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# On Concentration of Real Solutions for Fractional Helmholtz Equation

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**Abstract.** This paper studies the nonlinear fractional Helmholtz equation

$$(-\Delta)^s u - k^2 u = Q(x)|u|^{p-2}u, \quad \text{in } \mathbb{R}^N,\ N \geq 3, \tag{0.1}$$

where $\frac{N}{N+1} < s < \frac{N}{2}$, $\frac{2(N+1)}{N-1} < p < \frac{2N}{N-2s}$ are two real exponents, and the coefficient $Q(x)$ is a bounded continuous, nonnegative function that satisfies the condition

$$\limsup_{|x|\to\infty} Q(x) < \sup_{x\in\mathbb{R}^N} Q(x). \tag{0.2}$$

For sufficiently large $k>0$, the existence of real-valued solutions to (0.1) is established. Furthermore, as $k\to\infty$, it is shown that the sequence of solutions associated with the ground states of a dual equation concentrates, after rescaling, at global maximum points of the function $Q(x)$.

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**Key Words**: Fractional Helmholtz equation; concentration of solutions; dual variational method; Lusternik-Schnirelman category.

## 1 Introduction and main results

In this paper, we are concerned with the nonlinear fractional Helmholtz equation

$$(-\Delta)^s u - k^2 u = Q(x)|u|^{p-2}u, \quad \text{in } \mathbb{R}^N,\ N \geq 3, \tag{1.1}$$

where $\frac{N}{N+1}<s<\frac{N}{2}$, $\frac{2(N+1)}{N-1}<p<\frac{2N}{N-2s}$ are two real exponents, $Q(x)$ is a bounded continuous function.

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When $s=1$, the Helmholtz equation

$$-\Delta u-k^2u=f(x,u),\quad \text{in } \mathbb{R}^N, \tag{1.2}$$

has garnered significant attention due to its importance in scattering and optics. The main feature of this problem is that the parameter $k^2>0$ is contained in the essential spectrum of negative Laplace $-\Delta$. A common method for detecting the existence of weak solutions is the linking argument, that is finding the critical point of the corresponding functional. However, it is challenging to find an appropriate space where the associated functional of (1.2) can be well-defined, this difficulty arises because the the oscillating solutions with slow decay typically do not belong to $H^1(\mathbb{R}^N)$. Consequently, the direct variational approach is not applicable.

To address this issue, a dual invariant method has been proposed, which is based on the "Limiting Absorption Principle". By constructing the auxiliary problems

$$-\Delta u-(\lambda+i\varepsilon)u=f(x,u),\quad \text{in } \mathbb{R}^N, \tag{1.3}$$

one can obtain the boundedness estimate for the resolvent operator

$$\mathcal{R}_{\lambda,\varepsilon}=(-\Delta-(\lambda+i\varepsilon))^{-1},$$

and as $\varepsilon\to 0^+$, one can also obtain the boundedness estimate for the resolvent $\mathcal{R}_\lambda=(-\Delta-\lambda)^{-1}$, see [1, Theorem 6] or see [2–6]. Based on the boundedness estimate, Evéquoz and Weth [7] ([8]) set up a dual variational framework for (1.2). Correspondingly, the nontrivial real-valued solutions of Eq. (1.2) with $f(x,u)=Q(x)|u|^{p-2}u$ are detected via the mountain pass argument, where $Q(x)$ is a positive weight function, For further details on other cases, see [9–13]. By a similar way, Shen and the second author [14] obtained real valued solutions for the fractional Helmholtz Eq. (1.1) under the conditions $0<k^2<+\infty$ and $Q(x)$ being either periodic or decaying.

Recently, Evéquoz [15] examined (1.2) in a limiting case and obtain some surprising results regarding its solutions. Specifically, if $Q$ is assumed to be a bounded continuous function, Eq. (1.2) still admits a real-valued solution for sufficiently large $k$. Furthermore, the solutions tend to concentrate at the global maximum points of the function $Q(x)$ as the frequency $\lambda$ approaches infinity. Actually, concentrating solutions have long been a significant topic in the study of the nonlinear Schrödinger equation

$$-\varepsilon^2\Delta u+V(x)u=Q(x)|u|^{p-2}u,\quad \text{in } \mathbb{R}^N, \tag{1.4}$$

where $V(x)\geq 0$ is a potential function. Typically, we can investigate the concentration behavior of solutions through two approaches. One is the Lyapunov-Schmidt reduction scheme proposed by Floer and Weinstein [16], which has been further extended and combined with variational arguments by Ambrosetti et al. [17–20], see also for example [21, 22] for multibump solutions. Another one is the purely variational approach initiated by Rabinowitz [23], which is mainly relayed by del Pino and Felmer [24–27]. More precisely, under the global condition

$$\liminf_{|x|\to\infty} V(x) > \inf_{x\in\mathbb{R}^N} V(x), \tag{1.5}$$

it was proved in [28] that a ground state (i.e., positive least-energy solution) of (1.4) exists for small $\varepsilon>0$. In the limit $\varepsilon\to 0$, Wang [29] showed that sequences of ground states concentrate at a global minimum point $x_0$ of $V$ and converge, after rescaling, toward the ground state of the limit problem

$$-\Delta u + V(x_0)u = |u|^{p-2}u, \quad \text{in } \mathbb{R}^N. \tag{1.6}$$

These results are also extended to the fractional Shrödinger equation

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u = f(x,u), \quad \text{in } \mathbb{R}^N. \tag{1.7}$$

Redears can also consult the works [30,31] and [32,33] for more information.

Motivated by these works, we explore in this paper the existence and concentration phenomena of (1.1) as $k\to\infty$. As previously mentioned, the structure of the Helmholtz equation is highly complex. There are no known results regarding the uniqueness or non-degeneracy of real-valued solutions, which means that classical reduction methods may not be applicable for constructing concentrating solutions. This compels us to consider an alternative approach proposed by Rabinowitz. Indeed, the variational method cannot be directly adapted to our case, as there is no natural concept of a ground state associated with the direct variational formulation.

Follow the idea in [15], we define the dual ground state for (1.1) as follow. Setting $\varepsilon=k^{-1}$, $u_\varepsilon(x)=\varepsilon^{\frac{2s}{p-2}}u(\varepsilon x)$ and $Q_\varepsilon(x)=Q(\varepsilon x)$, $x\in\mathbb{R}^N$, (1.1) can be rewritten as

$$(-\Delta)^s u_\varepsilon - u_\varepsilon = Q_\varepsilon(x)|u_\varepsilon|^{p-2}u_\varepsilon, \quad \text{in } \mathbb{R}^N. \tag{1.8}$$

Furthermore, setting $v=Q^{\frac{1}{p'}}|u_\varepsilon|^{p-2}u_\varepsilon$, we are led to consider the integral equation

$$|v|^{p-2}v = Q_\varepsilon^{\frac{1}{p}}\left[\mathbf{R}^s * \left(Q_\varepsilon^{\frac{1}{p}} v\right)\right], \tag{1.9}$$

where $p'=\frac{p}{p-1}$ and $\mathbf{R}^s$ denotes the real part of the fractional Helmholtz resolvent operator, see [14]. The solutions of this integral equation are critical points of the so-called dual energy functional $J_\varepsilon : L^{p'}(\mathbb{R}^N)\to\mathbb{R}$ given by

$$J_\varepsilon(v) = \frac{1}{p'}\int_{\mathbb{R}^N}|v|^{p'}\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} Q_\varepsilon^{\frac{1}{p}} v\mathbf{R}^s\left(Q_\varepsilon^{\frac{1}{p}} v\right)\mathrm{d}x. \tag{1.10}$$

Furthermore, every critical point $v$ of $J_\varepsilon$ gives rise to a strong solution $u$ of (1.1) with $k=\frac{1}{\varepsilon}$, by setting

$$u(x) = k^{\frac{2s}{p-2}}\mathbf{R}^s\left(Q_\varepsilon^{\frac{1}{p}} v\right)(kx), \quad \text{in } \mathbb{R}^N. \tag{1.11}$$

This correspondence allows us to define a notion of ground state for (1.1) as follows. If $\varepsilon=\frac{1}{k}$, and $v$ is a nontrivial critical point for $J_\varepsilon$ at the mountain pass level, the function $u$ given by (1.11) will be called a *dual ground state* of (1.1).

Apparently, once we obtain the existence and concentration of $v$, we can subsequently determine the existence and concentration of $u$, up to rescaling, of sequences of dual ground states. This leads us to our main result.

