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Free Transport Equation and Hyperbolic Schrödinger Equation via Wigner Transformation

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Abstract. In this work, we study the regularity of the Cauchy problem for the free transport equation, and by using the inverse Wigner transformation, we reduce this problem to the Cauchy problem of a class of linear homogeneous hyperbolic Schrödinger equation. We prove firstly the analytical smoothing effect of Cauchy problem for Schrödinger type equation if the initial datum is exponential decay. Finally we prove the directional propagation of the exponential decay and also analytic regularity for free transport equation.

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Key Words: Free ransport equation; hyperbolic Schrödinger equation; analytical smoothing effect; exponential decay.

1 Introduction and main result

We consider the following Cauchy problem for free transport equation

$$\begin{cases}(\partial_t+v\cdot\nabla_x)f=0,\\ f|_{t=0}=f_0\in L^2\big(\mathbb{R}^{2n}_{x,v}\big).\end{cases}\tag{1.1}$$

It is well known that the transport equation transports the singularity and regularity of the initial datum which follows the characteristic $x\pm tv$. In this work, we prove the following results.

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Theorem 1.1. *Assume that* $e^{2|x|^2-\frac{1}{2}\Delta_v}f_0\in L^2\big(\mathbb{R}^{2n}_{x,v}\big)$*, then the Cauchy problem* (1.1) *admits a unique solution* $f(t,x,v)$*, which satisfies: For any* $0<T$*, there exists* $A>0$*, such that for any* $\alpha\in\mathbb{N}^n$*,*

$$\|(x-tv)^\alpha f(t)\|^2_{L^2(\mathbb{R}^{2n}_{x,v})}\le A^{|\alpha|+1}\alpha!\,,$$
$$\|(\partial_v+t\partial_x)^\alpha f(t)\|^2_{L^2(\mathbb{R}^{2n}_{x,v})}\le A^{|\alpha|+1}\alpha!\,.$$

This implies that the solution of the Cauchy problem (1.1) is exponential decay in the direction $x-tv$, and analytic in the direction $\partial_v+t\partial_x$. Our motivation is to study the Cauchy problem of spatially inhomogeneous kinetic equation:

$$\begin{cases}(\partial_t+v\cdot\nabla_x)f=Q(f,\,f),\\ f|_{t=0}=f_0,\end{cases}$$

where the collision operator $Q(f,\,f)$ is Boltzmann operator or Landau operator ([1,2]). For the spatially inhomogeneous problem, the kinetic derivation $(\partial_t+v\cdot\nabla_x)$ is the main difficulty for the analysis of above Cauchy problem, since the nonlinear terms is difficult to study by using characteristic method, so in this work, we will transform the free transport equation to a hyperbolic type Schrödinger equation. For this way, we define the inverse Wigner transformation as following

$$u(x,y)=\mathcal{W}^{-1}(f)(x,y)=\int_{\mathbb{R}^n}f\left(\frac{x+y}{2},v\right)e^{iv\,\cdot\,(x-y)}\mathrm{d}v,$$

and the usual Wigner transformation

$$f(x,v)=\mathcal{W}(u)(x,v)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}u\left(x+\frac{x'}{2},x-\frac{x'}{2}\right)e^{-iv\,\cdot\,x'}\mathrm{d}x'.$$

Then the Cauchy problem (1.1) reduces to the following Cauchy problem

$$\begin{cases}i\partial_t u(t,x,y)+\dfrac{1}{2}\Delta_x u(t,x,y)-\dfrac{1}{2}\Delta_y u(t,x,y)=0,\\ u|_{t=0}=u_0(x,y).\end{cases}\tag{1.2}$$

This is a special case of the following Schrödinger equation,

$$\begin{cases}i\partial_t u(t,x,y)+\delta\dfrac{1}{2}\Delta_x u(t,x,y)+\delta'\dfrac{1}{2}\Delta_y u(t,x,y)=\lambda F(x,u,\partial u),\\ u(0,x,y)=\varphi(x,y).\end{cases}\tag{1.3}$$

When $\delta=\delta'=1$, Eq. (1.3) reduces to the following type nonlinear Schrödinger equations

$$\begin{cases}i\partial_t u(t,x)+\dfrac{1}{2}\Delta_x u(t,x)=\lambda F(x,u,\partial u), & \text{in } (\mathbb{R}\times\mathbb{R}^n),\\ u(0,x)=\varphi(x), & \text{in } \mathbb{R}^n.\end{cases}\tag{1.4}$$

