\|_Y>c$. For any $H\in\Sigma_\alpha$, $\gamma(H\cap Z_{n-1})\le n-1$. On the other hand, we have $\operatorname{codim}W_n=n-1$. Hence for $H\in\Sigma_\alpha$ with $\gamma(H)\ge n$, $H\cap W_n$ is non-empty. Indeed, since $H=(H\cap Z_{n-1})\cup(H\cap W_n)$, it follows from Proposition 3.9 (iii) that

$$n\le\gamma(H)\le\gamma(H\cap Z_{n-1})+\gamma(H\cap W_n)\le n-1+\gamma(H\cap W_n),$$

so $\gamma(H\cap W_n)\ge 1$. Hence $H\cap W_n\ne\emptyset$. For $n\ge n_0$, using (3.17), we have

$$\begin{aligned}
c_{(n,\alpha)} &= \inf_{H\in\Sigma_\alpha,\gamma(H)\ge n}\sup_{u\in H}\widetilde{\Phi}(u)\\
&= \inf_{H\in\Sigma_\alpha,\gamma(H)\ge n}\max\left\{\sup_{u\in H\cap(Y\setminus Z_{n-1})}\widetilde{\Phi}(u),\ \sup_{u\in H\cap Z_{n-1}}\widetilde{\Phi}(u)\right\}\\
&\ge \inf_{H\in\Sigma_\alpha,\gamma(H)\ge n}\sup_{u\in H\cap(Y\setminus Z_{n-1})}\widetilde{\Phi}(u)\ge \inf_{H\in\Sigma_\alpha,\gamma(H)\ge n}\sup_{u\in H\cap W_n}\widetilde{\Phi}(u)\\
&\ge \inf_{H\in\Sigma_\alpha,\gamma(H)\ge n}\sup_{u\in H\cap W_n}\frac{k_0}{p^+}\|u\|_Y^{p^-}\ge\frac{k_0}{p^+}c^{p^-}.
\end{aligned}$$

Thus we get (3.21). □

We are in a position to state the following theorem.

Theorem 3.1. *Assume that* (A.0)-(A.5) *and* (g.0)-(g.2) *hold and fix* $\alpha>0$. *Then for every* $n\in\mathbb{N}$, $c_{(n,\alpha)}$ *defined by* (3.17) *is a critical value of* $\widetilde{\Phi}$ *with respect to the submanifold* M_α *such that*

$$0<c_{(n,\alpha)}\le c_{(n+1,\alpha)}<\infty\quad\text{and}\quad c_{(n,\alpha)}\to\infty\ \text{ as } n\to\infty.$$

Moreover, the problem (1.1) *has infinitely many eigenpair sequences* $\{(u_{(n,\alpha)},\lambda_{(n,\alpha)})\}$ *such that*

$$K(\pm u_{(n,\alpha)})=\alpha,\Phi(\pm u_{(n,\alpha)})=c_{(n,\alpha)}\quad\text{and}\quad 0<\lambda_{(n,\alpha)}\to\infty\ \text{ as } n\to\infty.$$

Proof. Taking Proposition 3.10, (3.10)-(3.11) and Proposition 3.12 into consideration, it suffices to derive that $\lambda_{(n,\alpha)}\to\infty$ as $n\to\infty$. It follows from (g.2) that

$$\langle K'(u_{(n,\alpha)}),u_{(n,\alpha)}\rangle_{Y^*,Y}\le r^+K(u_{(n,\alpha)})=r^+\alpha.$$

Hence

$$\lambda_{(n,\alpha)}=\frac{\langle\Phi'(u_{(n,\alpha)}),u_{(n,\alpha)}\rangle_{Y^*,Y}}{\langle K'(u_{(n,\alpha)}),u_{(n,\alpha)}\rangle_{Y^*,Y}}\ge\frac{\langle\Phi'(u_{(n,\alpha)}),u_{(n,\alpha)}\rangle_{Y^*,Y}}{r^+\alpha}. \tag{3.22}$$