**Theorem 1.1.** *Let $N\geq 3$, $\frac{N}{N+1}<s<\frac{N}{2}$, $\frac{2(N+1)}{N-1}<p<\frac{2N}{N-2s}$ and consider a function $Q(x)$ satisfying*

(Q0) *$Q(x)$ is continuous, bounded and $Q(x)\geq 0$ on $\mathbb{R}^N$;*

(Q1) $Q_\infty:=\limsup_{|x|\to\infty} Q(x)<Q_0:=\sup_{x\in\mathbb{R}^N} Q(x)$.

(i) *There is $k_0>0$ such that for all $k>k_0$, the problem* (1.1) *admits a dual ground state.*

(ii) *Let $(k_n)_n\subset(k_0,\infty)$ satisfy $\lim\limits_{n\to\infty}k_n=\infty$ and consider for each $n$, a dual ground state $u_n$ of*

$$(-\Delta)^s u-k_n^2u=Q(x)|u|^{p-2}u, \quad \text{in } \mathbb{R}^N. \tag{1.12}$$

*Then there is a maximum point $x_0$ of $Q$, a dual ground state $u_0$ of*

$$(-\Delta)^s u-u=Q_0|u|^{p-2}u, \quad \text{in } \mathbb{R}^N, \tag{1.13}$$

*and a sequence $(x_n)_n\subset\mathbb{R}^N$ such that (up to a subsequence) $\lim\limits_{n\to\infty}x_n=x_0$ and*

$$k_n^{-\frac{2}{p-2}}u_n\Big(\frac{\cdot}{k_n}+x_n\Big)\to u_0, \quad \text{in } L^p\big(\mathbb{R}^N\big), \text{ as } n\to\infty. \tag{1.14}$$

Owing to the assumption on $Q(x)$, we demonstrate that for sufficiently small $\varepsilon$, the dual energy functional is strictly lower than the minimum of all possible energy levels at infinity; this is detailed in Section 2. Consequently, we demonstrate that the dual energy functional fulfills the Palais-Smale condition, thereby proving the first part of the theorem, as outlined in Section 3. The proof of the second part relies on a representation lemma and is completed in Section 4.

Follow the same idea in [15, Theorem 2], we can also obtain the multiplicity result for (1.1). Let $M=\{x\in\mathbb{R}^N: Q(x)=Q_0\}$ denotes the set of maximum points of $Q$, and for $\delta>0$ we let $M_\delta=\{x\in\mathbb{R}^N:\mathrm{dist}(x,M)\leq\delta\}$. Also, for a closed subset $Y$ of a metric space $X$ we denote by $\mathrm{cat}_X(Y)$ the Lusternik-Schnirelmann category of $Y$ with respect to $X$, i.e., the least number of closed contractible sets in $X$ which cover $Y$.

**Theorem 1.2.** *Let $N\geq 3$, $\frac{N}{N+1}<s<\frac{N}{2}$, $\frac{2(N+1)}{N-1}<p<\frac{2N}{N-2s}$ and consider a function $Q$ satisfying (Q0) and (Q1). For every $\delta>0$, there exists $k(\delta)>0$ such that* (1.1) *has at least* $\mathrm{cat}_{M_\delta}(M)$ *nontrivial solutions for all $k>k_\delta$.*

The proof of the aforementioned result hinges on topological arguments akin to those developed by Cingolani and Lazzo [34] for (1.4) (see also [35]). These arguments are rooted in the ideas of Benci, Cerami and Passaseo [36, 37], particularly in their work on problems defined on bounded domains. The crux of the matter lies in constructing two maps whose composition is homotopic to the inclusion $M \hookrightarrow M_\delta$. For a detailed exposition, we refer the reader to [15, Theorem 2].

We close the introduction by some notations. For $1 \le q \le \infty$, we write $||\cdot||_q$ instead $||\cdot||_{L^q(\mathbb{R}^N)}$ for the standard norm of the Lebesgue space $L^q(\mathbb{R}^N)$. In addition, for $r>0$ and $x \in \mathbb{R}^N$, we denote by $B_r(x)$ the open ball in $\mathbb{R}^N$ of radius $r$ centered at $x$, and let $B_r = B_r(0)$.

# 2 The variational framework

## 2.1 Dual functional

Before comparing the energy functional, let us revisit some properties of the dual functiona (1.10). Given that $p'<2$ and the kernel of the operator $\mathbf{R}^s$ is positive near the origin, the geometry of the functional $J_\varepsilon$ exhibits a mountain pass structure:

$$\exists\, \alpha>0 \text{ and } \rho>0 \text{ such that } J_\varepsilon(v) \ge \alpha>0, \quad \forall\, v \in L^{p'}(\mathbb{R}^N) \text{ with } ||v||_{p'} = \rho. \tag{2.1}$$

$$\exists v_0 \in L^{p'}(\mathbb{R}^N) \text{ such that } ||v_0||_{p'} > \rho \text{ and } J_\varepsilon(v_0) < 0. \tag{2.2}$$

As a consequence, the Nehari set associated to $J_\varepsilon$:

$$\mathcal{N}_\varepsilon := \left\{ v \in L^{p'}(\mathbb{R}^N) \setminus \{0\} : J'_\varepsilon(v)v = 0 \right\}, \tag{2.3}$$

is not empty. More precisely, by (2.1), the set

$$U_\varepsilon^+ := \left\{ v \in L^{p'}(\mathbb{R}^N) : \int_{\mathbb{R}^N} Q_\varepsilon^{\frac{1}{p}} v \mathbf{R}^s \left( Q_\varepsilon^{\frac{1}{p}} v \right) \mathrm{d}x > 0 \right\} \tag{2.4}$$

is not empty and for each $v \in U_\varepsilon^+$ there is a unique $t_v > 0$ such that $t_v v \in \mathcal{N}_\varepsilon$ holds. It is given by

$$t_v^{2-p'} = \frac{\displaystyle\int_{\mathbb{R}^N} |v|^{p'} \mathrm{d}x}{\displaystyle\int_{\mathbb{R}^N} Q_\varepsilon^{\frac{1}{p}} v \mathbf{R}^s \left( Q_\varepsilon^{\frac{1}{p}} v \right) \mathrm{d}x}. \tag{2.5}$$

In addition, $t_v$ is the unique maximum point of $t \mapsto J_\varepsilon(tv)$, $t \ge 0$. Using (2.1), we obtain in particular

$$c_\varepsilon := \inf_{\mathcal{N}_\varepsilon} J_\varepsilon = \inf_{v \in U_\varepsilon^+} J_\varepsilon(t_v v) > 0. \tag{2.6}$$

Moreover, for every $v \in \mathcal{N}_\varepsilon$ we have $c_\varepsilon \le J_\varepsilon(v) = \left(\frac{1}{p'} - \frac{1}{2}\right) ||v||_{p'}^{p'}$. Hence, 0 is isolated in the set $\left\{ v \in L^{p'}(\mathbb{R}^N) : J'_\varepsilon(v_n) \to 0 \right\}$ and as a consequence, the $C^1$-submanifold $\mathcal{N}_\varepsilon$ of $L^{p'}(\mathbb{R}^N)$ is complete.

We recall that $(v_n)_n \subset L^{p'}(\mathbb{R}^N)$ is termed a Palais-Smale sequence, or a (PS)-sequence, for $J_\varepsilon$ if $(J_\varepsilon(v_n))_n$ is bounded and $J'_\varepsilon(v_n)\to 0$ as $n\to\infty$. Also, for $d>0$, we say that $(v_n)_n$ is a $(\mathrm{PS})_d$-sequence for $J_\varepsilon$ if it is a (PS)-sequence and if $J_\varepsilon \to d$ as $n\to\infty$. The following properties hold ([14, Sect.2]).

**Lemma 2.1.** *Let $(v_n)_n\subset L^{p'}(\mathbb{R}^N)$ be a Palais-Smale sequence for $J_\varepsilon$. Then $(v_n)_n$ is bounded and there exists $v\in L^{p'}(\mathbb{R}^N)$ such that $J'_\varepsilon(v)=0$ and, up to a subsequence, $v_n \rightharpoonup v$ weakly in $L^{p'}(\mathbb{R}^N)$ and $J_\varepsilon(v)\le \liminf_{n\to\infty} J_\varepsilon(v_n)$.*

*Moreover, for every bounded and measurable set $B\subset\mathbb{R}^N$, $1_B v_n \to 1_B v$ strongly in $L^{p'}(\mathbb{R}^N)$.*

As a consequence, we obtain the following characterization of the infimum $c_\varepsilon$ of $J_\varepsilon$ over the Nehari manifold $\mathcal{N}_\varepsilon$ ([14, Sect.4]).