The Schrödinger equation arises in various physical applications, such as an axially nonuniform plasma channel, high power laser beams involved in laser plasma experiments and nonlinear optics ([3–5]). There are many papers on the Cauchy problem for nonlinear Schrödinger equations. Existence of a solution on the Cauchy problem for nonlinear Schrödinger equations has been investigated. We refer the reader to [6,7] and their references. H.Z. Cong, L.F. Mi and Y.F. Shi have investigated the long-time stability of solutions for nonlinear Schrödinger equation in [8]. Smoothing of a solution on the cauchy problem for Schrödinger equations has been obtained. We refer the reader to [9,10] and their references.

In recent decades, analyticity of solutions to the nonlinear Schrödinger equations (1.4) has been obtained by many researchers. We refer the reader to [11–21]. In particular, in [11], N. Hayashi and S. Saitoh gained two identities in a certain class of analytic functions and the analyticity of solutions of the equation for $t \neq 0$, when the initial functions decay exponentially as $|x| \to \infty$ and $n=1$ corresponding homogeneous Schrödinger equation in (1.4). Replacing $\lambda F(x,u,\partial u)$ with $|u|^2 u$ in (1.4), N. Hayashi and S. Saitoh gained existence and analyticity in space variables of the solutions for $t \in \mathbb{R} \backslash \{0\}$ in [12] if $\varphi(x) \in H^{2n+1,2}(\mathbb{R}_x^n)$ decay exponentially as $|x| \to \infty$ and $n \geq 2$. Replacing $\lambda F(x,u,\partial u)$ with $\lambda |u|^{2p} u$ in (1.4), $\lambda \in C$ and $p \in \mathbb{N}$, N. Hayashi and K. Kato investigated that there exists a unique solution $u(t,x)$ of equation and that $u(t,x)$ is analytic in time and space variables and has an analytic continuation when the initial functions φ satisfies $||e^{|x|^2}\varphi||_{H^{[\frac{n}{2}]+1}} < \infty$ in [13]. Replacing $\lambda F(x,u,\partial u)$ with $if(u,\partial u)$ in (1.4), H. Chihara proved that the solution of semilinear Schrödinger equations with gauge invariant nonlinearity becomes real-analytic in the space variable and a Gevrey function of order 2 in the time variable except in the initial plane when the initial data decays exponentially in [19]. Replacing $\lambda F(x,u,\partial u)$ with $|u|^2 u$ and $n=1$ in (1.4), A. Tesfahun gained that the uniform radius of spatial analyticity $\sigma(t)$ of solutions at time t to the cubic nonlinear Schrödinger equations on the circle cannot decay faster than $1/t$ as $t \to \infty$, when initial data is analytic with fixed radius σ_0 in [21].

In [22], Cao and Guo gained the existence and smoothness of solution for homogeneous Schrödinger equation, when $\delta = 2$ and $\delta' = -2$ to the problem (1.3). C.E.Kenig, G.Ponce and L.Vega obtained the existence and smoothness of solution for nonlinear Schrödinger equation in [10], when $\delta = 1$, $\delta' = -1$ for the problem (1.3).

Inspired [11,13,23], in this paper, we will study the analyticity of solutions for the problem (1.2), then apply it to study the solutions of free transport equation, and we prove Theorem 1.1, by using the following analytic smoothing effect of Schrödinger type equations which is independently interesting.

Theorem 1.2. *Assume that* $e^{|x|^2+|y|^2}u_0 \in L^2\left(\mathbb{R}^{2n}_{x,y}\right)$. *Then the Cauchy problem* (1.2) *admits a unique solution* $u(t,x,y)$ *which is analytic on* $\mathbb{R}^n_x \times \mathbb{R}^n_y$ *for* $t \neq 0$. *Moreover, it can be analytically* *extended to* $\mathbb{C}^{2n}$*, and we have that, for* $t\in\mathbb{R}\backslash\{0\}$*, the analytic continuation satisfy*

$$\begin{aligned}&\frac{1}{(2t^2\pi)^n}\int_{\mathbb{R}^{4n}} e^{-\frac{w^2+\theta^2}{2t^2}} e^{\frac{2(xw-y\theta)}{t}}|u(t,(x,y)+i(w,\theta))|^2\mathrm{d}x\mathrm{d}y\mathrm{d}w\mathrm{d}\theta\\ =&\int_{\mathbb{R}^{2n}} e^{2(|x|^2+|y|^2)}|u_0(x,y)|^2\mathrm{d}x\mathrm{d}y.\end{aligned}\tag{1.5}$$

