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Regularity Criteria for 3D Liquid Crystal Flows in Besov Space

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Abstract. This note is devoted to investigating regularity criteria for 3D liquid crystal flows in Besov space.

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1 Introduction

We will consider the following simplified version of the nematic Ericksen-Leslie model for liquid crystal flows:

$$\begin{cases} u_t+(u\cdot\nabla)u+\nabla P=\nu\Delta u-\lambda\nabla\cdot(\nabla d\otimes\nabla d), & \text{in } R^3\times(0,T),\\ d_t+(u\cdot\nabla)d=\gamma(\Delta d-f(d)), & \text{in } R^3\times(0,T),\\ \nabla\cdot u=0, & \text{in } R^3\times(0,T),\\ u(x,0)=u_0(x), d(x,0)=d_0(x), & \text{in } R^3, \end{cases} \tag{1.1}$$

where u is the velocity field, P is the scalar pressure and d represents the macroscopic molecular orientation field of the liquid crystal materials. The (i,j)th entry of $\nabla d\otimes\nabla d$ is given by $\nabla_{x_i}d\cdot\nabla_{x_j}d$ for $1\le i,j\le 3$. Moreover, $f(d)=\frac{1}{\eta^2}(|d|^2-1)d$. Without loss of generality, we assume that they are all one, since ν,λ,γ and η are positive constants. We set $\nabla_h=(\partial_{x_1},\partial_{x_2})$ as the horizontal gradient operator, $\Delta_h=\partial_{x_1}^2+\partial_{x_2}^2$ as the horizontal

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Laplacian, Δ and ∇ are the usual Laplacian and the gradient operators respectively. Here we use the classical notations

$$(u\cdot\nabla)d=\sum_{i=1}^{3}u_i\partial_{x_i}d,\quad \nabla\cdot u=\sum_{i=1}^{3}\partial_{x_i}u_i,$$

and for sake of simplicity, we denote ∂_{x_i} by ∂_i. The hydrodynamic theory for liquid crystals was derived by Ericksen and Leslie ([1,2]) in the 1960's. Lin and Liu [3] proved a global existence theorem of weak solutions for the simplified Ericksen-Leslie equations, and the local well-posed results for strong solutions are also established.

When the orientation field d equals a constant, the above equations become the incompressible Navier-Stokes equations. Many regularity results on the solutions to the three-dimensional Navier-Stokes equations have been well studied, where they proved the strong solution can not blow up, see e.g., [4–13] and the references therein. Similarly, there are also some interesting results for liquid crystal system, see [3, 14–28]. Recently, Zhao, Wang and Wang [25] has established the regularity of the weak solutions to 3D liquid crystal equations as follows:

$$u_h\in L^p\big(0,T;L^q\big(\mathbb{R}^3\big)\big),\quad \frac{2}{p}+\frac{3}{q}\le\frac{1}{2},\ q\ge 6. \tag{1.2}$$

Zhao and Li [26] showed the following regularity criterion for the liquid crystal system (1.1) and that is

$$u_3,\nabla_h d\in L^p\big(0,T;L^q\big(\mathbb{R}^3\big)\big),\quad \frac{2}{p}+\frac{3}{q}\le\frac{3}{4}+\frac{1}{2q},\ q>\frac{10}{3}. \tag{1.3}$$

Motivated by their ideas, we extend their regular results in Besov Space. Our main results can be stated in the following:

Theorem 1.1. *Let $u_0\in H^1\big(R^3\big)$, $d_0\in H^2\big(R^3\big)$, (u,d) be a strong solution of (1.1) on $[0,T)$ for some $0<T<\infty$. Suppose that one of the following conditions is ture:*

(i) u and d satisfies the following condition

$$\nabla u_h,\nabla\partial_3 d\in L^{\frac{6}{5-2s}}\big(0,T;\dot{B}^{-s}_{\infty,\infty}\big(R^3\big)\big),\quad \text{with } 0<s<1. \tag{1.4}$$

(ii) u and d satisfies the following condition

$$\nabla u_3,\nabla_h\nabla d\in L^{\frac{8}{5-2s}}\big(0,T;\dot{B}^{-s}_{\infty,\infty}\big(R^3\big)\big),\quad \text{with } 0<s<1. \tag{1.5}$$

Then (u,d) is regular up to time T.

2 Preliminaries

We begin this section with some notations and Lemmas, which is useful for us to prove the main results. In order to define Besov spaces, we first introduce the Littlewood-Paley decomposition theory. Let $\mathcal{S}(R^3)$ be the Schwartz class of rapidly decreasing function, given $f\in\mathcal{S}(R^3)$, its Fourier transformation $\mathcal{F}f=\hat{f}$ is defined by

$$\hat{f}(\xi)=\int_{R^3}e^{-ix\cdot\xi}f(x)\mathrm{d}x,$$

and its inverse Fourier transform $\mathcal{F}^{-1}f=\check{f}$ is defined by

$$\check{f}(x)=(2\pi)^{-3}\int_{R^3}e^{ix\cdot\xi}f(\xi)\mathrm{d}\xi.$$

More generally, the Fourier transform of any $f\in\mathcal{S}'(R^3)$, the space of tempered distributions, is given by

$$\langle\hat{f},g\rangle=\langle f,\hat{g}\rangle,$$

for any $g\in\mathcal{S}(R^3)$. The Fourier transform is a bounded linear bijection from $\mathcal{S}'$ to $\mathcal{S}'$ whose inverse is also bounded. We fix the notation

$$\mathcal{S}_h=\left\{\phi\in\mathcal{S},\int_{R^3}\phi(x)x^{\gamma}\mathrm{d}x=0,|\gamma|=0,1,2,\cdots\right\}.$$

Its dual is given by

$$\mathcal{S}_h'=\frac{\mathcal{S}'}{\mathcal{S}_h^{\perp}}=\frac{\mathcal{S}'}{\mathcal{P}},$$

where $\mathcal{P}$ is the space of polynomial. In other words, two distributions in $\mathcal{S}_h'$ are identified as the same if their difference is a polynomial. Let us choose two nonnegative radial functions $\chi,\varphi\in\mathcal{S}(R^3)$ supported in $\mathfrak{B}=\{\xi\in R^3:|\xi|\le 4/3\}$ and $\mathfrak{C}=\{\xi\in R^3:3/4\le|\xi|\le 8/3\}$ respectively, such that

$$\sum_{j\in Z}\varphi(2^{-j}\xi)=1,\qquad\forall\xi\in R^3\backslash\{0\},$$

and

$$\chi(\xi)+\sum_{j\ge 0}\varphi(2^{-j}\xi)=1,\qquad\forall\xi\in R^3.$$

Let $h=\mathcal{F}^{-1}\varphi$ and $\tilde{h}=\mathcal{F}^{-1}\chi$, and then we define the homogeneous dyadic blocks $\dot{\Delta}_j$ and the homogeneous low-frequency cut-off operator $\dot{S}_j$ as follows:

$$\dot{\Delta}_j u=\varphi(2^{-j}D)u=2^{3j}\int_{R^3}h(2^jy)u(x-y)\mathrm{d}y,$$

and

$$\dot{S}_j u=\chi(2^{-j}D)u=2^{3j}\int_{R^3}\tilde{h}(2^j y)u(x-y)\mathrm{d}y.$$

Informally, $\dot{\Delta}_j$ is a frequency projection to the annulus $\{|\xi|\sim 2^j\}$, while $\dot{S}_j$ is a frequency projection to the ball $\{|\xi|\sim 2^j\}$. And one can easily verify that $\dot{\Delta}_j\dot{\Delta}_k f=0$ if $|j-k|\geqslant 2$. Especially for any $f\in L^2(R^3)$, we have the Littlewood-Paley decomposition:

$$f=\sum_{j=-\infty}^{+\infty}\dot{\Delta}_j f.$$

We now give the definitions of Besov spaces. Let $s\in R, p,q\in[1,\infty]$, the homogeneous Besov space $\dot{B}^s_{p,q}(R^3)$ is defined by the full-dyadic decomposition. We say that $f\in\dot{B}^s_{p,q}(R^3)$, if $f\in\mathcal{S}_h'$ and

$$\sum_{j=-\infty}^{+\infty}(2^{js}\|\dot{\Delta}_j f\|_{L^p})^q<\infty,$$

with the norm

$$\|f\|_{\dot{B}^s_{p,q}}=\begin{cases}\left(\displaystyle\sum_{j=-\infty}^{+\infty}2^{qjs}\|\dot{\Delta}_j f\|^q_{L^p}\right)^{\frac{1}{q}}, & 1\le q<\infty,\\ \displaystyle\sup_{j\in Z}2^{js}\|\dot{\Delta}_j f\|_{L^p}, & q=\infty.\end{cases}\tag{2.1}$$

It is of interest to note that the homogeneous Besov space $\dot{B}^s_{2,2}(R^3)$ is equivalent to the homogeneous Sobolev space $\dot{H}^s(R^3)$.