**Lemma 2.2.** (i) *$c_\varepsilon$ coincides with the mountain pass level, i.e.,*

$$c_\varepsilon = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J_\varepsilon(\gamma(t)),$$

*where*

$$\Gamma=\Big\{\gamma\in C\Big([0,1],L^{p'}(\mathbb{R}^N)\Big):\gamma(0)=0,\text{ and } J(\gamma(1))<0\Big\}.$$

(ii) *If $c_\varepsilon$ is attained, then $c_\varepsilon=\min\Big\{J_\varepsilon(v):v\in L^{p'}(\mathbb{R}^N)\setminus\{0\},\ J'_\varepsilon(v)=0\Big\}$.*

(iii) *If $Q_\varepsilon$ is constant or $\mathbb{Z}^N-$periodic, then $c_\varepsilon$ is attained.*

## 2.2 Energy compare

Consider the functional

$$J_0(v):=\frac{1}{p'}\int_{\mathbb{R}^N}|v|^{p'}\mathrm{d}x-\frac{1}{2}\int_{\mathbb{R}^N}Q_0^{\frac{1}{p}}v\mathbf{R}^s\left(Q_0^{\frac{1}{p}}v\right)\mathrm{d}x,\ \ v\in L^{p'}(\mathbb{R}^N) \tag{2.7}$$

and the corresponding Nehari manifold

$$\mathcal{N}_0:=\Big\{v\in L^{p'}(\mathbb{R}^N)\setminus\{0\}:J'_0(v)v=0\Big\}, \tag{2.8}$$

associated to the limit problem

$$(-\Delta)^s u-u=Q_0|u|^{p-2}u,\quad \text{in } \mathbb{R}^N. \tag{2.9}$$

Lemma 2.2 implies that the level $c_0:=\inf\limits_{\mathcal{N}_0} J_0$ is attained and coincides with the least-energy level, i.e.,

$$c_0=\inf\Big\{J_0(v):v\in L^{p'}(\mathbb{R}^N),v\neq 0\ \text{ and } J'_0(v)=0\Big\}. \tag{2.10}$$

Denote the set of maximum points of $Q$ by

$$M:=\Big\{x\in\mathbb{R}^N:Q(x)=Q_0\Big\}. \tag{2.11}$$

It then follows from (Q0) and (Q1) that $M \neq \emptyset$. We start by studying the projection on the Nehari manifold of truncation of of translated and rescaled ground states of $J_0$. Take a cut-off function $\eta \in C_c^\infty(\mathbb{R}^N)$, $0 \le \eta \le 1$, such that $\eta \equiv 1$ in $B_1(0)$ and $\eta \equiv 0$ in $\mathbb{R}^N \setminus B_2(0)$. For $y \in M$, $\varepsilon > 0$ we let

$$\varphi_{\varepsilon,y}(x) := \eta(\varepsilon x - y) w(x - \varepsilon^{-1} y), \tag{2.12}$$

where $w \in L^{p'}(\mathbb{R}^N)$ is some fixed least-energy critical point of $J_0$.

**Lemma 2.3.** *There is $\varepsilon^* > 0$ such that for all $0 < \varepsilon \le \varepsilon^*$, $y \in M$, a unique $t_{\varepsilon,y} > 0$ satisfying $t_{\varepsilon,y}\varphi_{\varepsilon,y} \in \mathcal{N}_\varepsilon$ exists. Moreover,*

$$\lim_{\varepsilon \to 0^+} J_\varepsilon(t_{\varepsilon,y}\varphi_{\varepsilon,y}) = c_0, \quad \text{uniformly for } y \in M. \tag{2.13}$$

*Proof.* Since $M$ is compact and $Q$ is continuous by assumption, $Q(y+\varepsilon\cdot)\eta(\varepsilon\cdot)w \to Q_0 w$ in $L^{p'}(\mathbb{R}^N)$ as $\varepsilon \to 0^+$, uniformly with respect to $y \in M$. Consequently, as $\varepsilon \to 0^+$,

$$\begin{aligned}\int_{\mathbb{R}^N} Q_\varepsilon^{\frac{1}{p}} \varphi_{\varepsilon,y} \mathbf{R}^s \left( Q_\varepsilon^{\frac{1}{p}} \varphi_{\varepsilon,y} \right) \mathrm{d}x &= \int_{\mathbb{R}^N} Q^{\frac{1}{p}}(y+\varepsilon z)\eta(\varepsilon z) w(z) \mathbf{R}^s \left( Q^{\frac{1}{p}}(y+\varepsilon\cdot) \right) \eta(\varepsilon\cdot) w)(x) \mathrm{d}z \\ &\to \int_{\mathbb{R}^N} Q_0^{\frac{1}{p}} w \mathbf{R}^s \left( Q_0^{\frac{1}{p}} w \right) \mathrm{d}z = \left( \frac{1}{p'} - \frac{1}{2} \right)^{-1} c_0 > 0,\end{aligned} \tag{2.14}$$

uniformly for $y \in M$. Therefore, for all $y \in M$ and $\varepsilon > 0$ small enough, we deduce that $\varphi_{\varepsilon,y} \in U_\varepsilon^+$, this implies the first assertion with $t_{\varepsilon,y}$ given by (2.5). In addition, for all $y \in M$,

$$\int_{\mathbb{R}^N} |\varphi_{\varepsilon,y}|^{p'} \mathrm{d}x = \int_{\mathbb{R}^N} |\eta(\varepsilon z) w(z)|^{p'} \mathrm{d}z \to \int_{\mathbb{R}^N} |w|^{p'} \mathrm{d}z = \left( \frac{1}{p'} - \frac{1}{2} \right)^{-1} c_0, \quad \text{as } \varepsilon \to 0^+. \tag{2.15}$$

As a consequence, $t_{\varepsilon,y} \to 1$ as $\varepsilon \to 0^+$, uniformly for $y \in M$, and we obtain $J_\varepsilon(t_{\varepsilon,y}\varphi_{\varepsilon,y}) \to c_0$ as $\varepsilon \to 0^+$, uniformly for $y \in M$. The second assertion follows. □

**Lemma 2.4.** *For all $\varepsilon > 0$ there holds $c_0 \le c_\varepsilon$. Moreover, $\lim_{\varepsilon \to 0^+} c_\varepsilon = c_0$.*

*Proof.* Consider $v_\varepsilon \in \mathcal{N}_\varepsilon$ and set $v_0 := \left( \frac{Q_\varepsilon}{Q_0} \right)^{\frac{1}{p}} v_\varepsilon$. Notice that $|v_0| \le |v_\varepsilon|$ a.e. on $\mathbb{R}^N$. Since $v_\varepsilon \in U_\varepsilon^+$, we find

$$\int_{\mathbb{R}^N} Q_0^{\frac{1}{p}} v_0 \mathbf{R}^s \left( Q_0^{\frac{1}{p}} v_0 \right) \mathrm{d}x = \int_{\mathbb{R}^N} Q_\varepsilon^{\frac{1}{p}} v_\varepsilon \mathbf{R}^s \left( Q_\varepsilon^{\frac{1}{p}} v_\varepsilon \right) \mathrm{d}x > 0, \tag{2.16}$$

i.e., $v_0 \in U_0^+$. By (2.5) we deduce

$$t_\varepsilon^{2-p'} = \frac{\int_{\mathbb{R}^N} |v_0|^{p'} \mathrm{d}x}{\int_{\mathbb{R}^N} Q_0^{\frac{1}{p}} v_0 \mathbf{R}^s \left( Q_0^{\frac{1}{p}} v_0 \right) \mathrm{d}x} \le \frac{\int_{\mathbb{R}^N} |v_\varepsilon|^{p'} \mathrm{d}x}{\int_{\mathbb{R}^N} Q_0^{\frac{1}{p}} v_\varepsilon \mathbf{R}^s \left( Q_\varepsilon^{\frac{1}{p}} v_\varepsilon \right) \mathrm{d}x} = 1. \tag{2.17}$$

This implies that $t_\varepsilon v_0 \in \mathcal{N}_0$. Follow the definition of the dual functional, we yield that

$$c_0 \le J_0(t_\varepsilon v_0) = \left(\frac{1}{p'} - \frac{1}{2}\right) t_\varepsilon^{p'} \int_{\mathbb{R}^N} |v_0|^{p'} \mathrm{d}x \le \left(\frac{1}{p'} - \frac{1}{2}\right) \int_{\mathbb{R}^N} |v_\varepsilon|^{p'} \mathrm{d}x = J_\varepsilon(v_\varepsilon). \tag{2.18}$$