Our results is different from [11], [12] and [13] in signal between δ with $\delta^{'}$, but we also obtain the analyticity of solutions and the corresponding identity for homogeneous hyperbolic Schrödinger equation when $\delta>0$ and $\delta^{'}<0$.

The article is organized as follows: In Section 2 we prove the existence and analyticity of solution for Cauchy problem (1.2). In Section 3 we prove Theorem 1.1. In Section 4, we extend the solution to the complex space and prove that it is holomorphic, and prove the identity (1.5).

2 Existence and analyticity of solution

We study firstly the existence and analyticity of solution for Cauchy problem (1.2). To simplify the notation, we take $n=2$.

Proposition 2.1. *Assume that* $e^{|x|^2+|y|^2}u_0\in L^2(\mathbb{R}^4)$*, then the Cauchy problem* (1.2) *admits a unique solution* $u(t,x,y)$ *which is analytic on* $\mathbb{R}^2_x\times\mathbb{R}^2_y$ *for* $t\neq 0$*.*

Proof. **Existence of smooth solution:** The Cauchy problem (1.2) admits a unique solution $u(t,x,y)$ and $u(t,x,y)\in C^\infty(\mathbb{R},C^\infty(\mathbb{R}^4_{xy}))$, when $e^{|x|^2+|y|^2}u_0\in L^2(\mathbb{R}^4)$.

Since $e^{|(x,y)|^2}u_0\in L^2(\mathbb{R}^4_{xy})$, we gain that

$$\sum_{n=0}^{\infty}\int_{\mathbb{R}^4}\frac{2^n|(x,y)|^{2n}}{n!}|u_0|^2\mathrm{d}x\mathrm{d}y=\int_{\mathbb{R}^4}e^{2|(x,y)|^2}|u_0|^2\mathrm{d}x\mathrm{d}y<+\infty,$$

therefore there exists $(c_n)\in\ell^1$, such that for any $n\in\mathbb{N}$,

$$\int_{\mathbb{R}^4}|(x,y)|^{2n}|u_0|^2\mathrm{d}x\mathrm{d}y\leq\frac{c_n n!}{2^n}.$$

Thus, one has, for any $n\in\mathbb{N}$,

$$|(x,y)|^n u_0\in L^2(\mathbb{R}^4).$$

By using the results of [22], the Cauchy problem (1.2) admits a unique solution $u(t,x,y)\in C^\infty(\mathbb{R}\backslash\{0\},C^\infty(\mathbb{R}^4_{xy}))$.

Analyticity of solution: we prove that the solution $u(t,x,y)$ to the Cauchy problem (1.2) is analytic on $\mathbb{R}^2_x\times\mathbb{R}^2_y$ for $t\neq 0$, when $e^{|x|^2+|y|^2}u_0\in L^2(\mathbb{R}^4)$. For this end, we prove now the following Proposition. □

Proposition 2.2. *Assume that $u(t,x,y)$ is a solution of Eq.* (1.2) *and $e^{|x|^2+|y|^2}u_0\in L^2(\mathbb{R}^4)$, then it holds that, for $t\neq 0$,*

$$\sum_{|\alpha|+|\beta|=0}^{\infty}\frac{(2t^2)^{(|\alpha|+|\beta|)}}{\alpha!\beta!}\int_{\mathbb{R}^4}\left|\partial_{xy}^{(\alpha,\beta)}\left(e^{-i\frac{|x|^2-|y|^2}{2t}}u(t,x,y)\right)\right|^2\mathrm{d}x\mathrm{d}y$$
$$=\int_{\mathbb{R}^4}e^{2(|x|^2+|y|^2)}|u_0(x,y)|^2\mathrm{d}x\mathrm{d}y.$$

