

# Spectral Properties of the Sub-Laplacian on 2-step Stratified Lie Groups without the Moore-Wolf Condition

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**Abstract.** In this paper, we consider the structure of the  $L^2$ -spectrum of the sub-Laplacian on 2-step stratified Lie groups by using the theory of unitary irreducible representations and Hermit functions. We extend the results for Heisenberg group and H-type Lie group to more general 2-step stratified Lie groups without the Moore-Wolf condition.

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## 1 Introduction

Harmonic analysis on nilpotent Lie groups is by now classical matter that goes back to the first half of the 20th century (see e.g. [1] for a self-contained presentation). In fact, as is known to all, harmonic analysis on nilpotent Lie groups plays a power role in contemporary investigations of linear PDEs. Since Folland and Stein's work in [2], some subjects concerning invariant operators on nilpotent Lie groups have been paid more and more attention. Many important results have been obtained, see [3–7] and their references.

In the Euclidean case  $\mathbb{R}^n$ , the Fourier transform of an integrable function  $f$  may thus be seen as the function on  $(\mathbb{R}^n)^*$  (usually identified to  $\mathbb{R}^n$ ) given by

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$$\mathcal{F}(f)(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi \cdot x} f(x) dx.$$

The Fourier transform on Euclidean has a number of noteworthy properties, we first recall that it changes convolution products into products of functions, namely

$$\mathcal{F}(f * g) = \mathcal{F}f \cdot \mathcal{F}g, \quad \forall f, g \in L^1(G).$$

What's more, we have the inverse Fourier transform

$$\forall x \in \mathbb{R}^n, \quad f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\xi \cdot x} \widehat{f}(\xi) d\xi,$$

and the Fourier-Plancherel formulae identity

$$\int_{\mathbb{R}^n} |f(x)|^2 dx = \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 d\xi.$$

Among the numerous additional properties of the Fourier transform on  $\mathbb{R}^n$ , let us just underline that it allows to "diagonalize" the Laplace operator, namely for all smooth compactly supported functions, we have

$$\mathcal{F}(\Delta f)(\xi) = -|\xi|^2 \widehat{f}(\xi). \quad (1.1)$$

For noncommutative groups, the simplest example (apart from  $\mathbb{R}^n$ ) of a nilpotent Lie group is the Heisenberg group, and the harmonic analysis there is a very well researched topic. We do not intend to make an overview of the subject here, but we refer to the books of Stein [8] and Thangavelu [9] for an introduction to the harmonic analysis on the Heisenberg group and for the historic development of the area. If we consider the Heisenberg group  $\mathbb{H}^d = \mathbb{R}^{2d+1}$  whose elements  $w \in \mathbb{R}^{2d+1}$  can be written  $w = (x, y, s)$  with  $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$ , endowed with the following product law:

$$w \cdot w' = (x, y, s) \cdot (x', y', s') = (x + x', y + y', s + s' - 2x \cdot y' + 2y \cdot x')$$

where for  $x, x' \in \mathbb{R}^d$ ,  $x \cdot x'$  denotes the Euclidean scalar product of the vectors  $x$  and  $x'$ . Equipped with the standard differential structure of the manifold  $\mathbb{R}^{2d+1}$ , the set  $\mathbb{H}^d$  is a noncommutative Lie group with identity  $(0, 0)$ .

Let us next recall the definition of the sub-Laplacian on the Heisenberg group, that will play a fundamental role as the Laplacian in  $\mathbb{R}^n$ . Being a real Lie group, the Heisenberg group may be equipped with a linear space of left invariant vector fields, that is vector fields commuting with any left translation  $\tau_w(w') = w \cdot w'$ . It is well known that this linear space has dimension  $2d + 1$  and is generated by the vector fields

$$S = \partial_s, \quad X_j = \partial_{x_j} + 2y_j \partial_s \quad \text{and} \quad Y_j = \partial_{y_j} - 2x_j \partial_s, \quad 1 \leq j \leq d.$$