Since $v_\varepsilon \in \mathcal{N}_\varepsilon$ was arbitrarily chosen, we conclude that $c_0 \le \inf_{\mathcal{N}_\varepsilon} = c_\varepsilon$. On the other hand, Lemma 2.3 gives for $y \in M$, $c_\varepsilon \le J_\varepsilon(t_{\varepsilon,y}\varphi_{\varepsilon,y}) \to c_0$ as $\varepsilon \to 0^+$. Hence, $\lim_{\varepsilon \to 0^+} c_\varepsilon = c_0$, as claimed. □

Now, consider the energy functional $J_\infty : L^{p'}(\mathbb{R}^N) \to \mathbb{R}$ given by

$$J_\infty(v) = \frac{1}{p'} \int_{\mathbb{R}^N} |v|^{p'} \mathrm{d}x - \frac{1}{2} \int_{\mathbb{R}^N} Q_\infty^{\frac{1}{p}} v \mathbf{R}^s \left(Q_\infty^{\frac{1}{p}} v\right) \mathrm{d}x, \quad v \in L^{p'}(\mathbb{R}^N). \tag{2.19}$$

The corresponding Nehari manifold

$$\mathcal{N}_\infty := \left\{ v \in L^{p'}(\mathbb{R}^N) \setminus \{0\} : J'_\infty(v)v = 0 \right\}, \tag{2.20}$$

has the same structure as $\mathcal{N}_\varepsilon$ and, since $Q_\infty$ is constant, Lemma 2.2 implies that $c_\infty := \inf_{\mathcal{N}_\infty} J_\infty$ is attained and coincides with the least energy level for nontrivial critical points of $J_\infty$.

**Proposition 2.1.** There is $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$, $c_\varepsilon < c_\infty$.

*Proof.* By Lemma 2.4 and Condition (Q1), there is $\varepsilon_0 > 0$ such that $c_\varepsilon < c_\infty$ for all $0 < \varepsilon < \varepsilon_0$. □

## 3 Exsitence of dual ground ststes

In this section, we prove the $(PS)_{c_\varepsilon}$ condition for the energy functional $J_\varepsilon$. Since the resolvent Helmholtz operator is not positive definite, we must carefully address the nonlocal interactions between functions with disjoint supports.

**Lemma 3.1.** *There exists a constant $C = C(N,p) > 0$ such that for any $R > 0$, $r \ge 1$ and $u,v \in L^{p'}(\mathbb{R}^N)$ with* $\mathrm{supp}(u) \subset B_R$ *and* $\mathrm{supp}(v) \subset \mathbb{R}^N \setminus B_{R+r}$,

$$\left| \int_{\mathbb{R}^N} u \mathbf{R}^s v \, \mathrm{d}x \right| \le C r^{-\lambda_p} ||u||_{p'} ||v||_{p'}, \tag{3.1}$$

*where* $\lambda_p = \frac{N-1}{2} - \frac{N+1}{p}$.

*Proof.* Let $\mathcal{R}^s$ denote the resolvent of the Fractional Helmholtz eqaution, which is given by the convolution with the kernel $K(x)$ (see [38] and [14] for more details). Since $\mathbf{R}^s$ is the real part of $\mathcal{R}^s$ and since $u,v$ are real-valued, we prove the lemma for the nonlocal term $\int_{\mathbb{R}^N} v \mathcal{R}^s u \mathrm{d}x$. By density, it suffices to prove the estimate for Schwartz function.

Let $M_{R+r}:=\mathbb{R}^N\backslash B_{R+r}$ and let $u,v\in\mathcal{S}(\mathbb{R}^N)$ be such that $\mathrm{supp}(u)\subset B_R$ and $\mathrm{supp}(v)\subset M_{R+r}$. The symmetry of the operator $\mathcal{R}^s$ and Hölder's inequality gives

$$\Big|\int_{\mathbb{R}^N} u\mathcal{R}^s v\mathrm{d}x\Big| = \Big|\int_{\mathbb{R}^N} v\mathcal{R}^s u\mathrm{d}x\Big| \le ||v||_{p'}||K*u||_{L^p(M_{R+r})}. \tag{3.2}$$

Then, it suffices to estimate the second factor on the RHS. For this, we decompose $K$ as follow. Fix $\psi\in\mathcal{S}(\mathbb{R}^N)$ such that $\widehat{\psi}\in\mathcal{C}_c^\infty(\mathbb{R}^N)$ is radial, $0\le\widehat{\psi}\le 1$, $\widehat{\psi}(\xi)=1$ for $\big||\xi|-1\big|\le\frac{1}{6}$ and $\widehat{\psi}(\xi)=0$ for $\big||\xi|-1\big|\ge\frac{1}{4}$. Writing $K=K_1+K_2$ with

$$K_1:=(2\pi)^{-\frac{N}{2}}(\psi*K), \quad K_2:=K-K_1. \tag{3.3}$$

It follows from the estimate in [14] that

$$|K_1(x)|\le C_0\Big(1+|x|\Big)^{\frac{1-N}{2}}, \qquad x\in\mathbb{R}^N, \tag{3.4}$$

$$|K_2(x)|\le C_0|x|^{-N}, \qquad x\neq 0. \tag{3.5}$$

Since the support of $u$ is contained in $B_R$, we find

$$\begin{aligned}||K_2*u||_{L^p(M_{R+r})} &\le \left[\int_{|x|\ge R+r}\left(\int_{|y|\le R}|K_2(x-y)||u(y)|\mathrm{d}y\right)^p\mathrm{d}x\right]^{\frac{1}{p}}\\ &\le \left[\int_{\mathbb{R}^N}\left(\int_{|x-y|\ge r}|K_2(x-y)||u(y)|\mathrm{d}y\right)^p\mathrm{d}x\right]^{\frac{1}{p}}\\ &= \big\|(1_{M_r}|K_2|*|u|)\big\|_p \le \big\|1_{M_r}K_2\big\|_{\frac{p}{2}}\big\|u\big\|_{p'}.\end{aligned} \tag{3.6}$$

Moreover, (3.5) gives

$$||1_{M_r}K_2||_{\frac{p}{2}} \le C_0\Big(\omega_N\int_r^\infty s^{N-1-\frac{Np}{2}}\mathrm{d}s\Big)^{\frac{2}{p}} \le Cr^{-\frac{N(p-2)}{p}} \le Cr^{-\lambda_p}, \tag{3.7}$$

since $r\ge 1$, and therefore

$$||K_2*u||_{L^p(M_{R+r})} \le Cr^{-\lambda_p}||u||_{p'}. \tag{3.8}$$

It remains to prove the estimate for $K_1$. Fix a radial function $K\in\mathcal{S}\big(\mathbb{R}^N\big)$ such that $\widehat{K}\in\mathcal{C}_c^\infty\big(\mathbb{R}^N\big)$ is radial, $0\le\widehat{K}\le 1$, $\widehat{K}(\xi)=1$ for $\big||\xi|-1\big|\le\frac{1}{4}$ and $\widehat{K}(\xi)=0$ for $\big||\xi|-1\big|\ge\frac{1}{2}$, let $\tilde{u}:=K*u\in\mathcal{S}\big(\mathbb{R}^N\big)$, we then have $K_1*u=(2\pi)^{-\frac{N}{2}}(K_1*\tilde{u})$, since $\widehat{K_1}\widehat{K}=\widehat{K_1}$ by construction. Now write

$$K_1*\tilde{u}=\Big[1_{B_{\frac{r}{2}}}K_1\Big]*\tilde{u}+\Big[1_{M_{\frac{r}{2}}}K_1\Big]*\tilde{u} \tag{3.9}$$

and let $g_r:=\Big[1_{B_{\frac{r}{2}}}K_1\Big]*K$. Since $\mathrm{supp}(u)\subset B_R$, we find as above

$$||1_{M_r}g_r||_{\frac{p}{2}}^{\frac{p}{2}} \le C_0^{\frac{p}{2}}\int_{|x|\ge r}\left(\int_{|y|\le\frac{r}{2}}|K(x-y)|\mathrm{d}y\right)^{\frac{p}{2}}\mathrm{d}x$$