Proof. We define the operator $P^\alpha P_\beta$ by

$$P^\alpha P_\beta u=(x+it\partial_x)^\alpha(-y+it\partial_y)^\beta u, \tag{2.1}$$

we have then

$$P^\alpha P_\beta u(t,x,y)=e^{i\frac{|x|^2-|y|^2}{2t}}(it)^{|\alpha|+|\beta|}\partial_{xy}^{(\alpha,\beta)}\left(e^{-i\frac{|x|^2-|y|^2}{2t}}u(t,x,y)\right). \tag{2.2}$$

Letting

$$L=i\partial_t+\frac{1}{2}\Delta_x-\frac{1}{2}\Delta_y,$$

we can easily gain

$$[L,P^\alpha P_\beta]=LP^\alpha P_\beta-P^\alpha P_\beta L=0. \tag{2.3}$$

Since u is a solution of the problem (1.2), by using (2.1) and (2.3), we obtain that $P^\alpha P_\beta u$ is also a solution of the following Cauchy problem

$$\begin{cases} i\partial_t\left(P^\alpha P_\beta u\right)+\dfrac{1}{2}\Delta_x\left(P^\alpha P_\beta u\right)-\frac{1}{2}\Delta_y\left(P^\alpha P_\beta u\right)=0,\\ \left(P^\alpha P_\beta u\right)\Big|_{t=0}=x^\alpha(-y)^\beta u_0(x,y). \end{cases} \tag{2.4}$$

Integrating both sides of (2.4) multiplied by $\overline{P^\alpha P_\beta u}$, we have

$$\begin{aligned}\int_{\mathbb{R}^4}\Big(&i\partial_t(P^\alpha P_\beta u(t,x,y))\cdot\overline{P^\alpha P_\beta u(t,x,y)}+\Big(\frac{1}{2}\Delta_x(P^\alpha P_\beta u(t,x,y)\Big)\\ &-\frac{1}{2}\Delta_y(P^\alpha P_\beta u(t,x,y)))\cdot\overline{P^\alpha P_\beta u(t,x,y)}\Big)\mathrm{d}x\mathrm{d}y=0.\end{aligned} \tag{2.5}$$

Taking the imaginary part in (2.5), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|P^\alpha P_\beta u(t)\|_{L^2}^2=0,$$

which implies

$$\|P^\alpha P_\beta u(t)\|_{L^2}^2=\|P^\alpha P_\beta u\|_{L^2}^2\Big|_{t=0},\quad \forall t\in\mathbb{R}. \tag{2.6}$$

Owing to (2.2) and (2.6), one has

$$\begin{aligned}
&\|P^\alpha P_\beta u(t)\|_{L^2}^2=\int_{\mathbb{R}^4}|it|^{2(|\alpha|+|\beta|)}\left|\partial_{xy}^{(\alpha,\beta)}\left(e^{-i\frac{|x|^2-|y|^2}{2t}}u(t,x,y)\right)\right|^2\mathrm{d}x\mathrm{d}y\\
=&|t|^{2(|\alpha|+|\beta|)}\int_{\mathbb{R}^4}\left|\partial_{xy}^{(\alpha,\beta)}\left(e^{-i\frac{|x|^2-|y|^2}{2t}}u(t,x,y)\right)\right|^2\mathrm{d}x\mathrm{d}y=\|P^\alpha P_\beta u\|_{L^2}^2\Big|_{t=0}\\
=&\int_{\mathbb{R}^4}x^{2\alpha}y^{2\beta}|u_0(x,y)|^2\mathrm{d}x\mathrm{d}y.
\end{aligned}\tag{2.7}$$

Multiplying both sides of (2.7) by $\frac{2^{|\alpha|}2^{|\beta|}}{\alpha!\beta!}$, we have

$$\begin{aligned}
&\frac{2^{|\alpha|}2^{|\beta|}|t|^{2(|\alpha|+|\beta|)}}{\alpha!\beta!}\int_{\mathbb{R}^4}\left|\partial_{xy}^{(\alpha,\beta)}\left(e^{-i\frac{|x|^2-|y|^2}{2t}}u(t,x,y)\right)\right|^2\mathrm{d}x\mathrm{d}y\\
=&\int_{\mathbb{R}^4}\frac{(2x_1^2)^{\alpha_1}}{\alpha_1!}\frac{(2x_2^2)^{\alpha_2}}{\alpha_2!}\frac{(2y_1^2)^{\beta_1}}{\beta_1!}\frac{(2y_2^2)^{\beta_2}}{\beta_2!}|u_0|^2(x,y)\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