Assume that $\lambda_{(n,\alpha)}\le M$ for all $n\in\mathbb{N}$. Then by (3.22), $\langle\Phi'(u_{(n,\alpha)}),u_{(n,\alpha)}\rangle_{Y^*,Y}\le Mr^+\alpha=:c_2$. From (A.3), we have

$$\widetilde{\rho}_{(p(\cdot),h_1(\cdot))}(u_{(n,\alpha)})=\int_\Omega h_1(x)|\nabla u_{(n,\alpha)}(x)|^{p(x)}\mathrm{d}x\le\frac{1}{k_0}\langle\Phi'(u_{(n,\alpha)}),u_{(n,\alpha)}\rangle_{Y^*,Y}\le\frac{c_2}{k_0}.$$

In particular, $\|u_{(n,\alpha)}\|_Y\le c_3$ for all $n\in\mathbb{N}$ with some constant $c_3>0$. Then from Lemma 3.1 (i),

$$c_{(n,\alpha)}=\Phi(u_{(n,\alpha)})\le c\Big(2\|h_0\|_{L^{p'(\cdot)}(\Omega)}\|u_{(n,\alpha)}\|_Y+\widetilde{\rho}_{(p(\cdot),h_1(\cdot))}(u_{(n,\alpha)})\Big)\le c_4$$

for some constant $c_4>0$. This contradicts to Proposition 3.12. □

Remark 3.4. We do not know whether the problem (1.1) only has eigenvalue sequence of the form $\{\lambda_{(n,\alpha)}\}$.

4 The infimum of all the eigenvalues

In this section, we consider the infimum of all the eigenvalues of the problem (1.1). We show that there exist two cases where the infimum is equal to zero or positive according to the hypoetheses on the variable exponent.

Put $\Lambda=\{\lambda;\lambda$ is an eigenvalue of the problem (1.1)$\}$ and $\lambda_*=\inf\Lambda$. For a subset $A\subset\overline{\Omega}$ and $\delta>0$, put $B(A,\delta)=\{x\in\mathbb{R}^N;\mathrm{dist}(x,A)<\delta\}$, $B_\Omega(A,\delta)=B(A,\delta)\cap\Omega$ and $B_{\Gamma_2}(A,\delta)=B(A,\delta)\cap\Gamma_2$.

Here, for $x_0\in\overline{\Omega}$, if $A=\{x_0\}$, then we write $B(\{x_0\},\delta)$, $B_\Omega(\{x_0\},\delta)$ and $B_{\Gamma_2}(\{x_0\},\delta)$ by simply $B(x_0,\delta)$, $B_\Omega(x_0,\delta)$ and $B_{\Gamma_2}(x_0,\delta)$, respectively. Assume that (A.0)-(A.5) and (g.0)-(g.2) hold.

Lemma 4.1. *For $\delta,\alpha>0$, if we define*

$$\beta_\delta(u)=\int_{B_\Omega(\overline{\Gamma_2},\delta)}h_1(x)|\nabla u(x)|^{p(x)}\mathrm{d}x \quad \text{for } u\in Y,$$

then

$$\beta_{(\delta,\alpha)}:=\inf_{u\in M_\alpha}\beta_\delta(u)>0.$$

Proof. First we note that $Y\hookrightarrow Y\big(B_\Omega(\overline{\Gamma_2},\delta)\big)$ and β_δ is a modular on $Y\big(B_\Omega(\overline{\Gamma_2},\delta)\big)$. Assume that $\beta_{(\delta,\alpha)}=0$. Then there exist $\{u_n\}\subset M_\alpha$ such that $\beta_\delta(u_n)\to 0$ as $n\to\infty$. Hence $\|u_n\|_{Y(B_\Omega(\overline{\Gamma_2},\delta))}\to 0$ as $n\to\infty$, where $\|u\|_{Y(B_\Omega(\overline{\Gamma_2},\delta))}=\inf\{\tau>0;\beta_\delta\left(\frac{u}{\tau}\right)\le 1\}$.

