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On Concentration of Real Solutions for Fractional Helmholtz Equation

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Abstract. This paper studies the nonlinear fractional Helmholtz equation

$$(-\Delta)^s u - k^2 u = Q(x)|u|^{p-2}u, \quad \text{in } \mathbb{R}^N,\ N \geq 3, \tag{0.1}$$

where $\frac{N}{N+1} < s < \frac{N}{2}$, $\frac{2(N+1)}{N-1} < p < \frac{2N}{N-2s}$ are two real exponents, and the coefficient $Q(x)$ is a bounded continuous, nonnegative function that satisfies the condition

$$\limsup_{|x|\to\infty} Q(x) < \sup_{x\in\mathbb{R}^N} Q(x). \tag{0.2}$$

For sufficiently large $k>0$, the existence of real-valued solutions to (0.1) is established. Furthermore, as $k\to\infty$, it is shown that the sequence of solutions associated with the ground states of a dual equation concentrates, after rescaling, at global maximum points of the function $Q(x)$.

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1 Introduction and main results

In this paper, we are concerned with the nonlinear fractional Helmholtz equation

$$(-\Delta)^s u - k^2 u = Q(x)|u|^{p-2}u, \quad \text{in } \mathbb{R}^N,\ N \geq 3, \tag{1.1}$$

where $\frac{N}{N+1} < s < \frac{N}{2}$, $\frac{2(N+1)}{N-1} < p < \frac{2N}{N-2s}$ are two real exponents, $Q(x)$ is a bounded continuous function.

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When $s=1$, the Helmholtz equation

$$-\Delta u-k^2u=f(x,u),\quad \text{in } \mathbb{R}^N, \tag{1.2}$$

has garnered significant attention due to its importance in scattering and optics. The main feature of this problem is that the parameter $k^2>0$ is contained in the essential spectrum of negative Laplace $-\Delta$. A common method for detecting the existence of weak solutions is the linking argument, that is finding the critical point of the corresponding functional. However, it is challenging to find an appropriate space where the associated functional of (1.2) can be well-defined, this difficulty arises because the the oscillating solutions with slow decay typically do not belong to $H^1(\mathbb{R}^N)$. Consequently, the direct variational approach is not applicable.

To address this issue, a dual invariant method has been proposed, which is based on the "Limiting Absorption Principle". By constructing the auxiliary problems

$$-\Delta u-(\lambda+i\varepsilon)u=f(x,u),\quad \text{in } \mathbb{R}^N, \tag{1.3}$$

one can obtain the boundedness estimate for the resolvent operator

$$\mathcal{R}_{\lambda,\varepsilon}=(-\Delta-(\lambda+i\varepsilon))^{-1},$$

and as $\varepsilon\to 0^+$, one can also obtain the boundedness estimate for the resolvent $\mathcal{R}_\lambda=(-\Delta-\lambda)^{-1}$, see [1, Theorem 6] or see [2–6]. Based on the boundedness estimate, Evéquoz and Weth [7] ([8]) set up a dual variational framework for (1.2). Correspondingly, the nontrivial real-valued solutions of Eq. (1.2) with $f(x,u)=Q(x)|u|^{p-2}u$ are detected via the mountain pass argument, where $Q(x)$ is a positive weight function, For further details on other cases, see [9–13]. By a similar way, Shen and the second author [14] obtained real valued solutions for the fractional Helmholtz Eq. (1.1) under the conditions $0<k^2<+\infty$ and $Q(x)$ being either periodic or decaying.

Recently, Evéquoz [15] examined (1.2) in a limiting case and obtain some surprising results regarding its solutions. Specifically, if Q is assumed to be a bounded continuous function, Eq. (1.2) still admits a real-valued solution for sufficiently large k. Furthermore, the solutions tend to concentrate at the global maximum points of the function $Q(x)$ as the frequency λ approaches infinity. Actually, concentrating solutions have long been a significant topic in the study of the nonlinear Schrödinger equation

$$-\varepsilon^2\Delta u+V(x)u=Q(x)|u|^{p-2}u,\quad \text{in } \mathbb{R}^N, \tag{1.4}$$

where $V(x)\geq 0$ is a potential function. Typically, we can investigate the concentration behavior of solutions through two approaches. One is the Lyapunov-Schmidt reduction scheme proposed by Floer and Weinstein [16], which has been further extended and combined with variational arguments by Ambrosetti et al. [17–20], see also for example [21, 22] for multibump solutions. Another one is the purely variational approach