JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **39**, No. 1, pp. 91-106

doi: 10.4208/jpde.v39.n1.5
March 2026

Regularity Criteria for 3D Liquid Crystal Flows in Besov Space

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Received 13 November 2023; Accepted 4 June 2024

Abstract. This note is devoted to investigating regularity criteria for 3D liquid crystal flows in Besov space.

AMS Subject Classifications: 76A15, 35Q35

Chinese Library Classifications: O175.2

Key Words: Regularity criteria; Liquid crystal flows; Navier-Stokes equations; Besov space.

1 Introduction

We will consider the following simplified version of the nematic Ericksen-Leslie model for liquid crystal flows:

$$\begin{cases} u_t+(u\cdot\nabla)u+\nabla P=\nu\Delta u-\lambda\nabla\cdot(\nabla d\otimes\nabla d), & \text{in } R^3\times(0,T),\\ d_t+(u\cdot\nabla)d=\gamma(\Delta d-f(d)), & \text{in } R^3\times(0,T),\\ \nabla\cdot u=0, & \text{in } R^3\times(0,T),\\ u(x,0)=u_0(x),d(x,0)=d_0(x), & \text{in } R^3, \end{cases} \tag{1.1}$$

where u is the velocity field, P is the scalar pressure and d represents the macroscopic molecular orientation field of the liquid crystal materials. The (i,j)th entry of $\nabla d\otimes\nabla d$ is given by $\nabla_{x_i}d\cdot\nabla_{x_j}d$ for $1\le i,j\le 3$. Moreover, $f(d)=\frac{1}{\eta^2}(|d|^2-1)d$. Without loss of generality, we assume that they are all one, since ν,λ,γ and η are positive constants. We set $\nabla_h=(\partial_{x_1},\partial_{x_2})$ as the horizontal gradient operator, $\Delta_h=\partial_{x_1}^2+\partial_{x_2}^2$ as the horizontal

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Laplacian, Δ and ∇ are the usual Laplacian and the gradient operators respectively. Here we use the classical notations

$$(u\cdot\nabla)d=\sum_{i=1}^{3}u_i\partial_{x_i}d,\quad \nabla\cdot u=\sum_{i=1}^{3}\partial_{x_i}u_i,$$

and for sake of simplicity, we denote ∂_{x_i} by ∂_i. The hydrodynamic theory for liquid crystals was derived by Ericksen and Leslie ([1,2]) in the 1960's. Lin and Liu [3] proved a global existence theorem of weak solutions for the simplified Ericksen-Leslie equations, and the local well-posed results for strong solutions are also established.

When the orientation field d equals a constant, the above equations become the incompressible Navier-Stokes equations. Many regularity results on the solutions to the three-dimensional Navier-Stokes equations have been well studied, where they proved the strong solution can not blow up, see e.g., [4–13] and the references therein. Similarly, there are also some interesting results for liquid crystal system, see [3, 14–28]. Recently, Zhao, Wang and Wang [25] has established the regularity of the weak solutions to 3D liquid crystal equations as follows:

$$u_h\in L^p\left(0,T;L^q\left(\mathbb{R}^3\right)\right),\quad \frac{2}{p}+\frac{3}{q}\le\frac{1}{2},\ q\ge 6. \tag{1.2}$$

Zhao and Li [26] showed the following regularity criterion for the liquid crystal system (1.1) and that is

$$u_3,\nabla_h d\in L^p\left(0,T;L^q\left(\mathbb{R}^3\right)\right),\quad \frac{2}{p}+\frac{3}{q}\le\frac{3}{4}+\frac{1}{2q},\ q>\frac{10}{3}. \tag{1.3}$$

Motivated by their ideas, we extend their regular results in Besov Space. Our main results can be stated in the following:

Theorem 1.1. *Let $u_0\in H^1\left(R^3\right)$, $d_0\in H^2\left(R^3\right)$, (u,d) be a strong solution of (1.1) on $[0,T)$ for some $0<T<\infty$. Suppose that one of the following conditions is ture:*

(i) u and d satisfies the following condition

$$\nabla u_h,\nabla\partial_3 d\in L^{\frac{6}{5-2s}}\left(0,T;\dot{B}^{-s}_{\infty,\infty}\left(R^3\right)\right),\quad \text{with } 0<s<1. \tag{1.4}$$

(ii) u and d satisfies the following condition

$$\nabla u_3,\nabla_h\nabla d\in L^{\frac{8}{5-2s}}\left(0,T;\dot{B}^{-s}_{\infty,\infty}\left(R^3\right)\right),\quad \text{with } 0<s<1. \tag{1.5}$$

Then (u,d) is regular up to time T.