On the other hand, we have

$$\int_{\partial B_\Omega(\overline{\Gamma_2},\delta))}G(x,u_n(x))\mathrm{d}\sigma_x\ge\int_{\Gamma_2}G(x,u_n(x))\mathrm{d}\sigma_x=K(u_n)=\alpha>0.$$

Since K is continuous on $Y\big(B_\Omega(\overline{\Gamma_2},\delta)\big)$, we can see that $K(u_n)\to K(0)=0$. This is a contradiction. □

Lemma 4.2. *For $\alpha>0$, let u_0 be an eigenfunction associated with $\lambda_{(1,\alpha)}$. Then*

$$\Phi(u_0)=c_{(1,\alpha)}=\inf\{\Phi(u);u\in M_\alpha\}.$$

Proof. Put $b_\alpha=\inf\{\Phi(u);u\in M_\alpha\}$. Since $c_{(1,\alpha)}=\inf_{H\in\Sigma_\alpha,\gamma(H)\ge 1}\sup_{u\in H}\widetilde{\Phi}(u)$, if $u\in H$ and $H\in\Sigma_\alpha\subset M_\alpha$ with $\gamma(H)\ge 1$, then $\widetilde{\Phi}(u)=\Phi(u)\ge b_\alpha$. Thus $c_{(1,\alpha)}\ge b_\alpha$.

By the definition of b_α, there exists a sequence $\{u_n\}\subset M_\alpha$ such that $b_\alpha=\lim_{n\to\infty}\Phi(u_n)$. For large n, $b_\alpha+1\ge\Phi(u_n)\ge k_0\|u_n\|_Y^{p^+}\wedge\|u_n\|_Y^{p^-}$. Thus $\{u_n\}$ is bounded in Y. So there exist a subsequence $\{u_{n'}\}$ of $\{u_n\}$ and $u_*\in Y$ such that $u_{n'}\to u_*$ weakly in Y. Thereby, $\Phi(u_*)\le\liminf_{n'\to\infty}\Phi(u_{n'})=b_\alpha$. Since M_α is a weakly closed subset of Y, $u_*\in M_\alpha$, so $\Phi(u_*)\ge b_\alpha$. Thus we have $\Phi(u_*)=b_\alpha$. By (A.5), $\Phi(\pm u_*)=b_\alpha$. Let $H_0=\{\pm u_*\}$, then clearly $\gamma(H_0)=1$. Therefore, $c_{(1,\alpha)}\le\sup_{u\in H_0}\Phi(u)=b_\alpha$. Thus we have $c_{(1,\alpha)}=b_\alpha$. □

Remark 4.1. We assume the following more restrictive condition instead of (A.3). (A.3′) $k_0h_1(x)|\boldsymbol{\xi}|^{p(x)} \le \boldsymbol{a}(x,\boldsymbol{\xi})\cdot\boldsymbol{\xi} = p(x)A(x,\boldsymbol{\xi})$ for a.e. $x\in\Omega$ and all $\boldsymbol{\xi}\in\mathbb{R}^N$.

We note that (i) in Example 3.1 satisfies this condition, but (ii) does not satisfy this condition.

Under (A.0)-(A.2), (A.3′), (A.4)-(A.5) and (g.0)-(g.2), we have

$$\lambda_{(n+1,\alpha)} \ge \frac{p^-r^-}{p^+r^+}\lambda_{(n,\alpha)}. \tag{4.1}$$

In particular, if $p(x)=p=\text{const.}$ and $r(x)=r=\text{const.}$, then $\lambda_{(n+1,\alpha)}\ge\lambda_{(n,\alpha)}$.

Proof. Let u_n be the eigenfuntion associated with the eigenvalue $\lambda_{(n,\alpha)}$ $n=1,\cdots$. From the hypotheses (A.3′), (g.2) and Theorem 3.1, we have

$$\begin{aligned}\lambda_{(n+1,\alpha)} &= \frac{\langle\Phi'(u_{n+1}),u_{n+1}\rangle_{Y^*,Y}}{\langle K'(u_{n+1}),u_{n+1}\rangle_{Y^*,Y}} = \frac{\int_\Omega \boldsymbol{a}(x,\nabla u_{n+1}(x))\cdot\nabla u_{n+1}(x)\mathrm{d}x}{\int_{\Gamma_2} g(x,u_{n+1}(x))u_{n+1}(x)\mathrm{d}\sigma_x}\\ &= \frac{\int_\Omega p(x)A(x,\nabla u_{n+1}(x))\mathrm{d}x}{\int_{\Gamma_2} r(x)G(x,u_{n+1}(x))\mathrm{d}\sigma_x} \ge \frac{p^-\Phi(u_{n+1})}{r^+K(u_{n+1})} = \frac{p^-}{r^+\alpha}c_{(n+1,\alpha)} \ge \frac{p^-}{r^+\alpha}c_{(n,\alpha)}.\end{aligned}$$