Lemma 2.1. ([29]) *Let $\mathfrak{B}$ be a ball and $\mathfrak{C}$ an annulus. A constant C exists such that for any nonnegative integer k, and couple (p,q) in $[1,\infty]^2$ with $1\le p\le q$, and any function u of $L^p(R^d)$, we have*

$$\begin{aligned}&\text{supp }\hat{u}\subset\lambda\mathfrak{B}\Rightarrow\sup_{|\alpha|=k}\|\partial^\alpha u\|_{L^q}\le C^{k+1}\lambda^{k+d\left(\frac{1}{p}-\frac{1}{q}\right)}\|u\|_{L^p},\\ &\text{supp }\hat{u}\subset\lambda\mathfrak{C}\Rightarrow C^{-k-1}\lambda^k\|u\|_{L^p}\le\sup_{|\alpha|=k}\|\partial^\alpha u\|_{L^p}\le C^{k+1}\lambda^k\|u\|_{L^p}.\end{aligned}\tag{2.2}$$

By the energy inequality, for the Leray-Hopf weak solutions to (1.1) (For example, see [21]), we have

$$\|u\|_{L^\infty(0,T;L^2)}+\|u\|_{L^2(0,T;H^1)}+\|d\|_{L^\infty(0,T;H^1)}+\|d\|_{L^2(0,T;H^2)}\le C,\tag{2.3}$$

for all $0<t<T$.

3 Proof of the main result

Proof of Theorem 1.1. Firstly, we deal with (i). By (3.8) in [25], we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla u\|_{L^2}^2+\|\Delta d\|_{L^2}^2\right)+\|\nabla\Delta d\|_{L^2}^2+\|\Delta u\|_{L^2}^2\\
=&-2\int_{R^3}\sum_{i,k=1}^{3}\partial_k u_i\partial_k\partial_i d\cdot\Delta d\,\mathrm{d}x+\int_{R^3}\nabla(|d|^2d)\cdot\nabla\Delta d\,\mathrm{d}x+\|\Delta d\|_{L^2}^2\\
&+\int_{R^3}\sum_{i=1}^{2}\sum_{j,k=1}^{3}u_i\partial_i u_j\cdot\partial_k\partial_k u_j\mathrm{d}x+\int_{R^3}\sum_{j=1}^{2}\sum_{k=1}^{3}\partial_3\partial_k u_3 u_j\partial_k u_j\mathrm{d}x\\
&+\int_{R^3}\sum_{j=1}^{2}\sum_{k=1}^{3}\partial_k u_3 u_j\partial_3\partial_k u_j\mathrm{d}x+\int_{R^3}\sum_{i,j=1}^{2}\sum_{k=1}^{3}u_i\partial_i\partial_k u_j\partial_k u_j\mathrm{d}x\\
&-\int_{R^3}\sum_{j=1}^{2}\sum_{k=1}^{3}u_j\partial_j\partial_k u_3\partial_k u_3\mathrm{d}x=I_1+I_2+\cdots+I_8.
\end{aligned}\tag{3.1}
$$

Next we come to estimate every term.

$$
\begin{aligned}
I_1=&-2\int_{R^3}\sum_{i=1}^{2}\partial_k u_i\partial_k\partial_i d\cdot\Delta d\,\mathrm{d}x-2\int_{R^3}\partial_k u_3\partial_k\partial_3 d\cdot\Delta d\,\mathrm{d}x\\
\le& C\int_{R^3}\sum_{i=1}^{2}|u_i||\nabla^3 d||\nabla^2 d|\mathrm{d}x+C\int_{R^3}|\nabla^2 u||\partial_3 d||\nabla^2 d|\mathrm{d}x+C\int_{R^3}|\nabla u||\partial_3 d||\nabla\Delta d|\mathrm{d}x\\
\le& C\int_{R^3}|u_h||\nabla^3 d||\nabla^2 d|\mathrm{d}x+C\int_{R^3}|\nabla^2 u||\partial_3 d||\nabla^2 d|\mathrm{d}x+C\int_{R^3}|\nabla u||\partial_3 d||\nabla\Delta d|\mathrm{d}x.
\end{aligned}\tag{3.2}
$$

Just as (3.12) in [25], similarly for the term I_2, we have

$$
I_2=\int_{R^3}|d|^2\nabla d\nabla\Delta d\,\mathrm{d}x+2\int_{R^3}(d\cdot\nabla d)(d\cdot\nabla\Delta d)\mathrm{d}x\le\frac{1}{10}\|\nabla\Delta d\|_{L^2}^2+C\|\Delta d\|_{L^2}^2.\tag{3.3}
$$

For the term I_4-I_8,

$$
I_4+\cdots+I_8\le C\int_{R^3}|u_h||\nabla u||\nabla^2 u|\mathrm{d}x.\tag{3.4}
$$

Combining (3.2)-(3.4), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla u\|_{L^2}^2+\|\Delta d\|_{L^2}^2\right)+\|\nabla\Delta d\|_{L^2}^2+\|\Delta u\|_{L^2}^2\\
\le& C\int_{R^3}|u_h|\left(|\nabla^3 d||\nabla^2 d|+|\nabla u||\nabla^2 u|\right)\mathrm{d}x+C\int_{R^3}|\partial_3 d|\left(|\nabla^2 u||\nabla^2 d|+|\nabla u||\nabla\Delta d|\right)\mathrm{d}x\\
&+\frac{1}{10}\|\nabla\Delta d\|_{L^2}^2+C\|\Delta d\|_{L^2}^2=II_1+II_2+II_3+II_4.
\end{aligned}\tag{3.5}
$$

We estimate II_1 and II_2 by the method just as [13, Theorem 1.1]. For II_1, by the Littlewood-Paley decomposition, we decompose u_h as follows:

$$u_h=\sum_{j=-\infty}^{+\infty}\dot{\Delta}_j u_h=\sum_{j\leq[\delta]-1}\dot{\Delta}_j u_h+\sum_{j\geq[\delta]}\dot{\Delta}_j u_h, \tag{3.6}$$

where δ is a real number determined later, and $[\cdot]$ denotes the integer part of δ. Therefore, we have

$$\begin{aligned} II_1=&C\int_{R^3}\sum_{j\leq[\delta]-1}|\dot{\Delta}_j u_h|(|\nabla^3 d||\nabla^2 d|+|\nabla u||\nabla^2 u|)\mathrm{d}x\\ &+C\int_{R^3}\sum_{j\geq[\delta]}|\dot{\Delta}_j u_h|(|\nabla^3 d||\nabla^2 d|+|\nabla u||\nabla^2 u|)\mathrm{d}x=II_{11}+II_{12}. \end{aligned}\tag{3.7}$$