According to the assumption $e^{|x|^2+|y|^2}u_0\in L^2(\mathbb{R}^4)$, one has

$$\begin{aligned}
&\sum_{|\alpha|+|\beta|=0}^{\infty}\frac{(2t^2)^{(|\alpha|+|\beta|)}}{\alpha!\beta!}\int_{\mathbb{R}^4}\left|\partial_{xy}^{(\alpha,\beta)}\left(e^{-i\frac{|x|^2-|y|^2}{2t}}u(t,x,y)\right)\right|^2\mathrm{d}x\mathrm{d}y\\
=&\sum_{|\alpha|+|\beta|=0}^{\infty}\int_{\mathbb{R}^4}\frac{(2x_1^2)^{\alpha_1}}{\alpha_1!}\frac{(2x_2^2)^{\alpha_2}}{\alpha_2!}\frac{(2y_1^2)^{\beta_1}}{\beta_1!}\frac{(2y_2^2)^{\beta_2}}{\beta_2!}|u_0(x,y)|^2\mathrm{d}x\mathrm{d}y\\
=&\int_{\mathbb{R}^4}\sum_{|\alpha|+|\beta|=0}^{\infty}\frac{(2x_1^2)^{\alpha_1}}{\alpha_1!}\frac{(2x_2^2)^{\alpha_2}}{\alpha_2!}\frac{(2y_1^2)^{\beta_1}}{\beta_1!}\frac{(2y_2^2)^{\beta_2}}{\beta_2!}|\varphi(x,y)|^2\mathrm{d}x\mathrm{d}y\\
=&\int_{\mathbb{R}^4}e^{2(|x|^2+|y|^2)}|u_0(x,y)|^2\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

This completes the proof of Proposition 2.2. □

End of proof of Proposition 2.1

Since $e^{|x|^2+|y|^2}u_0(x,y)\in L^2\left(\mathbb{R}^4_{xy}\right)$, there exists a constant $M>0$ such that

$$\int_{\mathbb{R}^4}e^{2(|x|^2+|y|^2)}|u_0|^2(x,y)\mathrm{d}x\mathrm{d}y\le M.$$

Using Proposition 2.2, we have that

$$\sum_{|\alpha|+|\beta|=0}^{\infty}\frac{(2t^2)^{(|\alpha|+|\beta|)}}{\alpha!\beta!}\int_{\mathbb{R}^4}\left|\partial_{xy}^{(\alpha,\beta)}\left(e^{-i\frac{|x|^2-|y|^2}{2t}}u(t,x,y)\right)\right|^2\mathrm{d}x\mathrm{d}y<M.$$

Therefore, there exists $(c_{\alpha,\beta})\in\ell^1$, such that

$$\sum_{|\alpha|+|\beta|=0}^{\infty} c_{\alpha,\beta}\le M,$$

and for $\alpha,\beta\in\mathbb{N}^2$, it holds that

$$\int_{\mathbb{R}^4}\left|\partial_{xy}^{(\alpha,\beta)}\left(e^{-i\frac{|x|^2-|y|^2}{2t}}u(t,x,y)\right)\right|^2 \mathrm{d}x\mathrm{d}y<\frac{c_{\alpha,\beta}\alpha!\beta!}{(2t^2)^{(|\alpha|+|\beta|)}}. \tag{2.8}$$

Using (2.2) and (2.8), we have

$$\|P^\alpha P_\beta u(t)\|_{L^2}^2\le\frac{c_{\alpha,\beta}\alpha!\beta!}{2^{|\alpha|+|\beta|}}. \tag{2.9}$$

Owing to (2.8), we get

$$\left\|\partial_x^\alpha\partial_y^\beta\left(e^{-i\frac{|x|^2-|y|^2}{2t}}u\right)\right\|_{L^2}\le\frac{c_{\alpha,\beta}^{\frac{1}{2}}(\alpha!\beta!)^{\frac{1}{2}}}{(\sqrt{2}t)^{|\alpha|+|\beta|}},\qquad \forall t\neq 0. \tag{2.10}$$

On the other hand, $e^{i\frac{|x|^2-|y|^2}{2t}}$ is analytic on $\mathbb{R}_x^2\times\mathbb{R}_y^2$ for $t\neq 0$, then

$$u(t,x,y)=e^{i\frac{|x|^2-|y|^2}{2t}}\left(e^{-i\frac{|x|^2-|y|^2}{2t}}u(t,x,y)\right)$$

is analytical on $\mathbb{R}_x^2\times\mathbb{R}_y^2$ for $t\neq 0$. We end the proof of Proposition 2.1.