On the other hand, from (g.2) we have

$$\begin{aligned}c_{(n,\alpha)} &= \alpha\frac{\Phi(u_n)}{K(u_n)} = \alpha\frac{\int_\Omega A(x,\nabla u_n(x))\mathrm{d}x}{\int_{\Gamma_2} G(x,u_n(x))\mathrm{d}\sigma_x} \ge \alpha\frac{\int_\Omega \frac{1}{p(x)}\boldsymbol{a}(x,\nabla u_n(x))\cdot\nabla u_n(x)\mathrm{d}x}{\int_{\Gamma_2}\frac{1}{r(x)}g(x,u_n(x))u_n(x)\mathrm{d}\sigma_x}\\ &\ge \frac{r^-\alpha}{p^+}\frac{\langle\Phi'(u_n),u_n\rangle_{Y^*,Y}}{\langle K'(u_n),u_n\rangle_{Y^*,Y}} = \frac{r^-\alpha}{p^+}\lambda_{(n,\alpha)}.\end{aligned}$$

Thus we obtain the conclusive estimate (4.1). □

From now on, we suppose that more restrictive assumptions on the given function g.

(g.0′) and (g.0) holds with $r(x)=p(x)$.

(g.2′) and (g.2) holds with $r(x)=p(x)$, that is, $0<g(x,t)t=p(x)G(x,t)$ for σ-a.e. $x\in\Gamma_2$ and all $0\neq t\in\mathbb{R}$.

For example, a function $g(x,t)=b(x)|t|^{p(x)-2}t$ with a function $b(x)$ satisfying the condition in (g.0) with $r(x)=p(x)$ verifies (g.0′), (g.1) and (g.2′).

Theorem 4.1. *Assume that* (A.0)-(A.5), (g.0′), (g.1) *and* (g.2′) *hold, moreover, suppose that there exists* $\delta>0$ *such that* $p(x)=p=\text{const.}$ *for all* $x\in B_\Omega(\overline{\Gamma_2},\delta)$. *Then we have* $\lambda_*>0$.

Proof. Let u be the eigenfunction associated with λ of the problem (1.1). Then $K(u)>0$. In fact, let $K(u)=0$. Since $\langle \Phi'(u),u\rangle_{Y^*,Y}=\lambda\langle K'(u),u\rangle_{Y^*,Y}$, it follows from (g.2′) that

$$\int_\Omega \boldsymbol{a}(x,\nabla u(x))\cdot\nabla u(x)\mathrm{d}x=\lambda\int_{\Gamma_2} g(x,u(x))u(x)\mathrm{d}\sigma_x=\lambda p\int_{\Gamma_2} G(x,u(x))\mathrm{d}\sigma_x=\lambda pK(u)=0.$$

Hence from (A.3) we have

$$0=\int_\Omega \boldsymbol{a}(x,\nabla u(x))\cdot\nabla u(x)\mathrm{d}x\geq k_0\int_\Omega h_1(x)|\nabla u(x)|^{p(x)}\mathrm{d}x.$$

Thus we have $\nabla u(x)=\mathbf{0}$ a.e. $x\in\Omega$. From Proposition 2.14, we have $u=0$ a.e. in Ω. This is a contradiction.