In what follows, we estimate II_{11} and II_{12}. For II_{11}, by the Hölder's and Young's inequalities, as well as Lemma 2.1, we have

$$\begin{aligned} II_{11}\leq&C\sum_{j\leq[\delta]-1}\|\dot{\Delta}_j u_h\|_{L^\infty}(\|\nabla\Delta d\|_{L^2}\|\Delta d\|_{L^2}+\|\nabla u\|_{L^2}\|\Delta u\|_{L^2})\\ \leq&C\left(\sum_{j\leq[\delta]-1}2^{\frac{3}{2}j}\right)\|u_h\|_{L^2}(\|\nabla\Delta d\|_{L^2}\|\Delta d\|_{L^2}+\|\nabla u\|_{L^2}\|\Delta u\|_{L^2})\\ \leq&C2^{\frac{3}{2}([\delta]-1)}\|u_h\|_{L^2}(\|\nabla\Delta d\|_{L^2}\|\Delta d\|_{L^2}+\|\nabla u\|_{L^2}\|\Delta u\|_{L^2})\\ \leq&C2^{\frac{3}{2}[\delta]}\|u_h\|_{L^2}(\|\nabla\Delta d\|_{L^2}\|\Delta d\|_{L^2}+\|\nabla u\|_{L^2}\|\Delta u\|_{L^2})\\ \leq&C2^{3\delta}\|u_h\|_{L^2}^2\big(\|\Delta d\|_{L^2}^2+\|\nabla u\|_{L^2}^2\big)+\frac{1}{10}\|\nabla\Delta d\|_{L^2}^2+\frac{1}{8}\|\Delta u\|_{L^2}^2, \end{aligned}\tag{3.8}$$

in the last inequality, we haved used the fact that $[\delta]\leq\delta$. As to II_{12}, we take the same strategy to II_{11}, by the definition of norm of the Besov space, for any $0<\varepsilon<1$, we have

$$\begin{aligned} II_{12}\leq&C\sum_{j\geq[\delta]}\|\dot{\Delta}_j u_h\|_{L^\infty}(\|\nabla\Delta d\|_{L^2}\|\Delta d\|_{L^2}+\|\nabla u\|_{L^2}\|\Delta u\|_{L^2})\\ \leq&C\sum_{j\geq[\delta]}2^{-\varepsilon j}2^{(-1+\varepsilon)j}\|\dot{\Delta}_j\nabla u_h\|_{L^\infty}(\|\nabla\Delta d\|_{L^2}\|\Delta d\|_{L^2}+\|\nabla u\|_{L^2}\|\Delta u\|_{L^2})\\ \leq&C\left(\sum_{j\geq[\delta]}2^{-\varepsilon j}\right)\|\nabla u_h\|_{\dot{B}^{-1+\varepsilon}_{\infty,\infty}}(\|\nabla\Delta d\|_{L^2}\|\Delta d\|_{L^2}+\|\nabla u\|_{L^2}\|\Delta u\|_{L^2})\\ \leq&C2^{-2\varepsilon[\delta]}\|\nabla u_h\|_{\dot{B}^{-1+\varepsilon}_{\infty,\infty}}^2\big(\|\Delta d\|_{L^2}^2+\|\nabla u\|_{L^2}^2\big)+\frac{1}{10}\|\nabla\Delta d\|_{L^2}^2+\frac{1}{8}\|\Delta u\|_{L^2}^2\\ \leq&C2^{-2\varepsilon\delta}\|\nabla u_h\|_{\dot{B}^{-1+\varepsilon}_{\infty,\infty}}^2\big(\|\Delta d\|_{L^2}^2+\|\nabla u\|_{L^2}^2\big)+\frac{1}{10}\|\nabla\Delta d\|_{L^2}^2+\frac{1}{8}\|\Delta u\|_{L^2}^2, \end{aligned}\tag{3.9}$$

in the last inequality, we have used the fact that $\delta-1<[\delta]$. Inserting (3.8) and (3.9) into (3.7) to obtain

$$\begin{aligned} II_1 \leq & C2^{-2\varepsilon\delta}\|\nabla u_h\|^2_{\dot{B}^{-1+\varepsilon}_{\infty,\infty}}\big(\|\Delta d\|^2_{L^2}+\|\nabla u\|^2_{L^2}\big)+C2^{3\delta}\|u_h\|^2_{L^2}\big(\|\Delta d\|^2_{L^2}+\|\nabla u\|^2_{L^2}\big) \\ & +\frac{1}{5}\|\nabla\Delta d\|^2_{L^2}+\frac{1}{4}\|\Delta u\|^2_{L^2}. \end{aligned} \tag{3.10}$$

As to II_2, we take the same strategy to II_1, using the Littlewood-Paley decomposition, we decompose $\partial_3 d$ as follows:

$$\partial_3 d=\sum_{j=-\infty}^{+\infty}\dot{\Delta}_j\partial_3 d=\sum_{j\leq[\eta]-1}\dot{\Delta}_j\partial_3 d+\sum_{j\geq[\eta]}\dot{\Delta}_j\partial_3 d, \tag{3.11}$$

where η is a real number determined later, and $[\cdot]$ denotes the integer part of η. Therefore, we have

$$\begin{aligned} II_2 = & C\int_{R^3}\sum_{j\leq[\eta]-1}|\dot{\Delta}_j\partial_3 d|\big(|\nabla u||\nabla\Delta d|+|\nabla^2 u||\nabla^2 d|\big)\mathrm{d}x \\ & +C\int_{R^3}\sum_{j\geq[\eta]}|\dot{\Delta}_j\partial_3 d|\big(|\nabla u||\nabla\Delta d|+|\nabla^2 u||\nabla^2 d|\big)\mathrm{d}x \\ = & II_{21}+II_{22}. \end{aligned} \tag{3.12}$$

In what follows, we estimate II_{21} and II_{22}. For II_{21}, by the Hölder's and Young's inequalities, as well as Lemma 2.1, we have

$$\begin{aligned} II_{21} = & \sum_{j\leq[\eta]-1}\|\dot{\Delta}_j\partial_3 d\|_{L^\infty}\big(\|\nabla u\|_{L^2}\|\nabla\Delta d\|_{L^2}+\|\Delta u\|_{L^2}\|\Delta d\|_{L^2}\big) \\ \leq & C\left(\sum_{j\leq[\eta]-1}2^{\frac{3}{2}j}\right)\|\partial_3 d\|_{L^2}\big(\|\nabla u\|_{L^2}\|\nabla\Delta d\|_{L^2}+\|\Delta u\|_{L^2}\|\Delta d\|_{L^2}\big) \\ \leq & C2^{\frac{3}{2}([\eta]-1)}\|\partial_3 d\|_{L^2}\big(\|\nabla u\|_{L^2}\|\nabla\Delta d\|_{L^2}+\|\Delta u\|_{L^2}\|\Delta d\|_{L^2}\big) \\ \leq & C2^{\frac{3}{2}[\eta]}\|\partial_3 d\|_{L^2}\big(\|\nabla u\|_{L^2}\|\nabla\Delta d\|_{L^2}+\|\Delta u\|_{L^2}\|\Delta d\|_{L^2}\big) \\ \leq & C2^{3\eta}\|\partial_3 d\|^2_{L^2}\big(\|\nabla u\|^2_{L^2}+\|\Delta d\|^2_{L^2}\big)+\frac{1}{10}\|\nabla\Delta d\|^2_{L^2}+\frac{1}{8}\|\Delta u\|^2_{L^2}, \end{aligned} \tag{3.13}$$

in the last inequality, we have used the fact that $[\eta]\leq\eta$. As to II_{22}, by the definition of norm of the Besov space, for any $0<\varepsilon<1$, we have

$$\begin{aligned} II_{22} \leq & C\sum_{j\geq[\eta]}\|\dot{\Delta}_j\partial_3 d\|_{L^\infty}\big(\|\nabla u\|_{L^2}\|\nabla\Delta d\|_{L^2}+\|\Delta u\|_{L^2}\|\Delta d\|_{L^2}\big) \\ \leq & C\sum_{j\geq[\eta]}2^{-\varepsilon j}2^{(-1+\varepsilon)j}\|\dot{\Delta}_j\nabla\partial_3 d\|_{L^\infty}\big(\|\nabla u\|_{L^2}\|\nabla\Delta d\|_{L^2}+\|\Delta u\|_{L^2}\|\Delta d\|_{L^2}\big) \end{aligned}$$

