JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **39**, No. 1, pp. 107-125

doi: 10.4208/jpde.v39.n1.6
March 2026

Free Transport Equation and Hyperbolic Schrödinger Equation via Wigner Transformation

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Received 9 December 2023 ; Accepted 9 October 2024

Abstract. In this work, we study the regularity of the Cauchy problem for the free transport equation, and by using the inverse Wigner transformation, we reduce this problem to the Cauchy problem of a class of linear homogeneous hyperbolic Schrödinger equation. We prove firstly the analytical smoothing effect of Cauchy problem for Schrödinger type equation if the initial datum is exponential decay. Finally we prove the directional propagation of the exponential decay and also analytic regularity for free transport equation.

AMS Subject Classifications: 32D15, 35Q99

Chinese Library Classifications: O175.27

Key Words: Free ransport equation; hyperbolic Schrödinger equation; analytical smoothing effect; exponential decay.

1 Introduction and main result

We consider the following Cauchy problem for free transport equation

$$\begin{cases} (\partial_t + v\cdot\nabla_x)f = 0, \\ f|_{t=0} = f_0 \in L^2\left(\mathbb{R}^{2n}_{x,v}\right). \end{cases} \tag{1.1}$$

It is well known that the transport equation transports the singularity and regularity of the initial datum which follows the characteristic $x\pm tv$. In this work, we prove the following results.

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Theorem 1.1. *Assume that* $e^{2|x|^2-\frac{1}{2}\Delta_v}f_0\in L^2(\mathbb{R}^{2n}_{x,v})$, *then the Cauchy problem* (1.1) *admits a unique solution* $f(t,x,v)$, *which satisfies: For any* $0<T$, *there exists* $A>0$, *such that for any* $\alpha\in\mathbb{N}^n$,

$$\|(x-tv)^\alpha f(t)\|^2_{L^2(\mathbb{R}^{2n}_{x,v})}\le A^{|\alpha|+1}\alpha!\,,$$
$$\|(\partial_v+t\partial_x)^\alpha f(t)\|^2_{L^2(\mathbb{R}^{2n}_{x,v})}\le A^{|\alpha|+1}\alpha!\,.$$

This implies that the solution of the Cauchy problem (1.1) is exponential decay in the direction $x-tv$, and analytic in the direction $\partial_v+t\partial_x$. Our motivation is to study the Cauchy problem of spatially inhomogeneous kinetic equation:

$$\begin{cases}(\partial_t+v\cdot\nabla_x)f=Q(f,\,f),\\ f|_{t=0}=f_0,\end{cases}$$

where the collision operator $Q(f,\,f)$ is Boltzmann operator or Landau operator ([1,2]). For the spatially inhomogeneous problem, the kinetic derivation $(\partial_t+v\cdot\nabla_x)$ is the main difficulty for the analysis of above Cauchy problem, since the nonlinear terms is difficult to study by using characteristic method, so in this work, we will transform the free transport equation to a hyperbolic type Schrödinger equation. For this way, we define the inverse Wigner transformation as following

$$u(x,y)=\mathcal{W}^{-1}(f)(x,y)=\int_{\mathbb{R}^n}f\left(\frac{x+y}{2},v\right)e^{iv\,\cdot\,(x-y)}\mathrm{d}v,$$

and the usual Wigner transformation

$$f(x,v)=\mathcal{W}(u)(x,v)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}u\left(x+\frac{x'}{2},x-\frac{x'}{2}\right)e^{-iv\,\cdot\,x'}\mathrm{d}x'.$$

Then the Cauchy problem (1.1) reduces to the following Cauchy problem

$$\begin{cases}i\partial_t u(t,x,y)+\dfrac{1}{2}\Delta_x u(t,x,y)-\dfrac{1}{2}\Delta_y u(t,x,y)=0,\\ u|_{t=0}=u_0(x,y).\end{cases}\tag{1.2}$$

This is a special case of the following Schrödinger equation,

$$\begin{cases}i\partial_t u(t,x,y)+\delta\dfrac{1}{2}\Delta_x u(t,x,y)+\delta'\dfrac{1}{2}\Delta_y u(t,x,y)=\lambda F(x,u,\partial u),\\ u(0,x,y)=\varphi(x,y).\end{cases}\tag{1.3}$$

When $\delta=\delta'=1$, Eq. (1.3) reduces to the following type nonlinear Schrödinger equations

$$\begin{cases}i\partial_t u(t,x)+\dfrac{1}{2}\Delta_x u(t,x)=\lambda F(x,u,\partial u), & \text{in } (\mathbb{R}\times\mathbb{R}^n),\\ u(0,x)=\varphi(x), & \text{in } \mathbb{R}^n.\end{cases}\tag{1.4}$$