3 Free transport equation

We study now the Cauchy problem of the free transport equation (1.1).

If $u\in L^2(\mathbb{R}^{2n})$, we define the partial Fourier transform $\mathcal{F}_2$ of u with respect to the second variable as $\mathcal{F}_2u(x,\xi)=\int_{\mathbb{R}^n}u(x,y)e^{-i\xi y}\mathrm{d}y$ and the partial inverse Fourier transform $\mathcal{F}_2$ of u with respect to the second variable as $\mathcal{F}_2^{-1}u(x,\xi)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}u(x,y)e^{i\xi y}\mathrm{d}y$.

Lemma 3.1. *We have*

$$\mathcal{W}\circ\mathcal{W}^{-1}=\mathbf{Id}\ \text{ on }\ L^2(\mathbb{R}^{2n}_{x,v}),\quad \mathcal{W}^{-1}\circ\mathcal{W}=\mathbf{Id}\ \text{ on }\ L^2(\mathbb{R}^{2n}_{x,y}).$$

Proof. For any $f\in L^2(\mathbb{R}^{2n}_{x,v})$, by the inverse Wigner transformation and Fourier transformation, we have

$$\mathcal{W}\circ\mathcal{W}^{-1}(f)(x,v)=\mathcal{W}\left(\int_{\mathbb{R}^n}f\left(\frac{x+y}{2},\xi\right)e^{i\xi(x-y)}\mathrm{d}\xi\right)(x,v)$$

$$
\begin{aligned}
&=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} f\left(\frac{x+\frac{x'}{2}+x-\frac{x'}{2}}{2},\xi\right)e^{i\xi(x+\frac{x'}{2}-(x-\frac{x'}{2}))}\mathrm{d}\xi\right)e^{-ivx'}\mathrm{d}x'\\
&=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n} f(x,\xi)e^{i\xi x'}\mathrm{d}\xi\Big)e^{-ivx'}\mathrm{d}x'\\
&=\int_{\mathbb{R}^n}\mathcal{F}_2^{-1}f(x,x')e^{-ivx'}\mathrm{d}x'=f(x,v).
\end{aligned}
$$

So we have $\mathcal{W}\circ\mathcal{W}^{-1}=\mathbf{Id}$ on $L^2\big(\mathbb{R}^{2n}_{x,v}\big)$.

For any $u\in L^2\big(\mathbb{R}^{2n}_{x,y}\big)$, by the inverse Wigner transformation and Fourier transformation, we have

$$
\begin{aligned}
\mathcal{W}^{-1}\circ\mathcal{W}(u)(x,y)&=\mathcal{W}^{-1}\left(\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}u\left(x+\frac{x'}{2},x-\frac{x'}{2}\right)e^{-ivx'}\mathrm{d}x'\right)\\
&=\int_{\mathbb{R}^n}\left(\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}u\left(\frac{x+y}{2}+\frac{x'}{2},\frac{x+y}{2}-\frac{x'}{2}\right)e^{-ivx'}\mathrm{d}x'\right)e^{iv(x-y)}\mathrm{d}v\\
&=\int_{\mathbb{R}^n}\left(\frac{1}{(2\pi)^n}\mathcal{F}_2u\Big(\frac{x+y}{2}+\frac{v}{2},\frac{x+y}{2}-\frac{v}{2}\Big)\right)e^{iv(x-y)}\mathrm{d}v\\
&=u\left(\frac{x+y}{2}+\frac{x-y}{2},\frac{x+y}{2}-\frac{x-y}{2}\right)=u(x,y).
\end{aligned}
$$