$$\begin{aligned}
&\le C\left(\sum_{j\ge[\eta]}2^{-\varepsilon j}\right)\|\nabla\partial_3 d\|_{\dot B^{-1+\varepsilon}_{\infty,\infty}}\left(\|\nabla u\|_{L^2}\|\nabla\Delta d\|_{L^2}+\|\Delta u\|_{L^2}\|\Delta d\|_{L^2}\right)\\
&\le C2^{-2\varepsilon[\eta]}\|\nabla\partial_3 d\|^2_{\dot B^{-1+\varepsilon}_{\infty,\infty}}\left(\|\nabla u\|^2_{L^2}+\|\Delta d\|^2_{L^2}\right)+\frac{1}{10}\|\nabla\Delta d\|^2_{L^2}+\frac{1}{8}\|\Delta u\|^2_{L^2}\\
&\le C2^{-2\varepsilon\eta}\|\nabla\partial_3 d\|^2_{\dot B^{-1+\varepsilon}_{\infty,\infty}}\left(\|\nabla u\|^2_{L^2}+\|\Delta d\|^2_{L^2}\right)+\frac{1}{10}\|\nabla\Delta d\|^2_{L^2}+\frac{1}{8}\|\Delta u\|^2_{L^2}, \qquad (3.14)
\end{aligned}$$

in the last inequality, we have used the fact that $\eta-1<[\eta]$. Inserting (3.13) and (3.14) into (3.12) to obtain

$$\begin{aligned}
II_2\le& C2^{3\eta}\|\partial_3 d\|^2_{L^2}\left(\|\nabla u\|^2_{L^2}+\|\Delta d\|^2_{L^2}\right)+C2^{-2\varepsilon\eta}\|\nabla\partial_3 d\|^2_{\dot B^{-1+\varepsilon}_{\infty,\infty}}\left(\|\nabla u\|^2_{L^2}+\|\Delta d\|^2_{L^2}\right)\\
&+\frac{1}{5}\|\nabla\Delta d\|^2_{L^2}+\frac{1}{4}\|\Delta u\|^2_{L^2}. \qquad (3.15)
\end{aligned}$$

Then combining (3.5)-(3.15), we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla u\|^2_{L^2}+\|\Delta d\|^2_{L^2}\right)+\|\nabla\Delta d\|^2_{L^2}+\|\Delta u\|^2_{L^2}\\
\le& C2^{-2\varepsilon\delta}\|\nabla u_h\|^2_{\dot B^{-1+\varepsilon}_{\infty,\infty}}\left(\|\Delta d\|^2_{L^2}+\|\nabla u\|^2_{L^2}\right)+C2^{3\delta}\|u_h\|^2_{L^2}\left(\|\Delta d\|^2_{L^2}+\|\nabla u\|^2_{L^2}\right)\\
&+C2^{-2\varepsilon\eta}\|\nabla\partial_3 d\|^2_{\dot B^{-1+\varepsilon}_{\infty,\infty}}\left(\|\nabla u\|^2_{L^2}+\|\Delta d\|^2_{L^2}\right)+C2^{3\eta}\|\partial_3 d\|^2_{L^2}\left(\|\nabla u\|^2_{L^2}+\|\Delta d\|^2_{L^2}\right)\\
&+\frac{1}{2}\|\nabla\Delta d\|^2_{L^2}+C\|\Delta d\|^2_{L^2}+\frac{1}{2}\|\Delta u\|^2_{L^2}. \qquad (3.16)
\end{aligned}$$

Next, we choose δ such that

$$2^{\frac{3}{2}\delta}\|u_h\|_{L^2}\left(\|\Delta d\|_{L^2}+\|\nabla u\|_{L^2}\right)=2^{-\varepsilon\delta}\|\nabla u_h\|_{\dot B^{-1+\varepsilon}_{\infty,\infty}}\left(\|\Delta d\|_{L^2}+\|\nabla u\|_{L^2}\right),$$

then we have

$$C2^{-\varepsilon\delta}\|\nabla u_h\|_{\dot B^{-1+\varepsilon}_{\infty,\infty}}\left(\|\Delta d\|_{L^2}+\|\nabla u\|_{L^2}\right)\le C\|u_h\|_{L^2}^{\frac{\varepsilon}{\frac{3}{2}+\varepsilon}}\|\nabla u_h\|_{\dot B^{-1+\varepsilon}_{\infty,\infty}}^{\frac{\frac{3}{2}}{\frac{3}{2}+\varepsilon}}\left(\|\Delta d\|_{L^2}+\|\nabla u\|_{L^2}\right).$$

We choose η such that

$$2^{\frac{3}{2}\eta}\|\partial_3 d\|_{L^2}\left(\|\nabla u\|_{L^2}+\|\Delta d\|_{L^2}\right)=2^{-\varepsilon\eta}\|\nabla\partial_3 d\|_{\dot B^{-1+\varepsilon}_{\infty,\infty}}\left(\|\nabla u\|_{L^2}+\|\Delta d\|_{L^2}\right),$$

then we have

$$C2^{-\varepsilon\eta}\|\nabla\partial_3 d\|_{\dot B^{-1+\varepsilon}_{\infty,\infty}}\left(\|\nabla u\|_{L^2}+\|\Delta d\|_{L^2}\right)\le C\|\partial_3 d\|_{L^2}^{\frac{\varepsilon}{\frac{3}{2}+\varepsilon}}\|\nabla\partial_3 d\|_{\dot B^{-1+\varepsilon}_{\infty,\infty}}^{\frac{\frac{3}{2}}{\frac{3}{2}+\varepsilon}}\left(\|\nabla u\|_{L^2}+\|\Delta d\|_{L^2}\right).$$

Integrating in time, combing above inequalities and the energy inequality (2.3), (3.16) becomes

$$\|\nabla u\|^2_{L^2}+\|\Delta d\|^2_{L^2}+\int_0^t\left(\|\nabla\Delta d\|^2_{L^2}+\|\Delta u\|^2_{L^2}\right)\mathrm{d}\tau$$

$$\leq \|\nabla u_0\|_{L^2}^2+\|\Delta d_0\|_{L^2}^2+C+C\int_0^t \Bigg[\|\nabla u_h\|_{\dot{B}^{-1+\varepsilon}_{\infty,\infty}}^{\frac{3}{\frac{3}{2}+\varepsilon}}(\|\Delta d\|_{L^2}^2+\|\nabla u\|_{L^2}^2)$$
$$+\|\nabla\partial_3 d\|_{\dot{B}^{-1+\varepsilon}_{\infty,\infty}}^{\frac{3}{\frac{3}{2}+\varepsilon}}\big(\|\nabla u\|_{L^2}^2+\|\Delta d\|_{L^2}^2\big)\Bigg]\mathrm{d}\tau. \tag{3.17}$$

If we set $s=1-\varepsilon$, then we have

$$\frac{3}{\frac{3}{2}+\varepsilon}=\frac{6}{5-2s}, \quad \text{with } 0<s<1.$$

By Gronwall's inequality, we finally obtain

$$\|\nabla u\|_{L^2}^2+\|\Delta d\|_{L^2}^2\leq \big(\|\nabla u_0\|_{L^2}^2+\|\Delta d_0\|_{L^2}^2+C\big)$$
$$\times\exp\left[C\int_0^t\left(\|\nabla u_h\|_{\dot{B}^{-s}_{\infty,\infty}}^{\frac{6}{5-2s}}+\|\nabla\partial_3 d\|_{\dot{B}^{-s}_{\infty,\infty}}^{\frac{6}{5-2s}}\right)\mathrm{d}\tau\right], \tag{3.18}$$

by condition (1.4), we get the H^1 norm of the strong solution u and the H^2 norm of the strong solution d are bounded. This completes the proof of (i).