We show that there exists $t_0>0$ such that $u_1:=\frac{1}{t_0}u\in M_1$. Indeed, since $g(x,t)t=pG(x,t)$ for σ-a.e. $x\in\Gamma_2$ and all $t\in\mathbb{R}$, we can write $G(x,t)=G(x,1)|t|^p$ from (3.4). Hence $K(u)=\int_{\Gamma_2}G(x,1)|u(x)|^p\mathrm{d}\sigma_x$. Thus $K\left(\frac{u}{t}\right)=t^{-p}K(u)$ for $t>0$. Here we can see that $K\left(\frac{u}{t}\right)\to 0$ as $t\to\infty$ and $K\left(\frac{u}{t}\right)\to\infty$ as $t\to+0$. Since $K\left(\frac{u}{t}\right)$ is continuous with respect to $t\in(0,\infty)$, there exists $t_0>0$ such that $K\left(\frac{u}{t_0}\right)=1$, so $u_1:=\frac{u}{t_0}\in M_1$.

Now since $K(u_1)=1$, it follows from (A.3) and Lemma 4.1 that

$$\begin{aligned}\lambda&=\frac{\displaystyle\int_\Omega \boldsymbol{a}(x,\nabla u(x))\cdot\nabla u(x)\mathrm{d}x}{\displaystyle\int_{\Gamma_2} g(x,u(x))u(x)\mathrm{d}\sigma_x}\geq\frac{\displaystyle\int_{B_\Omega(\overline{\Gamma_2},\delta)} \boldsymbol{a}(x,\nabla u(x))\cdot\nabla u(x)\mathrm{d}x}{\displaystyle\int_{\Gamma_2} g(x,u(x))u(x)\mathrm{d}\sigma_x}\\
&\geq\frac{\displaystyle k_0\int_{B_\Omega(\overline{\Gamma_2},\delta)} h_1(x)|\nabla u(x)|^p\mathrm{d}x}{\displaystyle p\int_{\Gamma_2} G(x,u(x))\mathrm{d}\sigma_x}=\frac{\displaystyle k_0\int_{B_\Omega(\overline{\Gamma_2},\delta)} h_1(x)|t_0\nabla u_1(x)|^p\mathrm{d}x}{\displaystyle p\int_{\Gamma_2} G(x,t_0u_1(x))\mathrm{d}\sigma_x}\\
&=\frac{\displaystyle k_0t_0^p\int_{B_\Omega(\overline{\Gamma_2},\delta)} h_1(x)|\nabla u_1(x)|^p\mathrm{d}x}{\displaystyle pt_0^p\int_{\Gamma_2} G(x,u_1(x))\mathrm{d}\sigma_x}=\frac{k_0}{p}\int_{B_\Omega(\overline{\Gamma_2},\delta)} h_1(x)|\nabla u_1(x)|^p\mathrm{d}x=\frac{k_0}{p}\beta_{(\delta,1)}>0.\end{aligned}$$

Thus we have $\lambda_*=\inf\Lambda\geq\frac{k_0}{p}\beta_{(\delta,1)}>0$. □

From the absolute continuity of integral, we obtain the following lemma which is needed later.

Lemma 4.3. *Let $u\in Y$ be given. Then for any $\varepsilon>0$, there exists $\delta_0>0$ such that for any $0<\delta<\delta_0$,*

$$\beta_u(\delta):=\int_{B_\Omega(\overline{\Gamma_2},\delta)} A(x,\nabla u(x))\mathrm{d}x<\varepsilon.$$

Theorem 4.2. *Let* (A.0)-(A.5), (g.0′), (g.1) *and* (g.2′) *hold. Assume that there exist $\delta>0$ and $x_0\in\Gamma_2$ such that the following* (i)-(iii) *hold.*

(i) $p(x)=p=\text{const.}$ *for all* $x\in B_{\Gamma_2}(x_0,\delta)$.
(ii) $p(x)<p$ *for all* $x\in B_\Omega(x_0,\delta)$.
(iii) $h_1\in L^1(B_\Omega(x_0,\delta))$, *where* h_1 *is the function of* (A.1)-(A.3).
Then we have $\lim_{\alpha\to\infty}\lambda_{(1,\alpha)}=0$, *so* $\lambda_*=0$.