So we have $\mathcal{W}^{-1}\circ\mathcal{W}=\mathbf{Id}$ on $L^2\big(\mathbb{R}^{2n}_{x,y}\big)$. This completes the proof of Lemma 3.1. $\square$

Lemma 3.2. *We have, for any $u\in L^2(\mathbb{R}^{2n}_{xv})$,*

$$
\|\mathcal{W}(u)\|_{L^2(\mathbb{R}^{2n}_{xv})}=\frac{1}{(2\pi)^{\frac{n}{2}}}\|u\|_{L^2(\mathbb{R}^{2n}_{st})}.
$$

Proof. Assuming $\mathcal{W}(u),u_1\in\mathcal{S}\big(\mathbb{R}^{2n}_{xv}\big)$, we have

$$
\begin{aligned}
\int_{\mathbb{R}^{2n}_{xv}}\mathcal{W}(u)\overline{u_1(x,v)}\mathrm{d}x\mathrm{d}v&=\int_{\mathbb{R}^{2n}_{xv}}\left[\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n_{x'}}u\left(x+\frac{x'}{2},x-\frac{x'}{2}\right)e^{-ivx'}\mathrm{d}x'\overline{u_1(x,v)}\right]\mathrm{d}x\mathrm{d}v\\
&=\int_{\mathbb{R}^{2n}_{xv}}\int_{\mathbb{R}^n_{x'}}\frac{1}{(2\pi)^n}u\left(x+\frac{x'}{2},x-\frac{x'}{2}\right)e^{-ivx'}\overline{u_1(x,v)}\mathrm{d}x'\mathrm{d}x\mathrm{d}v.
\end{aligned}
$$

Let $x+\frac{x'}{2}=t$, $x-\frac{x'}{2}=s$, then $x=\frac{t+s}{2}$, $x'=t-s$ and $\frac{\partial(x',x)}{\partial(t,s)}=1$. By the above formula, we have that

$$
\begin{aligned}
&\int_{\mathbb{R}^{2n}_{xv}}\mathcal{W}(u)\overline{u_1(x,v)}\mathrm{d}x\mathrm{d}v\\
=&\int_{\mathbb{R}^{2n}_{st}}\int_{\mathbb{R}^n_v}\frac{1}{(2\pi)^n}u(t,s)e^{-iv(t-s)}\overline{u_1\Big(\frac{t+s}{2},v\Big)}\mathrm{d}s\mathrm{d}t\mathrm{d}v
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^{2n}_{st}} u(t,s)\mathrm{d}s\mathrm{d}t \int_{\mathbb{R}^n_v} \frac{1}{(2\pi)^n} e^{-iv(t-s)} \overline{u_1\Big(\frac{t+s}{2},v\Big)}\mathrm{d}v \\
&= \int_{\mathbb{R}^{2n}_{st}} u(t,s)\mathrm{d}s\mathrm{d}t \frac{1}{(2\pi)^n} \overline{\int_{\mathbb{R}^n_v} u_1\Big(\frac{t+s}{2},v\Big) e^{iv(t-s)}\mathrm{d}v} \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}_{st}} u(t,s)\overline{\mathcal{W}^{-1}(u_1)(t,s)}\mathrm{d}s\mathrm{d}t.
\end{aligned}
$$

Let's use $\mathcal{W}(u_1)$ instead of u_1 and Lemma 3.1, then it follows that

$$
\begin{aligned}
\int_{\mathbb{R}^{2n}_{xv}} \mathcal{W}(u)\overline{\mathcal{W}(u_1)(x,v)}\mathrm{d}x\mathrm{d}v &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}_{st}} u(t,s)\overline{\mathcal{W}^{-1}(\mathcal{W}(u_1))(t,s)}\mathrm{d}s\mathrm{d}t \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}_{st}} u(t,s)\overline{u_1(t,s)}\mathrm{d}s\mathrm{d}t.
\end{aligned}
$$

When $u = u_1$, it holds that

$$
\|\mathcal{W}(u)\|_{L^2} = \frac{1}{(2\pi)^{\frac{n}{2}}}\|u\|_{L^2}.
$$

Since $\mathcal{S}$ space is dense in L^2, we have finished the proof of Lemma 3.2. $\square$

Lemma 3.3. *The function $f(t,x.v)$ is a weak solution of Cauchy problem* (1.1) *if and only if their inverse Wigner transformation is a weak solution of the following Cauchy problem:*

$$
\begin{cases}
\Big(i\partial_t + \frac{1}{2}(\triangle_x - \triangle_y)\Big)u(t,x,y) = 0, \\
u|_{t=0} = u_0 = \mathcal{W}^{-1}(f_0)(x,y).
\end{cases}
$$

Where the definition of weak solution is in the usual distribution sense.