Now we prove(ii). By (3.24) in [26], we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\nabla_h u\|_{L^2}^2+\|\nabla_h\nabla d\|_{L^2}^2\big)+\|\nabla_h\nabla u\|_{L^2}^2+\|\nabla_h\Delta d\|_{L^2}^2$$
$$\leq C\int|u_3||\nabla u||\nabla_h\nabla u|\mathrm{d}x+C\int|\nabla_h d||\nabla_h\nabla^2 d||\nabla_h u|\mathrm{d}x$$
$$+C\int|u_3||\nabla_h\nabla^2 d||\nabla^2 d|\mathrm{d}x+C\int|u_3||\nabla_h\nabla d||\nabla_h\nabla^2 d|\mathrm{d}x$$
$$+C\int|\nabla_h d||\nabla^2 d||\nabla_h\nabla u|\mathrm{d}x+\int|\nabla u||\nabla_h d||\nabla_h\nabla^2 d|\mathrm{d}x$$
$$+C\int|\nabla_h\nabla u||\nabla_h d||\nabla_h\nabla d|\mathrm{d}x+\Big(\frac{1}{14}\|\nabla_h\Delta d\|_{L^2}^2+C\|\nabla_h\nabla d\|_{L^2}^2\Big)$$
$$\leq C\int|u_3|\big(|\nabla u||\nabla_h\nabla u|+|\nabla_h\nabla^2 d||\nabla^2 d|\big)\mathrm{d}x$$
$$+C\int|\nabla_h d|\big(|\nabla^2 d||\nabla_h\nabla u|+|\nabla u||\nabla_h\nabla^2 d|\big)\mathrm{d}x$$
$$+\Big(\frac{1}{10}\|\nabla_h\Delta d\|_{L^2}^2+C\|\nabla_h\nabla d\|_{L^2}^2\Big)$$
$$=J_1+J_2+J_3. \tag{3.19}$$

We estimate J_1 and J_2 just like the method of [13, Theorem 1.1]. For J_1, by the Littlewood-Paley decomposition, we decompose u_3 as follows:

$$u_3=\sum_{j=-\infty}^{+\infty}\dot{\Delta}_j u_3=\sum_{j\leq[\alpha]-1}\dot{\Delta}_j u_3+\sum_{j\geq[\alpha]}\dot{\Delta}_j u_3. \tag{3.20}$$

Where α is a real number determined later, and $[\cdot]$ denotes the integer part of α. Therefore, we have

$$\begin{aligned}J_1 =& C\sum_{j\le[\alpha]-1}\int_{R^3}|\dot{\Delta}_j u_3|\big(|\nabla u||\nabla_h\nabla u|+|\nabla_h\nabla^2 d||\nabla^2 d|\big)\mathrm{d}x\\&+C\sum_{j\ge[\alpha]}\int_{R^3}|\dot{\Delta}_j u_3|\big(|\nabla u||\nabla_h\nabla u|+|\nabla_h\nabla^2 d||\nabla^2 d|\big)\mathrm{d}x=J_{11}+J_{12}.\end{aligned}\tag{3.21}$$

In what follows, we estimate J_{11} and J_{12}. For J_{11}, using the Hölder's and Young's inequalities, as well as Lemma 2.1, we have

$$\begin{aligned}J_{11}\le& C\sum_{j\le[\alpha]-1}\|\dot{\Delta}_j u_3\|_{L^\infty}(\|\nabla u\|_{L^2}\|\nabla_h\nabla u\|_{L^2}+\|\nabla_h\Delta d\|_{L^2}\|\Delta d\|_{L^2})\\ \le& C\Bigg(\sum_{j\le[\alpha]-1}2^{\frac{3}{2}j}\Bigg)\|u_3\|_{L^2}(\|\nabla u\|_{L^2}\|\nabla_h\nabla u\|_{L^2}+\|\nabla_h\Delta d\|_{L^2}\|\Delta d\|_{L^2})\\ \le& C2^{\frac{3}{2}([\alpha]-1)}\|u_3\|_{L^2}(\|\nabla u\|_{L^2}\|\nabla_h\nabla u\|_{L^2}+\|\nabla_h\Delta d\|_{L^2}\|\Delta d\|_{L^2})\\ \le& C2^{\frac{3}{2}[\alpha]}\|u_3\|_{L^2}(\|\nabla u\|_{L^2}\|\nabla_h\nabla u\|_{L^2}+\|\nabla_h\Delta d\|_{L^2}\|\Delta d\|_{L^2})\\ \le& C2^{3\alpha}\|u_3\|_{L^2}^2\big(\|\nabla u\|_{L^2}^2+\|\Delta d\|_{L^2}^2\big)+\frac{1}{8}\|\nabla_h\nabla u\|_{L^2}^2+\frac{1}{10}\|\nabla_h\Delta d\|_{L^2}^2,\end{aligned}\tag{3.22}$$

in the last inequality, we have used the fact that $[\alpha]\le\alpha$. As to J_{12}, by the definition of norm of the Besov space, for any $0<\varepsilon<1$, we have

$$\begin{aligned}J_{12}\le& C\sum_{j\ge[\alpha]}\|\dot{\Delta}_j\ u_3\|_{L^\infty}(\|\nabla u\|_{L^2}\|\nabla_h\nabla u\|_{L^2}+\|\nabla_h\Delta d\|_{L^2}\|\Delta d\|_{L^2})\\ \le& C\sum_{j\ge[\alpha]}2^{-\varepsilon j}2^{(-1+\varepsilon)j}\|\dot{\Delta}_j\nabla u_3\|_{L^\infty}(\|\nabla u\|_{L^2}\|\nabla_h\nabla u\|_{L^2}+\|\nabla_h\Delta d\|_{L^2}\|\Delta d\|_{L^2})\\ \le& C\Bigg(\sum_{j\ge[\alpha]}2^{-\varepsilon j}\Bigg)\|\nabla u_3\|_{\dot{B}^{-1+\varepsilon}_{\infty,\infty}}(\|\nabla u\|_{L^2}\|\nabla_h\nabla u\|_{L^2}+\|\nabla_h\Delta d\|_{L^2}\|\Delta d\|_{L^2})\\ \le& C2^{-2\varepsilon[\alpha]}\|\nabla u_3\|^2_{\dot{B}^{-1+\varepsilon}_{\infty,\infty}}\big(\|\nabla u\|_{L^2}^2+\|\Delta d\|_{L^2}^2\big)+\frac{1}{8}\|\nabla_h\nabla u\|_{L^2}^2+\frac{1}{10}\|\nabla_h\Delta d\|_{L^2}^2\\ \le& C2^{-2\varepsilon\alpha}\|\nabla u_3\|^2_{\dot{B}^{-1+\varepsilon}_{\infty,\infty}}\big(\|\nabla u\|_{L^2}^2+\|\Delta d\|_{L^2}^2\big)+\frac{1}{8}\|\nabla_h\nabla u\|_{L^2}^2+\frac{1}{10}\|\nabla_h\Delta d\|_{L^2}^2,\end{aligned}\tag{3.23}$$

in the last inequality, we have used the fact that $\alpha-1<[\alpha]$. Inserting (3.22) and (3.23) into (3.21) to obtain

$$\begin{aligned}J_1\le& C2^{3\alpha}\|u_3\|_{L^2}^2\big(\|\nabla u\|_{L^2}^2+\|\Delta d\|_{L^2}^2\big)+C2^{-2\varepsilon\alpha}\|\nabla u_3\|^2_{\dot{B}^{-1+\varepsilon}_{\infty,\infty}}\big(\|\nabla u\|_{L^2}^2+\|\Delta d\|_{L^2}^2\big)\\&+\frac{1}{4}\|\nabla_h\nabla u\|_{L^2}^2+\frac{1}{5}\|\nabla_h\Delta d\|_{L^2}^2.\end{aligned}\tag{3.24}$$