$$\leq C\int_{|x|\geq r}\left(\int_{|y|\leq\frac{r}{2}}|x-y|^{-m}\mathrm{d}y\right)^{\frac{p}{2}}\mathrm{d}x\leq C\left|B_{\frac{r}{2}}\right|^{\frac{p}{2}}\int_{|x|\geq r}\left(|x|-\frac{r}{2}\right)^{-\frac{mp}{2}}\mathrm{d}x$$
$$=Cr^{\frac{(N-m)p}{2}+N}\int_{|z|\geq 1}\left(|z|-\frac{1}{2}\right)^{-\frac{mp}{2}}\mathrm{d}z=Cr^{\frac{(N-m)p}{2}+N}, \tag{3.10}$$

where $C$ is independent of $r$ and where $m$ may be fixed so large that $\frac{(m-N)p}{2}-N\geq\lambda_p$. As a consequence of [7, Proposition 3.3], we have moreover

$$\left\|\left[1_{M_{\frac{r}{2}}}K_1\right]*\tilde{u}\right\|_{L^p(M_{R+r})}\leq\left\|\left[1_{M_{\frac{r}{2}}}K_1*\tilde{u}\right]\right\|_p\leq Cr^{-\lambda_p}||\tilde{u}||_{p'}\leq Cr^{-\lambda_p}||u||_{p'} \tag{3.11}$$

and we conclude that

$$||K_1*u||_{L^p(M_{R+r})}\leq Cr^{-\lambda_p}||u||_{p'}. \tag{3.12}$$

Combining (3.2), (3.12) and (3.8) yields the claim. □

**Lemma 3.2.** *Let $\varepsilon>0$ and assume $Q_\infty>0$ and $c_\varepsilon<c_\infty$. Then $J_\varepsilon$ satisfies the Palais-Smale condition on $\mathcal{N}_\varepsilon$ at level below $c_\infty$, i.e., every sequence $(v_n)_n\subset\mathcal{N}_\varepsilon$ such that $J_\varepsilon(v_n)\to d<c_\infty$ and $(J_\varepsilon|\mathcal{N}_\varepsilon)'(v_n)\to 0$ as $n\to\infty$ has a convergent subsequence.*

*Proof.* Since $c_\varepsilon<c_\infty$, the set $\{v\in\mathcal{N}_\varepsilon: J_\varepsilon(v)<c_\infty\}$ is not empty. If $d<c_\varepsilon$, all is done. It remains to consider the case $c_\varepsilon\leq d<c_\infty$. Let $(v_n)_n$ be a $(\mathrm{PS})_d$-sequence for $J_\varepsilon|\mathcal{N}_\varepsilon$. Since $\mathcal{N}_\varepsilon$ is a natural constraint and a $C^1-$manifold, we find that $(v_n)_n$ is a $(PS)_d$-sequence for the unconstrained functional $J_\varepsilon$. Using Lemma 2.1, we obtain that (up to a subsequence) $v_n\rightharpoonup v$ and $1_{B_R}v_n\to 1_{B_R}v$ in $L^{p'}(\mathbb{R}^N)$ for all $R>0$, where $v\in L^{p'}(\mathbb{R}^N)$ is a critical point of $J_\varepsilon$ with $J_\varepsilon(v)\leq d$. In order to conclude that $v_n\to v$ strongly in $L^{p'}(\mathbb{R}^N)$, it suffices to show that

$$\forall\,\zeta>0,\ \exists\,R>0\ \text{ such that }\int_{|x|>R}|v_n|^{p'}\mathrm{d}x<\zeta,\quad\forall\,n. \tag{3.13}$$

We prove (3.13) by contradiction. Assuming that there exists a subsequence $(n_{n_k})_k$ and $\zeta_0>0$ such that

$$\int_{|x|>k}|v_{n_k}|^{p'}\mathrm{d}x\geq\zeta_0,\quad\forall\,k. \tag{3.14}$$

Firstly, for an annular region, we claim that

$$\forall\,\eta>0\ \text{ and }\ \forall\,R>0,\quad\exists\,r>R\ \text{ such that }\ \liminf_{n\to\infty}\int_{r<|x|<2r}|v_n|^{p'}\mathrm{d}x<\eta. \tag{3.15}$$

Otherwise, for every $m>R_0$, $n_0=n_0(m)$, we can find $\eta_0,R_0$ such that $\int_{m<|x|<2m}|v_n|^{p'}\mathrm{d}x\geq\eta_0$ for all $n\geq n_0$. Without loss of generality, we assume that $n_0(m+1)\geq n_0(m)$ for all $m$. Hence, for every $l\in\mathbb{N}$ there is $N_0=N_0(l)$ such that

$$\int_{\mathbb{R}^N}|v_n|^{p'}\mathrm{d}x\geq\sum_{k=0}^{l-1}\int_{2^k([R_0]+1)<|x|<2^{k+1}([R_0]+1)}|v_n|^{p'}\mathrm{d}x\geq l\eta_0,\quad\forall\,n\geq N_0. \tag{3.16}$$

Letting $l\to\infty$, we obtain a contradiction to the fact that $(v_n)_n$ is bounded and this gives (3.15). Now fix $0<\eta<\min\{1,(\zeta_0/3C_1)^{p'}\}$, where

$$C_1=2C(N,p)||Q||_\infty^{\frac{2}{p}}\max\left\{1,\sup_{k\in\mathbb{N}}||v_{n_k}||_{p'}^2\right\},$$

the constant $C(N,p)$ being chosen such that Lemma 3.1 holds and $||\mathbf{R}^s v||_p\le C(N,p)||v||_{p'}$ for all $u\in L^{p'}(\mathbb{R}^N)$. By definition of $Q_\infty$ and since $\varepsilon>0$ is fixed, there exists $R(\eta)>0$ such that

$$Q_\varepsilon\le Q_\infty+\eta,\quad |x|\ge R(\eta). \tag{3.17}$$

Also, from (3.15), we can find $r>\max\left\{R(\eta),\eta^{-\frac{1}{\lambda_p}}\right\}$ and a subsequence, still denoted by $(v_{n_k})_k$, such that

$$\int_{r<|x|<2r}|v_{n_k}|^{p'}\mathrm{d}x<\eta,\quad \forall\, k. \tag{3.18}$$

Setting $w_{n_k}:=1_{\{|x|\ge 2r\}}v_{n_k}$, we can write for all $k$

$$\begin{aligned}
&\left|J_\varepsilon'(v_{n_k})w_{n_k}-J_\varepsilon'(w_{n_k})w_{n_k}\right|\\
=&\left|\int_{|x|<r}Q_\varepsilon^{\frac{1}{p}}v_{n_k}\mathbf{R}^s\left(Q_\varepsilon^{\frac{1}{p}}w_{n_k}\right)\mathrm{d}x+\int_{r<|x|<2r}Q_\varepsilon^{\frac{1}{p}}v_{n_k}\mathbf{R}^s\left(Q_\varepsilon^{\frac{1}{p}}w_{n_k}\right)\mathrm{d}x\right|\\
\le& C(N,p)r^{-\lambda_p}||Q||_\infty^{\frac{2}{p}}||v_{n_k}||_{p'}^2+C(N,p)||Q||_\infty^{\frac{2}{p}}||v_{n_k}||_{p'}\left(\int_{r<|x|<2r}|v_{n_k}|^{p'}\mathrm{d}x\right)^{\frac{1}{p'}}\\
\le& C_1\eta^{\frac{1}{p'}},
\end{aligned} \tag{3.19}$$

using Lemma 3.1. In addition, by (3.14) and the definition of $w_{n_k}$, there holds

$$\int_{\mathbb{R}^N}|w_{n_k}|^{p'}\mathrm{d}x\ge\zeta_0,\quad \forall\, k\ge 2r. \tag{3.20}$$

Recalling our choice of $\eta$, we know that $C_1\eta^{\frac{1}{p'}}<\frac{\zeta_0}{3}$, and we find some $k_0=k_0(r,\eta,\zeta_0)\ge 2r$ such that

$$\begin{aligned}
\int_{\mathbb{R}^N}Q_\varepsilon^{\frac{1}{p}}w_{n_k}\mathbf{R}^s\left(Q_\varepsilon^{\frac{1}{p}}w_{n_k}\right)\mathrm{d}x&=\int_{\mathbb{R}^N}|w_{n_k}|^{p'}\mathrm{d}x-J_\varepsilon'(v_{n_k})w_{n_k}+\left[J_\varepsilon'(v_{n_k})w_{n_k}-J_\varepsilon'(w_{n_k})w_{n_k}\right]\\
&\ge\int_{\mathbb{R}^N}|w_{n_k}|^{p'}\mathrm{d}x-|J_\varepsilon'(v_{n_k})w_{n_k}|-C_1\eta^{\frac{1}{p'}}\ge\frac{\zeta_0}{2},\quad \forall k\ge k_0,
\end{aligned} \tag{3.21}$$

since $J_\varepsilon'(v_{n_k})w_{n_k}\to 0$ as $k\to\infty$.