Proof. Replacing $\delta>0$ with smaller one, if necessary, we may assume that $B(x_0,\delta)\cap\Gamma\subset\Gamma_2$. Choose $0\le u\in C^\infty(\overline{\Omega})$ such that $u(x)=1$ for $x\in B_\Omega(x_0,\delta/4)$ and $u(x)=0$ for $x\in\Omega\setminus B_\Omega(x_0,\delta/2)$. We note that it follows from (iii) that $u\in Y$. By Lemma 4.3, for any $\varepsilon>0$, there exists $\delta_0\in(0,\delta/4)$ such that for any $\delta_1\in(0,\delta_0)$,

$$\int_{B_\Omega(\overline{\Gamma_2},\delta_1)}A(x,\nabla u(x))\mathrm{d}x<\varepsilon/(2c),$$

where

$$c=\frac{p^+}{p^-\int_{B_{\Gamma_2}(x_0,\delta/2)}G(x,u(x))\mathrm{d}\sigma_x}>0.$$

Since $p\in C(\overline{\Omega})$, it follows from (ii) that for any $x\in B_\Omega(x_0,\delta/2)\setminus B(\Gamma_2,\delta_0)$, we can see that

$$p(x)-p\le p^+\left(\overline{B_\Omega(x_0,\delta/2)}\setminus B(\Gamma_2,\delta_0)\right)-p:=-\varepsilon_0<0.$$

We note that $p(x)=p$ on $\mathrm{supp}\,u\cap\Gamma_2$. If we define $h(t)=K(tu)=t^pK(u)$, then h is differentiable in $(0,\infty)$ and $h'(t)=pt^{p-1}K(u)>0$, so h is strictly monotone increasing and clearly $h(t)\to0$ as $t\to+0$ and $h(t)\to\infty$ as $t\to\infty$. Hence for any $\alpha>0$, there exists a unique $t(\alpha)>0$ such that $t(\alpha)u\in M_\alpha$. Clearly $t(\alpha)\to0$ as $\alpha\to+0$ and $t(\alpha)\to\infty$ as $\alpha\to\infty$. So there exists $\alpha_0>1$ such that for any $\alpha\in(\alpha_0,\infty)$, $\max\left\{1,\left(2\varepsilon^{-1}c\Phi(u)\right)^{1/\varepsilon_0}\right\}<t(\alpha)$. Let u_0 be the eigenfunction associated with $\lambda_{(1,\alpha)}$. Then we have

$$\lambda_{(1,\alpha)}=\frac{\int_\Omega \boldsymbol{a}(x,\nabla u_0(x))\cdot\nabla u_0(x)\mathrm{d}x}{\int_{\Gamma_2}g(x,u_0(x))u_0(x)\mathrm{d}\sigma_x}\le\frac{\int_\Omega p(x)A(x,\nabla u_0(x))\mathrm{d}x}{p^-\int_{\Gamma_2}G(x,u_0(x))\mathrm{d}\sigma_x}\le\frac{p^+\Phi(u_0)}{p^-K(u_0)}=\frac{p^+\Phi(u_0)}{p^-\alpha}.$$

By Lemma 4.2, since $\Phi(u_0)=c_{(1,\alpha)}=\inf\{\Phi(u);u\in M_\alpha\}$, we have $\Phi(u_0)\le\Phi(t(\alpha)u)$. Hence

$$\lambda_{(1,\alpha)}\le\frac{p^+\Phi(t(\alpha)u)}{p^-\alpha}=\frac{p^+\Phi(t(\alpha)u)}{p^-K(t(\alpha)u)}.$$

To simplify the symbols, put

$$B_1=B_\Omega\Big(x_0,\frac{\delta}{2}\Big)\setminus B_\Omega(\Gamma_2,\delta_1),\quad B_2=B_\Omega\Big(x_0,\frac{\delta}{2}\Big)\cap B_\Omega(\Gamma_2,\delta_1).$$