As to J_2, by the Littlewood-Paley decomposition, we decompose $\nabla_h d$ as follows:

$$\nabla_h d = \sum_{j=-\infty}^{+\infty} \dot{\Delta}_j \nabla_h d = \sum_{j\le[\beta]-1} \dot{\Delta}_j \nabla_h d + \sum_{j\ge[\beta]} \dot{\Delta}_j \nabla_h d. \tag{3.25}$$

Where β is a real number determined later, and $[\cdot]$ denotes the integer part of β. Therefore, we have

$$\begin{aligned} J_2 =& C \sum_{j\le[\beta]-1} \int_{R^3} |\dot{\Delta}_j \nabla_h d| \big(|\nabla^2 d||\nabla_h \nabla u| + |\nabla u||\nabla_h \nabla^2 d|\big) \mathrm{d}x \\ &+ C \sum_{j\ge[\beta]} \int_{R^3} |\dot{\Delta}_j \nabla_h d| \big(|\nabla^2 d||\nabla_h \nabla u| + |\nabla u||\nabla_h \nabla^2 d|\big) \mathrm{d}x = J_{21} + J_{22}. \end{aligned} \tag{3.26}$$

In what follows, we estimate J_{21} and J_{22}. For J_{21}, using the Hölder's and Young's inequalities, as well as Lemma 2.1, we have

$$\begin{aligned} J_{21} =& \sum_{j\le[\beta]-1} \|\dot{\Delta}_j \nabla_h d\|_{L^\infty} (\|\Delta d\|_{L^2} \|\nabla_h \nabla u\|_{L^2} + \|\nabla u\|_{L^2} \|\nabla_h \Delta d\|_{L^2}) \\ \le& C \Bigg(\sum_{j\le[\beta]-1} 2^{\frac{3}{2}j} \Bigg) \|\nabla_h d\|_{L^2} (\|\Delta d\|_{L^2} \|\nabla_h \nabla u\|_{L^2} + \|\nabla u\|_{L^2} \|\nabla_h \Delta d\|_{L^2}) \\ \le& C 2^{\frac{3}{2}([\beta]-1)} \|\nabla_h d\|_{L^2} (\|\Delta d\|_{L^2} \|\nabla_h \nabla u\|_{L^2} + \|\nabla u\|_{L^2} \|\nabla_h \Delta d\|_{L^2}) \\ \le& C 2^{\frac{3}{2}[\beta]} \|\nabla_h d\|_{L^2}^2 \big(\|\Delta d\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\big) + \frac{1}{8} \|\nabla_h \nabla u\|_{L^2}^2 + \frac{1}{10} \|\nabla_h \Delta d\|_{L^2}^2 \\ \le& C 2^{3\beta} \|\nabla_h d\|_{L^2}^2 \big(\|\Delta d\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\big) + \frac{1}{8} \|\nabla_h \nabla u\|_{L^2}^2 + \frac{1}{10} \|\nabla_h \Delta d\|_{L^2}^2, \end{aligned} \tag{3.27}$$

in the last inequality, we have used the fact that $[\beta] \le \beta$. As to J_{22}, by the definition of norm of the Besov space, for any $0 < \varepsilon < 1$, we have

$$\begin{aligned} J_{22} \le& C \sum_{j\ge[\beta]} \|\dot{\Delta}_j \nabla_h d\|_{L^\infty} (\|\Delta d\|_{L^2} \|\nabla_h \nabla u\|_{L^2} + \|\nabla u\|_{L^2} \|\nabla_h \Delta d\|_{L^2}) \\ \le& C \sum_{j\ge[\beta]} 2^{-\varepsilon j} 2^{(-1+\varepsilon)j} \|\dot{\Delta}_j \nabla_h \nabla d\|_{L^\infty} (\|\Delta d\|_{L^2} \|\nabla_h \nabla u\|_{L^2} + \|\nabla u\|_{L^2} \|\nabla_h \Delta d\|_{L^2}) \\ \le& C \Bigg(\sum_{j\ge[\beta]} 2^{-\varepsilon j} \Bigg) \|\nabla_h \nabla d\|_{\dot{B}^{-1+\varepsilon}_{\infty,\infty}} (\|\Delta d\|_{L^2} \|\nabla_h \nabla u\|_{L^2} + \|\nabla u\|_{L^2} \|\nabla_h \Delta d\|_{L^2}) \\ \le& C 2^{-2\varepsilon[\beta]} \|\nabla_h \nabla d\|_{\dot{B}^{-1+\varepsilon}_{\infty,\infty}}^2 \big(\|\Delta d\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\big) + \frac{1}{8} \|\nabla_h \nabla u\|_{L^2}^2 + \frac{1}{10} \|\nabla_h \Delta d\|_{L^2}^2 \\ \le& C 2^{-2\varepsilon\beta} \|\nabla_h \nabla d\|_{\dot{B}^{-1+\varepsilon}_{\infty,\infty}}^2 \big(\|\Delta d\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\big) + \frac{1}{8} \|\nabla_h \nabla u\|_{L^2}^2 + \frac{1}{10} \|\nabla_h \Delta d\|_{L^2}^2, \end{aligned} \tag{3.28}$$

in the last inequality, we have used the fact that $\beta - 1 < [\beta]$. Inserting (3.27) and (3.28) into (3.26) to obtain

$$J_2 \le C 2^{3\beta} \|\nabla_h d\|_{L^2}^2 \big(\|\Delta d\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\big) + C 2^{-2\varepsilon\beta} \|\nabla_h \nabla d\|_{\dot{B}^{-1+\varepsilon}_{\infty,\infty}}^2 \big(\|\Delta d\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\big)$$

$$+\frac{1}{4}\|\nabla_h\nabla u\|_{L^2}^2+\frac{1}{5}\|\nabla_h\Delta d\|_{L^2}^2. \tag{3.29}$$

Then combining (3.19)-(3.29), we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\nabla_h u\|_{L^2}^2+\|\nabla_h\nabla d\|_{L^2}^2\big)+\|\nabla_h\nabla u\|_{L^2}^2+\|\nabla_h\Delta d\|_{L^2}^2\\ \le&C2^{3\alpha}\|u_3\|_{L^2}^2\big(\|\nabla u\|_{L^2}^2+\|\Delta d\|_{L^2}^2\big)+C2^{-2\varepsilon\alpha}\|\nabla u_3\|_{\dot{B}^{-1+\varepsilon}_{\infty,\infty}}^2\big(\|\nabla u\|_{L^2}^2+\|\Delta d\|_{L^2}^2\big)\\ &+C2^{3\beta}\|\nabla_h d\|_{L^2}^2\big(\|\Delta d\|_{L^2}^2+\|\nabla u\|_{L^2}^2\big)+C2^{-2\varepsilon\beta}\|\nabla_h\nabla d\|_{\dot{B}^{-1+\varepsilon}_{\infty,\infty}}^2\big(\|\Delta d\|_{L^2}^2+\|\nabla u\|_{L^2}^2\big)\\ &+\frac{1}{2}\|\nabla_h\nabla u\|_{L^2}^2+\frac{1}{2}\|\nabla_h\Delta d\|_{L^2}^2+C\|\nabla_h\nabla d\|_{L^2}^2. \end{aligned}\tag{3.30}$$

Next, we choose α such that

$$2^{\frac{3}{2}\alpha}\|u_3\|_{L^2}(\|\nabla u\|_{L^2}+\|\Delta d\|_{L^2})=2^{-\varepsilon\alpha}\|\nabla u_3\|_{\dot{B}^{-1+\varepsilon}_{\infty,\infty}}(\|\nabla u\|_{L^2}+\|\Delta d\|_{L^2}),$$

then we have

$$C2^{-2\varepsilon\alpha}\|\nabla u_3\|_{\dot{B}^{-1+\varepsilon}_{\infty,\infty}}(\|\nabla u\|_{L^2}+\|\Delta d\|_{L^2})\le C\|u_3\|_{L^2}^{\frac{\varepsilon}{\frac{3}{2}+\varepsilon}}\|\nabla u_3\|_{\dot{B}^{-1+\varepsilon}_{\infty,\infty}}^{\frac{\frac{3}{2}}{\frac{3}{2}+\varepsilon}}(\|\nabla u\|_{L^2}+\|\Delta d\|_{L^2}).$$