For $k\geq k_0$, let now $\tilde{w}_k:=\left(\frac{Q_\varepsilon}{Q_\infty}\right)^{\frac{1}{p}}w_{n_k}$ and notice that $|\tilde{w}_k|\leq\left(1+\frac{\eta}{Q_\infty}\right)^{\frac{1}{p}}|w_{n_k}|$. In view of (3.21), there is $t_k^\infty>0$, for which $t_k^\infty\tilde{w}_k\in\mathcal{N}_\infty$ and there holds

$$\begin{aligned}\left(t_k^\infty\right)^{2-p'}&\leq\frac{\left(1+\frac{\eta}{Q_\infty}\right)^{p'-1}\int_{\mathbb{R}^N}|w_{n_k}|^{p'}\mathrm{d}x}{\int_{\mathbb{R}^N}Q_\varepsilon^{\frac{1}{p}}w_{n_k}\mathbf{R}^s\left(Q_\varepsilon^{\frac{1}{p}}w_{n_k}\right)\mathrm{d}x}\\&\leq\left(1+\frac{\eta}{Q_\infty}\right)^{p'-1}\left(1+\frac{|J'_\varepsilon(w_{n_k})w_{n_k}|+C_1\eta^{\frac{1}{p'}}}{\int_{\mathbb{R}^N}Q_\varepsilon^{\frac{1}{p}}w_{n_k}\mathbf{R}^s\left(Q_\varepsilon^{\frac{1}{p}}w_{n_k}\right)\mathrm{d}x}\right)\\&\leq\left(1+\frac{\eta}{Q_\infty}\right)^{p'-1}\left(1+\frac{2\left|J'_\varepsilon(w_{n_k})w_{n_k}+2C_1\eta^{\frac{1}{p'}}\right|}{\zeta_0}\right).\end{aligned}\tag{3.22}$$

Since $v_{n_k}\in\mathcal{N}_\varepsilon$, there holds

$$\int_{\mathbb{R}^N}|w_{n_k}|^{p'}\mathrm{d}x\leq\int_{\mathbb{R}^N}|v_{n_k}|^{p'}\mathrm{d}x=\left(\frac{1}{p'}-\frac{1}{2}\right)^{-1}J_\varepsilon(v_{n_k}).\tag{3.23}$$

Consequently, for all $k\geq k_0$,

$$\begin{aligned}c_\infty\leq J_\infty(t_k^\infty\tilde{w}_k)&\leq\left(\frac{1}{p'}-\frac{1}{2}\right)(t_k^\infty)^{p'}\left(1+\frac{\eta}{Q_\infty}\right)^{p'-1}\int_{\mathbb{R}^N}|w_{n_k}|^{p'}\mathrm{d}x\\&\leq\left(1+\frac{\eta}{Q_\infty}\right)^{\frac{2(p'-1)}{2-p'}}\left(1+\frac{2\left|J'_\varepsilon(w_{n_k})w_{n_k}+2C_1\eta^{\frac{1}{p'}}\right|}{\zeta_0}\right)^{\frac{p'}{2-p'}}J_\varepsilon(v_{n_k}).\end{aligned}\tag{3.24}$$

Letting $k\to\infty$, we find

$$c_\infty\leq\left(1+\frac{\eta}{Q_\infty}\right)^{\frac{2(p'-1)}{2-p'}}\left(1+\frac{2C_1\eta^{\frac{1}{p'}}}{\zeta_0}\right)^{\frac{p'}{2-p'}}d,\tag{3.25}$$

and letting $\eta\to0$ we obtain

$$c_\infty\leq d,\tag{3.26}$$

which contradicts the assumption $d<\infty$ and prove (3.13). From this, we conclude the strong convergence $v_n\to v$ in $L^{p'}(\mathbb{R}^N)$ and the assertion follows. □

**Proof of Theorem 1.1 (i).** Fix $\varepsilon_0$ in Proposition (2.1). For any $\varepsilon\leq\varepsilon_0$, using the fact that $\mathcal{N}_\varepsilon$ is a $C^1-$submanifold of $L^{p'}(\mathbb{R}^N)$, we obtain from Ekeland's variational principle the existence of Palais-Smale sequence for $J_\varepsilon$ on $\mathcal{N}_\varepsilon$, at level $c_\varepsilon$, and by Lemma 3.2, $c_\varepsilon$ is attained. □

## 4 Concentration of dual ground states

To show the concentration behaviour of the solutions of (1.1), we first establish a representation lemma for the Palais-Smale sequences of the functional $J_\varepsilon$. This lemma is inspired by the work of Benci and Cerami [39]. A crucial element, especially regarding the nonlocal quadratic part of the energy functional, is the nonvanishing theory developed in [14, Sect.4]. For simplicity, we omit the subscript $\varepsilon$.

**Lemma 4.1.** *Suppose $Q\equiv Q(0)>0$ on $\mathbb{R}^N$. Consider for some $d>0$, a $(PS)_d$-sequence $(v_n)_n\subset L^{p'}(\mathbb{R}^N)$ for $J$. Then there is an integer $m\geq 1$, critical points $w^{(1)},\cdots,w^{(m)}$ of $J$ and sequence $\left(x_n^{(1)}\right)_n,\cdots,\left(x_n^{(m)}\right)_n\subset\mathbb{R}^N$ such that (up to a subsequence)*

$$\begin{cases}\left\|v_n-\sum_{j=1}^m w^j\left(\cdot-x_n^j\right)\right\|_{p'}\to 0, & \text{as } n\to\infty,\\ \left|x_n^{(i)}-x_n^{(j)}\right|\to\infty, & \text{as } n\to\infty, \text{ if } i\neq j,\\ \sum_{j=1}^m J\left(w^{(j)}\right)=d. \end{cases} \tag{4.1}$$

*Proof.* For any bounded $(\mathrm{PS})_d$-sequenc $(v_n)_n$, we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} Q^{\frac{1}{p}}v_n\mathbf{R}^s\left(Q^{\frac{1}{p}}v_n\right)\mathrm{d}x=\frac{2p'}{2-p'}\lim_{n\to\infty}\left[J(v_n)-\frac{1}{p'}J'(v_n)v_n\right]=\frac{2p'd}{2-p'}>0. \tag{4.2}$$

It then follows from the nonvanishing theorem [14, Theorem 4.1] that there are $R,\zeta>0$ and a sequence $\left(x_n^{(1)}\right)_n$ such that, up to a subsequence,

$$\int_{B_r(x_n^{(1)})}|v_n|^{p'}\mathrm{d}x\geq\zeta>0,\quad\forall n. \tag{4.3}$$

Setting $v_n^{(1)}=v_n\left(\cdot+x_n^{(1)}\right)$, then by the invariance of the energy functional, $\left(v_n^1\right)_n$ is also a $(\mathrm{PS})_d$-sequence for $J$. By Lemma 2.1, going to a further subsequence, we may assume $v_n^{(1)}\rightharpoonup w^{(1)}$ weakly, $1_{B_R}v_n^{(1)}\to 1_{B_R}w^{(1)}$ strongly in $L^{p'}(\mathbb{R}^N)$, and

$$J(w^{(1)})\leq\lim_{n\to\infty}J\left(v_n^{(1)}\right)=d.$$

These properties and the definition of $v_n^{(1)}$ imply that $w^{(1)}$ is a nontrivial critical point of $J$. If $J\left(w^{(1)}\right)=d$, we obtain

$$\begin{aligned}\left(\frac{1}{p'}-\frac{1}{2}\right)||w^{(1)}||_{p'}^{p'}&=J\left(w^{(1)}\right)-\frac{1}{2}J'\left(w^{(1)}\right)w^{(1)}\\ &=d=\lim_{n\to\infty}\left[J(v_n)-\frac{1}{2}J'(v_n)v_n\right]=\left(\frac{1}{p'}-\frac{1}{2}\right)\lim_{n\to\infty}||v_n||_{p'}^{p'},\end{aligned} \tag{4.4}$$

i.e., $v_n^{(1)}\to w^{(1)}$ strongly in $L^{p'}(\mathbb{R}^N)$, then the lemma is proved.