Thus using (3.1), we have

$$\begin{aligned}
\lambda_{(1,\alpha)} \leq & \frac{p^+\int_\Omega A(x,t(\alpha)\nabla u(x))\mathrm{d}x}{p^-\int_{\Gamma_2} G(x,t(\alpha)u(x))\mathrm{d}\sigma_x} \\
= & \frac{p^+\int_{B_1} A(x,t(\alpha)\nabla u(x))\mathrm{d}x + p^+\int_{B_2} A(x,t(\alpha)\nabla u(x))\mathrm{d}x}{p^-\int_{B_{\Gamma_2}(x_0,\frac{\delta}{2})} G(x,t(\alpha)u(x))\mathrm{d}\sigma_x} \\
\leq & \frac{p^+\int_{B_1} t(\alpha)^{p(x)} A(x,\nabla u(x))\mathrm{d}x + p^+\int_{B_2} t(\alpha)^{p(x)} A(x,\nabla u(x))\mathrm{d}x}{p^- t(\alpha)^p \int_{B_{\Gamma_2}(x_0,\frac{\delta}{2})} G(x,u(x))\mathrm{d}\sigma_x} \\
= & \frac{p^+\int_{B_1} t(\alpha)^{p(x)-p} A(x,\nabla u(x))\mathrm{d}x + p^+\int_{B_2} t(\alpha)^{p(x)-p} A(x,\nabla u(x))\mathrm{d}x}{p^-\int_{B_{\Gamma_2}(x_0,\frac{\delta}{2})} G(x,u(x))\mathrm{d}\sigma_x} \\
\leq & ct(\alpha)^{-\varepsilon_0}\int_{B_\Omega(x_0,\frac{\delta}{2})\setminus B_\Omega(\Gamma_2,\delta_1)} A(x,\nabla u(x))\mathrm{d}x + c\int_{B_\Omega(\Gamma_1,\delta_1)} A(x,\nabla u(x))\mathrm{d}x \\
< & \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Therefore, $\lambda_{(1,\alpha)} < \varepsilon$ for all $\alpha > \alpha_0$. Since $\varepsilon > 0$ is arbitrary, we have $\lim_{\alpha\to\infty}\lambda_{(1,\alpha)} = 0$. □

We can also derive the next theorem.

Theorem 4.3. *Let* (A.0)-(A.5), (g.0′), (g.1) *and* (g.2′) *hold. Moreover, suppose that there exist an open subset V in Γ_2 with $\overline{V}\subset\Gamma_2$, $\delta>0$ and $\boldsymbol{\xi}\in\mathbb{R}^N\setminus\overline{V}$ such that for all $x\in V$, $I_x := \{x+\tau\boldsymbol{\xi}_x;\tau\in(0,\delta)\}\subset\Omega$, where $\boldsymbol{\xi}_x = \frac{\boldsymbol{\xi}-x}{|\boldsymbol{\xi}-x|}$ or $\frac{x-\boldsymbol{\xi}}{|\boldsymbol{\xi}-x|}$. Moreover, suppose that*

(A) $p^+(V) > p^+(\partial V)$.

(B) $p(x) > p(y)$ *for all* $x\in V, y\in I_x$.

(C) $h_1 \in L^1(\cup_{x\in V} I_x)$, *where h_1 is the function of* (A.1)-(A.3).

Then we have $\lim_{\alpha\to\infty}\lambda_{(1,\alpha)} = 0$, so $\lambda_* = 0$.

Proof. For $c,d>0$, put $p^c = \{x\in V; p(x) > p^+(\overline{V}) - c\}$ and $p^{c\times d} = \{x+\tau\boldsymbol{\xi}_x; x\in p^c, \tau\in(0,d)\}$, and $\varepsilon_0 = (p^+(V) - p^+(\partial V))/16 (>0)$. Then clearly $\overline{p^{4\varepsilon_0}} \subset p^{8\varepsilon_0}$. Choose a non-negative function $u\in C^\infty(\overline{\Omega})$ such that $u(x)=1$ for $x\in p^{4\varepsilon_0\times\delta/4}$ and $u(x)=0$ for $x\in\Omega\setminus p^{8\varepsilon_0\times\delta/2}$. Then it follows from (C) that $u\in Y$. For any $\varepsilon>0$, define $\delta_1 = \delta/8$. Then if $x\in p^{8\varepsilon_0\times\delta/2}\setminus p^{2\varepsilon_0\times\delta_1}$, then

$$p(x) - p^+(V) \leq p^+\left(p^{8\varepsilon_0\times\delta/2}\setminus p^{2\varepsilon_0\times\delta_1}\right) - p^+(V) := -2\varepsilon_1 < 0.$$