We choose β such that

$$2^{\frac{3}{2}\beta}\|\nabla_h d\|_{L^2}(\|\Delta d\|_{L^2}+\|\nabla u\|_{L^2})=2^{-\varepsilon\beta}\|\nabla_h\nabla d\|_{\dot{B}^{-1+\varepsilon}_{\infty,\infty}}(\|\Delta d\|_{L^2}+\|\nabla u\|_{L^2}),$$

then we have

$$C2^{-\varepsilon\beta}\|\nabla_h\nabla d\|_{\dot{B}^{-1+\varepsilon}_{\infty,\infty}}(\|\Delta d\|_{L^2}+\|\nabla u\|_{L^2})\le C\|\nabla_h d\|_{L^2}^{\frac{\varepsilon}{\frac{3}{2}+\varepsilon}}\|\nabla_h\nabla d\|_{\dot{B}^{-1+\varepsilon}_{\infty,\infty}}^{\frac{\frac{3}{2}}{\frac{3}{2}+\varepsilon}}(\|\Delta d\|_{L^2}+\|\nabla u\|_{L^2}).$$

Integrating in time, combing above inequalities and the energy inequality (2.3), we have

$$\begin{aligned}&\|\nabla_h u\|_{L^2}^2+\|\nabla_h\nabla d\|_{L^2}^2+\int_0^t\|\nabla_h\nabla u\|_{L^2}^2+\|\nabla_h\Delta d\|_{L^2}^2\mathrm{d}\tau\\ \le&C+C\int_0^t\|\nabla u_3\|_{\dot{B}^{-1+\varepsilon}_{\infty,\infty}}^{\frac{3}{\frac{3}{2}+\varepsilon}}\big(\|\nabla u\|_{L^2}^2+\|\Delta d\|_{L^2}^2\big)+\|\nabla_h\nabla d\|_{\dot{B}^{-1+\varepsilon}_{\infty,\infty}}^{\frac{3}{\frac{3}{2}+\varepsilon}}\big(\|\Delta d\|_{L^2}^2+\|\nabla u\|_{L^2}^2\big)\mathrm{d}\tau. \end{aligned}\tag{3.31}$$

Besides, referring to (3.34)-(3.38) in [26], we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\nabla u\|_{L^2}^2+\|\Delta d\|_{L^2}^2\big)+\|\Delta u\|_{L^2}^2+\|\nabla\Delta d\|_{L^2}^2\\ =&\int(u\cdot\nabla)u\cdot\Delta u\mathrm{d}x-2\sum_{i=1}^3\int\nabla u_i\partial_i\nabla d\cdot\Delta d\,\mathrm{d}x-\int\Delta f(d)\cdot\Delta d\,\mathrm{d}x\\ =&E_1+E_2+E_3, \end{aligned}\tag{3.32}$$

and

$$E_1 \le C\|\nabla_h u\|_{L^2}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla_h\nabla u\|_{L^2}\|\Delta u\|_{L^2}^{\frac{1}{2}}. \tag{3.33}$$

$$\begin{aligned} E_2 &\le C\int|\nabla_h d||\nabla^2 u||\nabla^2 d|\mathrm{d}x + C\int|\nabla_h d||\nabla u||\nabla\Delta d|\mathrm{d}x + C\int|u_3||\nabla^3 d||\nabla^2 d|\mathrm{d}x \\ &\le C\int|\nabla_h d|\big(|\nabla^2 u||\nabla^2 d| + |\nabla u||\nabla\Delta d|\big)\mathrm{d}x + C\int|u_3||\nabla^3 d||\nabla^2 d|\mathrm{d}x \\ &= E_{21}+E_{22}. \end{aligned} \tag{3.34}$$

$$E_3 \le \frac{1}{8}\|\nabla\Delta d\|_{L^2}^2 + C\|\Delta d\|_{L^2}^2. \tag{3.35}$$

For E_2, just like (3.20) and (3.25) using the Littlewood-Paley decomposition, we decompose $\nabla_h d$ and u_3 respectively,

$$\begin{aligned} E_{21} \le{}& 2^{3\lambda}\|\nabla_h d\|_{L^2}^2\big(\|\nabla^2 d\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\big) + 2^{-2\varepsilon\lambda}\|\nabla_h\nabla d\|_{\dot B_{\infty,\infty}^{-1+\varepsilon}}^2\big(\|\nabla^2 d\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\big) \\ &+ \frac{1}{2}\|\Delta u\|_{L^2}^2 + \frac{1}{8}\|\nabla\Delta d\|_{L^2}^2. \end{aligned} \tag{3.36}$$

$$E_{22} \le 2^{3\mu}\|u_3\|_{L^2}^2\|\Delta d\|_{L^2}^2 + 2^{-2\varepsilon\mu}\|\nabla u_3\|_{\dot B_{\infty,\infty}^{-1+\varepsilon}}^2\|\Delta d\|_{L^2}^2 + \frac{1}{4}\|\nabla\Delta d\|_{L^2}^2. \tag{3.37}$$

Combining (3.32)-(3.37), we obtain

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2\big) + \|\Delta u\|_{L^2}^2 + \|\nabla\Delta d\|_{L^2}^2 \\ \le{}& C\|\nabla_h u\|_{L^2}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla_h\nabla u\|_{L^2}\|\Delta u\|_{L^2}^{\frac{1}{2}} + 2^{3\lambda}\|\nabla_h d\|_{L^2}^2\big(\|\nabla^2 d\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\big) \\ &+ 2^{-2\varepsilon\lambda}\|\nabla_h\nabla d\|_{\dot B_{\infty,\infty}^{-1+\varepsilon}}^2\big(\|\nabla^2 d\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\big) + 2^{3\mu}\|u_3\|_{L^2}^2\|\Delta d\|_{L^2}^2 \\ &+ 2^{-2\varepsilon\mu}\|\nabla u_3\|_{\dot B_{\infty,\infty}^{-1+\varepsilon}}^2\|\Delta d\|_{L^2}^2 + \frac{1}{2}\big(\|\Delta u\|_{L^2}^2 + \|\nabla\Delta d\|_{L^2}^2\big) + C\|\Delta d\|_{L^2}^2. \end{aligned} \tag{3.38}$$

Then we choose λ such that

$$2^{\frac{3}{2}\lambda}\|\nabla_h d\|_{L^2}\big(\|\nabla^2 d\|_{L^2} + \|\nabla u\|_{L^2}\big) = 2^{-\varepsilon\lambda}\|\nabla_h\nabla d\|_{\dot B_{\infty,\infty}^{-1+\varepsilon}}\big(\|\nabla^2 d\|_{L^2} + \|\nabla u\|_{L^2}\big),$$

thus we have

$$C2^{-\varepsilon\lambda}\|\nabla_h\nabla d\|_{\dot B_{\infty,\infty}^{-1+\varepsilon}}\big(\|\nabla^2 d\|_{L^2} + \|\nabla u\|_{L^2}\big) \le C\|\nabla_h d\|_{L^2}^{\frac{\varepsilon}{\frac{3}{2}+\varepsilon}}\|\nabla_h\nabla d\|_{\dot B_{\infty,\infty}^{-1+\varepsilon}}^{\frac{\frac{3}{2}}{\frac{3}{2}+\varepsilon}}\big(\|\nabla^2 d\|_{L^2} + \|\nabla u\|_{L^2}\big).$$