Otherwise, $J(w^{(1)})<d$ and we set $v_n^{(2)}=v_n^{(1)}-w^{(1)}$. The weak convergence $v_n^{(1)}\rightharpoonup w^{(1)}$ then implies

$$\begin{aligned}\int_{\mathbb{R}^N} Q^{\frac{1}{p}}v_n^{(2)}\mathbf{R}^s\left(Q^{\frac{1}{p}}v_n^{(2)}\right)\mathrm{d}x=&\int_{\mathbb{R}^N} Q^{\frac{1}{p}}v_n^{(1)}\mathbf{R}^s\left(Q^{\frac{1}{p}}v_n^{(1)}\right)\mathrm{d}x\\&-\int_{\mathbb{R}^N} Q^{\frac{1}{p}}w_n^{(1)}\mathbf{R}^s\left(Q^{\frac{1}{p}}w_n^{(1)}\right)\mathrm{d}x+o(1),\end{aligned}\tag{4.5}$$

as $n\to\infty$. Moreover, by the Brézis-Lieb Lemma [40],

$$\int_{\mathbb{R}^N}\left|v_n^{(2)}\right|^{p'}\mathrm{d}x=\int_{\mathbb{R}^N}\left|v_n^{(1)}\right|^{p'}\mathrm{d}x-\int_{\mathbb{R}^N}|w^{(1)}|^{p'}\mathrm{d}x+o(1),\quad \text{as } n\to\infty.\tag{4.6}$$

These properties and the translation invariance of $J$ together give

$$J\left(v_n^{(2)}\right)=J\left(v_n^{(1)}\right)-J\left(w^{(1)}\right)+o(1)=d-J\left(w^{(1)}\right)+o(1),\quad \text{as } n\to\infty.\tag{4.7}$$

Since by Lemma 2.1, $1_{B_r}v_n^{(1)}\to 1_{B_r}w^{(1)}$ strongly in $L^{p'}(\mathbb{R}^N)$ for all $r>0$, we find

$$1_{B_r}\left|v_n^{(2)}\right|^{p'-2}v_n^{(2)}-1_{B_r}\left|v_n^{(1)}\right|^{p'-2}v_n^{(1)}+1_{B_r}|w^{(1)}|^{p'-2}w^{(1)}\to 0\ \text{ in } L^p\left(\mathbb{R}^N\right),\quad \text{as } n\to\infty.\tag{4.8}$$

On the other hand, since $\left||a|^{q-1}a-|b|^{q-1}b\right|\le 2^{1-q}|a-b|^q$ for all $a,b\in\mathbb{R}$ and $0<q<1$, it follows that

$$\int_{\mathbb{R}^N\setminus B_r}\left|\left|v_n^{(2)}\right|^{p'-2}v_n^{(2)}-\left|v_n^{(1)}\right|^{p'-2}v_n^{(1)}\right|^p\mathrm{d}x\le 2^{(2-p')p}\int_{\mathbb{R}^N\setminus B_r}|w^{(1)}|^{p'}\mathrm{d}x\to 0,\tag{4.9}$$

as $n\to\infty$, uniformly in $n$. The both estimates then give the strong convergence

$$\left|v_n^{(2)}\right|^{p'-2}v_n^{(2)}-\left|v_n^{(1)}\right|^{p'-2}v_n^{(1)}+\left|w^{(1)}\right|^{p'-2}w^{(1)}\to 0,\quad \text{in } L^p(\mathbb{R}^N),\text{ as } n\to\infty,\tag{4.10}$$

and therefore,

$$J'\left(v_n^{(2)}\right)=J'\left(v_n^{(1)}\right)-J'(w^{(1)})+o(1),\quad \text{as } n\to\infty.\tag{4.11}$$

This implies that $\left(v_n^{(2)}\right)_n$ is a (PS)-sequence for $J$ at level $d-J(w^{(1)})>0$. Thus, the nonvanishing theorem again gives the existence of $R_1$, $\zeta_1>0$ and of a sequence $(y_n)_n\subset\mathbb{R}^N$ such that, going to a subsequence

$$\int_{B_{R_1}(y_n)}\left|v_n^{(2)}\right|^{p'}\mathrm{d}x\ge\zeta_1>0,\qquad \forall n.\tag{4.12}$$

By Lemma 2.1, there is a critical point $w^{(2)}$ of $J$ such that (taking a further subsequence) $v_n^{(2)}(\cdot+y_n)\rightharpoonup w^{(2)}$ weakly and $1_Bv_n^{(2)}(\cdot+y_n)\to 1_Bw^{(2)}$ strongly in $L^{p'}(\mathbb{R}^N)$, for all bounded and measurable set $B\subset\mathbb{R}^N$. In particular, $w^{(2)}\neq 0$ and since $v_n^{(2)}\rightharpoonup 0$, we see that $|y_n|\to\infty$ as $n\to\infty$. Setting $x_n^{(2)}=x_n^{(1)}=y_n$, we obtain $\left|x_n^{(2)}-x_n^{(1)}\right|\to\infty$ as $n\to\infty$, and

$$v_n-\left(w^{(1)}\left(\cdot+x_n^{(1)}\right)+w_n^{(2)}\left(\cdot+x_n^{(2)}\right)\right)=v_n^{(2)}\left(\cdot+y_n-x_n^{(2)}\right)-w^{(2)}\left(\cdot-x_n^{(2)}\right)\rightharpoonup 0, \tag{4.13}$$

weakly in $L^{p'}(\mathbb{R}^N)$. In addition, the same argument as before show that

$$J\left(w^{(2)}\right)\leq\liminf_{n\to\infty}J\left(v_n^{(2)}\right)=d-J\left(w^{(1)}\right) \tag{4.14}$$

with equality if and only $v_n^{(2)}(\cdot+y_n)\to w^{(2)}$ strongly in $L^{p'}(\mathbb{R}^N)$. If the inequality is strict, we can iterate the procedure. Since for every nontrivial critical point $w$ of $J$ we have $J(w)\geq c=\inf_{\mathcal{N}}J>0$, the iterate has to stop after finitely many steps, and we obtain the desired result. □

**Proposition 4.1.** *Let $(\varepsilon_n)_n\subset(0,\infty)$ satisfy $\varepsilon_n\to 0$ as $n\to\infty$. Consider for each $n$ some $v_n\in\mathcal{N}_{\varepsilon_n}$ and assume that $J_{\varepsilon_n}(v_n)\to c_0$ as $n\to\infty$. Then, there is $x_0\in M$, a critical point $w_0$ of $J_0$ at level $c_0$ and a sequence $(y_n)_n\subset\mathbb{R}^N$ such that (up to a subsequence)*

$$\varepsilon_n y_n\to x_0 \quad\text{and}\quad ||v_n(\cdot+y_n)-w_0||_{p'}\to 0,\quad \text{as } n\to\infty. \tag{4.15}$$

*Proof.* For each $n\in\mathbb{N}$, set $v_{0,n}:=\left(\frac{Q_{\varepsilon_n}}{Q_0}\right)^{\frac{1}{p}}v_n$. It follows that $|v_{0,n}|\leq|v_n|$ a.e. on $\mathbb{R}^N$ and that

$$\int_{\mathbb{R}^N}Q_0^{\frac{1}{p}}v_{0,n}\mathbf{R}^s\left(Q_0^{\frac{1}{p}}v_{0,n}\right)\mathrm{d}x=\int_{\mathbb{R}^N}Q_{\varepsilon_n}^{\frac{1}{p}}v_n\mathbf{R}^s\left(Q_{\varepsilon_n}^{\frac{1}{p}}v_n\right)\mathrm{d}x>0. \tag{4.16}$$

Therefore, setting

$$t_{0,n}^{2-p'}=\frac{\displaystyle\int_{\mathbb{R}^N}|v_{0,n}|^{p'}\mathrm{d}x}{\displaystyle\int_{\mathbb{R}^N}Q_0^{\frac{1}{p}}v_{0,n}\mathbf{R}^s\left(Q_0^{\frac{1}{p}}v_{0,n}\right)\mathrm{d}x} \tag{4.17}$$

we find that $t_{0,n}v_{0,n}\in\mathcal{N}_0$ and $0<t_{0,n}\leq 1$. As a consequence, we can write

$$\begin{aligned} c_0\leq J_0(t_{0,n}v_{0,n})&=\left(\frac{1}{p'}-\frac{1}{2}\right)t_{0,n}^2\int_{\mathbb{R}^N}Q_0^{\frac{1}{p}}v_{0,n}\mathbf{R}^s\left(Q_0^{\frac{1}{p}}v_{0,n}\right)\mathrm{d}x\\ &=\left(\frac{1}{p'}-\frac{1}{2}\right)t_{0,n}^2\int_{\mathbb{R}^N}Q_{\varepsilon_n}^{\frac{1}{p}}v_n\mathbf{R}^s\left(Q_{\varepsilon_n}^{\frac{1}{p}}v_n\right)\mathrm{d}x\\ &=t_{0,n}^2J_{\varepsilon_n}(v_n)\leq J_{\varepsilon_n}(v_n)\to c_0,\quad \text{as } n\to\infty. \end{aligned} \tag{4.18}$$

In particular, we find

$$\lim_{n\to\infty}t_{0,n}=1, \tag{4.19}$$

and $(t_{0,n}v_{0,})_n\subset\mathcal{N}_0$ is thus a minimizing sequence for $J_0$ on $\mathcal{N}_0$. Using Ekeland's variational principle and the fact that $\mathcal{N}_0$ is a natural constraint, we obtain the existence of a $(\mathrm{PS})_{c_0}$-sequence $(w_n)_n\subset L^{p'}(\mathbb{R}^N)$ for $J_0$ with the property that $||v_{0,n}-w_n||_{p'}\to 0$, as $n\to\infty$.