Similarly as the proof of Theorem 4.2, there exists $\alpha_0>1$ such that for any $\alpha>\alpha_0$, there exists $t(\alpha)$ such that $K(t(\alpha)u)=\alpha$ and

$$\max\left\{1,\left(\frac{2\varepsilon^{-1}p^+\Phi(u)}{p^-\int_{p^{\varepsilon_1}}G(x,u(x))\mathrm{d}\sigma_x}\right)^{\frac{1}{\varepsilon_1}}\right\}\le t(\alpha).$$

Using again (3.1), we have

$$\begin{aligned}\frac{\int_{p^{8\varepsilon_0\times\frac{\delta}{2}}\setminus p^{2\varepsilon_0\times\delta_1}}A(x,t(\alpha)\nabla u(x))\mathrm{d}x}{\int_{\Gamma_2}G(x,t(\alpha)u(x))\mathrm{d}\sigma_x}&\le\frac{\int_{p^{8\varepsilon_0\times\frac{\delta}{2}}\setminus p^{2\varepsilon_0\times\delta_1}}t(\alpha)^{p(x)}A(x,\nabla u(x))\mathrm{d}x}{\int_{p^{\varepsilon_1}}t(\alpha)^{p(x)}G(x,u(x))\mathrm{d}\sigma_x}\\&\le\frac{\int_{p^{8\varepsilon_0\times\frac{\delta}{2}}\setminus p^{2\varepsilon_0\times\delta_1}}t(\alpha)^{p^+(V)-2\varepsilon_1}A(x,\nabla u(x))\mathrm{d}x}{\int_{p^{\varepsilon_1}}t(\alpha)^{p^+(V)-\varepsilon_1}G(x,u(x))\mathrm{d}\sigma_x}\\&\le t(\alpha)^{-\varepsilon_1}\frac{\Phi(u)}{\int_{p^{\varepsilon_1}}G(x,u(x))\mathrm{d}\sigma_x}\le\frac{p^-}{2p^+}\varepsilon.\end{aligned}$$

On the other hand, since $p^c\subset p^{c'}$ if $c<c'$, $p^{2\varepsilon_0\times\delta_1}\subset p^{4\varepsilon_0\times\delta_1}$. Hence if $x\in p^{2\varepsilon_0\times\delta_1}$, then $u(x)=1$, so $\nabla u(x)=\mathbf{0}$. Hence we have

$$\int_{p^{2\varepsilon_0\times\delta_1}}A(x,t(\alpha)\nabla u(x))\mathrm{d}x=0.$$

Thus we have

$$\begin{aligned}\lambda_{(1,\alpha)}\le\frac{p^+\Phi(t(\alpha)u)}{p^-K(t(\alpha)u)}&\le\frac{p^+\int_{p^{8\varepsilon_0\times\frac{\delta}{2}}\setminus p^{2\varepsilon_0\times\delta_1}}A(x,t(\alpha)\nabla u(x))\mathrm{d}x}{p^-\int_{\Gamma_2}G(x,t(\alpha)u(x))\mathrm{d}\sigma_x}\\&\quad+\frac{p^+\int_{p^{2\varepsilon_0\times\delta_1}}A(x,t(\alpha)\nabla u(x))\mathrm{d}x}{p^-\int_{\Gamma_2}G(x,t(\alpha)u(x))\mathrm{d}\sigma_x}\le\frac{\varepsilon}{2}<\varepsilon.\end{aligned}$$

Therefore, we have $\lim_{\alpha\to\infty}\lambda_{(1,\alpha)}=0$, so $\lambda_*=0$. □

Remark 4.2. (1) If $p(x)=p=$const. in Ω, then it is well known that $\lambda_*=\lambda_{(1,\alpha)}=\lambda_1$ and so λ_* is a principal eigenvalue.

(2) For a variable exponent $p(x)$, under some assumptions, $\lambda_*=0$. This means that under some assumptions, there does not exist a principal eigenvalue and the set of eigenvalues is not closed.

(3) For a variable exponent $p(x)$, under some assumptions, we have $\lambda_*>0$.

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