We choose μ such that

$$2^{\frac{3}{2}\mu}\|u_3\|_{L^2}\|\Delta d\|_{L^2} = 2^{-\varepsilon\mu}\|\nabla u_3\|_{\dot B_{\infty,\infty}^{-1+\varepsilon}}\|\Delta d\|_{L^2},$$

then we have

$$C2^{-\varepsilon\mu}\|\nabla u_3\|_{\dot B_{\infty,\infty}^{-1+\varepsilon}}\|\Delta d\|_{L^2} \le C\|u_3\|_{L^2}^{\frac{\varepsilon}{\frac{3}{2}+\varepsilon}}\|\nabla u_3\|_{\dot B_{\infty,\infty}^{-1+\varepsilon}}^{\frac{\frac{3}{2}}{\frac{3}{2}+\varepsilon}}\|\Delta d\|_{L^2}.$$

Integrating in time, combing above inequalities and (3.31), then (3.38) becomes

$$\begin{aligned}
&\|\nabla u\|_{L^2}^2+\|\Delta d\|_{L^2}^2+\int_0^t\|\Delta u\|_{L^2}^2+\|\nabla\Delta d\|_{L^2}^2\mathrm{d}\tau\\
\leq& C+\Big(\sup_{0\leq s\leq t}\|\nabla_h u\|_{L^2}\Big)\Big(\int_0^t\|\nabla u\|_{L^2}^2\mathrm{d}\tau\Big)^{\frac14}\times\Big(\int_0^t\|\nabla_h\nabla u\|_{L^2}^2\mathrm{d}\tau\Big)^{\frac12}\Big(\int_0^t\|\Delta u\|_{L^2}^2\mathrm{d}\tau\Big)^{\frac14}\\
&\quad+C\int_0^t\|\nabla_h\nabla d\|_{\dot{B}^{-1+\varepsilon}_{\infty,\infty}}^{\frac{3}{\frac32+\varepsilon}}\big(\|\nabla^2 d\|_{L^2}^2+\|\nabla u\|_{L^2}^2\big)\mathrm{d}\tau+C\int_0^t\|\nabla u_3\|_{\dot{B}^{-1+\varepsilon}_{\infty,\infty}}^{\frac{3}{\frac32+\varepsilon}}\|\Delta d\|_{L^2}^2\mathrm{d}\tau\\
\leq& C+C\Bigg\{\int_0^t\|\nabla u_3\|_{\dot{B}^{-1+\varepsilon}_{\infty,\infty}}^{\frac{3}{\frac32+\varepsilon}}\big(\|\nabla u\|_{L^2}^2+\|\Delta d\|_{L^2}^2\big)\mathrm{d}\tau\\
&\quad+\int_0^t\|\nabla_h\nabla d\|_{\dot{B}^{-1+\varepsilon}_{\infty,\infty}}^{\frac{3}{\frac32+\varepsilon}}\big(\|\nabla u\|_{L^2}^2+\|\Delta d\|_{L^2}^2\big)\mathrm{d}\tau\Bigg\}\Big(\int_0^t\|\Delta u\|_{L^2}^2\mathrm{d}\tau\Big)^{\frac14}\\
&\quad+C\int_0^t\|\nabla_h\nabla d\|_{\dot{B}^{-1+\varepsilon}_{\infty,\infty}}^{\frac{3}{\frac32+\varepsilon}}\big(\|\nabla^2 d\|_{L^2}^2+\|\nabla u\|_{L^2}^2\big)\mathrm{d}\tau+C\int_0^t\|\nabla u_3\|_{\dot{B}^{-1+\varepsilon}_{\infty,\infty}}^{\frac{3}{\frac32+\varepsilon}}\|\Delta d\|_{L^2}^2\mathrm{d}\tau. \qquad (3.39)
\end{aligned}$$

By the Hölder's and Young's inequalities, it follows that

$$\begin{aligned}
&\|\nabla u\|_{L^2}^2+\|\Delta d\|_{L^2}^2+\int_0^t\|\Delta u\|_{L^2}^2+\|\nabla\Delta d\|_{L^2}^2\mathrm{d}\tau\\
\leq& C+C\Bigg\{\int_0^t\|\nabla u_3\|_{\dot{B}^{-1+\varepsilon}_{\infty,\infty}}^{\frac{3}{\frac32+\varepsilon}}\big(\|\nabla u\|_{L^2}^2+\|\Delta d\|_{L^2}^2\big)\mathrm{d}\tau\Bigg\}^{\frac43}\\
&\quad+C\Bigg\{\int_0^t\|\nabla_h\nabla d\|_{\dot{B}^{-1+\varepsilon}_{\infty,\infty}}^{\frac{3}{\frac32+\varepsilon}}\big(\|\nabla u\|_{L^2}^2+\|\Delta d\|_{L^2}^2\big)\mathrm{d}\tau\Bigg\}^{\frac43}\\
&\quad+C\int_0^t\|\nabla_h\nabla d\|_{\dot{B}^{-1+\varepsilon}_{\infty,\infty}}^{\frac{3}{\frac32+\varepsilon}}\big(\|\nabla^2 d\|_{L^2}^2+\|\nabla u\|_{L^2}^2\big)\mathrm{d}\tau+C\int_0^t\|\nabla u_3\|_{\dot{B}^{-1+\varepsilon}_{\infty,\infty}}^{\frac{3}{\frac32+\varepsilon}}\|\Delta d\|_{L^2}^2\mathrm{d}\tau\\
&\quad+\frac14\int_0^t\|\Delta u\|_{L^2}^2\mathrm{d}\tau\\
\leq& C+C\int_0^t\Bigg(\|\nabla u_3\|_{\dot{B}^{-1+\varepsilon}_{\infty,\infty}}^{\frac{4}{\frac32+\varepsilon}}+\|\nabla_h\nabla d\|_{\dot{B}^{-1+\varepsilon}_{\infty,\infty}}^{\frac{4}{\frac32+\varepsilon}}\Bigg)\big(\|\nabla u\|_{L^2}^2+\|\Delta d\|_{L^2}^2\big)\mathrm{d}\tau\\
&\quad+C\int_0^t\|\nabla_h\nabla d\|_{\dot{B}^{-1+\varepsilon}_{\infty,\infty}}^{\frac{3}{\frac32+\varepsilon}}\big(\|\nabla^2 d\|_{L^2}^2+\|\nabla u\|_{L^2}^2\big)\mathrm{d}\tau+C\int_0^t\|\nabla u_3\|_{\dot{B}^{-1+\varepsilon}_{\infty,\infty}}^{\frac{3}{\frac32+\varepsilon}}\|\Delta d\|_{L^2}^2\mathrm{d}\tau. \qquad (3.40)
\end{aligned}$$

Thanks again to the energy inequality, we get

$$\begin{aligned}
&\|\nabla u\|_{L^2}^2+\|\Delta d\|_{L^2}^2+\int_0^t\|\Delta u\|_{L^2}^2+\|\nabla\Delta d\|_{L^2}^2\mathrm{d}\tau\\
\leq& C+\int_0^t\Bigg(\|\nabla u_3\|_{\dot{B}^{-1+\varepsilon}_{\infty,\infty}}^{\frac{4}{\frac32+\varepsilon}}+\|\nabla_h\nabla d\|_{\dot{B}^{-1+\varepsilon}_{\infty,\infty}}^{\frac{4}{\frac32+\varepsilon}}\Bigg)\big(\|\nabla u\|_{L^2}^2+\|\Delta d\|_{L^2}^2\big)\mathrm{d}\tau. \qquad (3.41)
\end{aligned}$$

If we set $s=1-\varepsilon$, then we have

$$\frac{4}{\frac{3}{2}+\varepsilon}=\frac{8}{5-2s}, \quad \text{with } 0<s<1.$$

Therefore, by Gronwall's inequality, we finally obtain

$$\|\nabla u\|_{L^2}^2+\|\Delta d\|_{L^2}^2 \leq C\exp\left(C\int_0^t \|\nabla u_3\|_{\dot{B}_{\infty,\infty}^{-s}}^{\frac{8}{5-2s}}+\|\nabla_h \nabla d\|_{\dot{B}_{\infty,\infty}^{-s}}^{\frac{8}{5-2s}} \mathrm{d}\tau\right), \tag{3.42}$$

by condition (1.5), then we completes the proof of (ii). □

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