By Lemma 4.1, there exists a critical point $w_0$ for $J_0$ at level $c_0$ and a sequence $(y_n)_n\subset\mathbb{R}^N$ such that (up to a subsequence) $||w_n(\cdot+y_n)-w_0||_{p'}\to 0$, as $n\to\infty$. Therefore,

$$v_{0,n}(\cdot+y_n)\to w_0 \quad \text{strongly in } L^{p'}(\mathbb{R}^N), \quad \text{as } n\to\infty. \tag{4.20}$$

We are going to show that $(\varepsilon_n y_n)_n$ is bounded. Suppose not, there exist a subsequence (which we still call $(\varepsilon_n y_n)_n$) such that $\lim\limits_{n\to\infty}|\varepsilon_n y_n|=\infty$. We consider the following two cases.

(1) $Q_\infty=0$. In this case, by the assumption on $Q$, we have $Q(\varepsilon_n\cdot+\varepsilon_n y_n)\to 0$, as $n\to\infty$, holds uniformly on bounded sets of $\mathbb{R}^N$. From the definition of $v_{0,n}$, we deduce that $v_{0,n}(\cdot+y_n)\rightharpoonup 0$ and therefore $w_0=0$, in contradiction to $J_0(w_0)=c_0>0$. Hence, $(\varepsilon_n y_n)_n$ is bounded in this case.

(2) $Q_\infty>0$. By the Fatou's lemma and the strong convergence $v_{0,n}(\cdot+y_n)\to w_0$, we deduce that

$$\begin{aligned}
c_0&=\lim_{n\to\infty}J_{\varepsilon_n}(v_n)=\lim_{n\to\infty}\left(\frac{1}{p'}-\frac{1}{2}\right)\int_{\mathbb{R}^N}|v_n|^{p'}\mathrm{d}x\\
&=\lim_{n\to\infty}\left(\frac{1}{p'}-\frac{1}{2}\right)\int_{\mathbb{R}^N}|v_n(x+y_n)|^{p'}\mathrm{d}x\\
&=\liminf_{n\to\infty}\left(\frac{1}{p'}-\frac{1}{2}\right)\int_{\mathbb{R}^N}\left(\frac{Q_0}{Q(\varepsilon_n x+\varepsilon_n y_n)}\right)^{p'-1}|v_{0,n}(x+y_n)|^{p'}\mathrm{d}x\\
&\geq\left(\frac{1}{p'}-\frac{1}{2}\right)\int_{\mathbb{R}^N}\left(\frac{Q_0}{Q_\infty}\right)^{p'-1}|w_0|^{p'}\mathrm{d}x\\
&=\left(\frac{Q_0}{Q_\infty}\right)^{p'-1}c_0,
\end{aligned}\tag{4.21}$$

and this contradicts (Q1). Therefore, $(\varepsilon_n y_n)_n$ is a bounded sequence, and we may assume (going to a subsequence) that $\varepsilon_n y_n\to x_0\in\mathbb{R}^N$.

Since $Q(\varepsilon_n x+\varepsilon_n y_n)\to Q_{x_0}$, as $n\to\infty$, uniformly on bounded set, the argument of case (1) above gives $Q(x_0)>0$ and, using the Dominated Convergence Theorem, we see that $Q(x_0)=Q_0$, since the following holds.

$$\begin{aligned}
c_0&=\lim_{n\to\infty}J_{\varepsilon_n}(v_n)=\lim_{n\to\infty}\left(\frac{1}{p'}-\frac{1}{2}\right)\int_{\mathbb{R}^N}|v_n|^{p'}\mathrm{d}x\\
&=\lim_{n\to\infty}\left(\frac{1}{p'}-\frac{1}{2}\right)\int_{\mathbb{R}^N}\left(\frac{Q_0}{Q(\varepsilon_n x+\varepsilon_n y_n)}\right)^{p'-1}|v_{0,n}(x+y_n)|^{p'}\mathrm{d}x\\
&=\left(\frac{1}{p'}-\frac{1}{2}\right)\int_{\mathbb{R}^N}\left(\frac{Q_0}{Q(x_0)}\right)^{p'-1}|w_0|^{p'}\mathrm{d}x=\left(\frac{Q_0}{Q_{x_0}}\right)^{p'-1}c_0.
\end{aligned}\tag{4.22}$$

Going back to the original sequence we obtain

$$v_n(\cdot+y_n)=\left(\frac{Q_0}{Q(\varepsilon_n+\varepsilon_n y_n)}\right)^{\frac{1}{p}} v_{0,n}(\cdot+y_n)\to\left(\frac{Q_0}{Q(x_0)}\right)^{\frac{1}{p}} w_0=w_0, \quad \text{as } n\to\infty, \tag{4.23}$$

strongly in $L^{p'}(\mathbb{R}^N)$, using again the Dominated Convergence Theorem. The proof is complete. □

**Proof of Theorem 1.1 (ii).** By (1.11), the dual ground state $u_n$ can be represented as

$$u_n(x)=k_n^{\frac{2s}{p-2}}\mathbf{R}^s\left(Q_{\varepsilon_n}^{\frac{1}{p}}v_n\right)(k_n x), \quad \text{in } \mathbb{R}^N, \tag{4.24}$$

where $\varepsilon_n=k_n^{-1}$ and $v_n\in L^{p'}(\mathbb{R}^N)$ is a least-energy critical point of $J_\varepsilon$, i.e., $J'_\varepsilon(v_n)=0$ and $J_{\varepsilon_n}(v_n)=c_{\varepsilon_n}$. By Lemma 2.4 and Proposition 4.1, there is $x_0\in M$ and a sequence $(y_n)_n\subset\mathbb{R}^N$ such that, as $n\to\infty$, $x_n:=\varepsilon_n y_n\to x_0$ and, going to a subsequence, $v(\cdot+y_n)\to w_0$ in $L^{p'}(\mathbb{R}^N)$ for some least-energy critical point $w_0$ of $J_0$. Therefore, for $x\in\mathbb{R}^N$,

$$k_n^{-\frac{2s}{p-2}}u_n\left(\frac{x}{k_n}+x_n\right)=\mathbf{R}^s\left(Q_{\varepsilon_n}^{\frac{1}{p}}v_n\right)(x+y_n)=\mathbf{R}^s\left(Q_{\varepsilon_n}^{\frac{1}{p}}(\cdot+y_n)v_n(\cdot+y_n)\right)(x). \tag{4.25}$$

On the other hand, by the continuity of $\mathbf{R}^s$ and the pointwise convergence $Q_{\varepsilon_n}(x+y_n)\to Q(x_0)=Q_0$ as $n\to\infty$ for all $x\in\mathbb{R}^N$, we have the following strong convergence

$$k_n^{-\frac{2s}{p-2}}u_n\left(\frac{x}{k_n}+x_n\right)\to\mathbf{R}^s\left(Q_0^{\frac{1}{p}}w_0\right), \qquad \text{in } L^p(\mathbb{R}^N). \tag{4.26}$$

Setting $u_0=\mathbf{R}^s\left(Q_0^{\frac{1}{p}}w_0\right)$, the properties $J_0(w_0)=c_0$ and $J'_0(w_0)=0$ imply that $u_0$ is a dual ground state solution of (2.9) and this conclude the proof. □

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