For the proof, we can refer [23, lemma 2.1], and we omit it here.

Lemma 3.4. *We have that $e^{2|x|^2 - \frac{1}{2}\Delta_v} f_0(x,v) \in L^2(\mathbb{R}^{2n}_{x,v})$ if and only if $e^{|x|^2+|y|^2}u_0(x,y) \in L^2(\mathbb{R}^{2n}_{x,y})$, where $f_0 = \mathcal{W}(u_0)$.*

Proof. By the inverse Wigner transformation, we have, for $k \in \mathbb{N}$,

$$
\begin{aligned}
&\mathcal{W}\Big(\big(|x|^2+|y|^2\big)^k u_0\Big)(x,v) \\
=& \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \bigg(\Big|x+\frac{x'}{2}\Big|^2 + \Big|x-\frac{x'}{2}\Big|^2\bigg)^k u_0\Big(x+\frac{x'}{2}, x-\frac{x'}{2}\Big) e^{-ivx'}\mathrm{d}x' \\
=& \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \bigg(2|x|^2 + \frac{|x'|^2}{2}\bigg)^k u_0\Big(x+\frac{x'}{2}, x-\frac{x'}{2}\Big) e^{-ivx'}\mathrm{d}x'
\end{aligned}
$$

$$
\begin{aligned}
&=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\sum_{j=0}^{k}C_k^j\left(\frac{|x'|^2}{2}\right)^j(2|x|^2)^{k-j}u_0\left(x+\frac{x'}{2},x-\frac{x'}{2}\right)e^{-ivx'}\mathrm{d}x'\\
&=\frac{1}{(2\pi)^n}\sum_{j=0}^{k}C_k^j(2|x|^2)^{k-j}\frac{1}{2^j}\int_{\mathbb{R}^n}(|x'|^2)^j u_0\left(x+\frac{x'}{2},x-\frac{x'}{2}\right)e^{-ivx'}\mathrm{d}x'\\
&=\frac{1}{(2\pi)^n}\sum_{j=0}^{k}C_k^j(2|x|^2)^{k-j}\frac{1}{2^j}\left(\sum_{|\alpha|=j}\int_{\mathbb{R}^n}(-1)^{|\alpha|}(-ix')^\alpha(-ix')^\alpha u_0\left(x+\frac{x'}{2},x-\frac{x'}{2}\right)e^{-ivx'}\mathrm{d}x'\right)\\
&=\frac{1}{(2\pi)^n}\sum_{j=0}^{k}C_k^j(2|x|^2)^{k-j}\frac{1}{2^j}\left(\sum_{|\alpha|=j}(-1)^j\frac{\partial^{2\alpha}}{\partial v^{2\alpha}}\int_{\mathbb{R}^n}u_0\left(x+\frac{x'}{2},x-\frac{x'}{2}\right)e^{-ivx'}\mathrm{d}x'\right)\\
&=\sum_{j=0}^{k}C_k^j(2|x|^2)^{k-j}\left(\sum_{|\alpha|=j}\left(\frac{-1}{2}\right)^j\frac{\partial^{2\alpha}}{\partial v^{2\alpha}}f_0(x,v)\right),
\end{aligned}
$$

then we have, for any $k\in\mathbb{N}$,

$$\mathcal{W}\Big((|x|^2+|y|^2)^k u_0\Big)(x,v)=\big(2|x|^2-\tfrac{1}{2}\Delta_v\big)^k f_0(x,v).$$

Proof of necessity : Since $e^{|x|^2+|y|^2}u_0(x,y)\in L^2$, we gain

$$
\begin{aligned}
\mathcal{W}\left(e^{|x|^2+|y|^2}u_0\right)(x,v)&=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}e^{|x+\frac{x'}{2}|+|x-\frac{x'}{2}|}u_0\left(x+\frac{x'}{2},x-\frac{x'}{2}\right)e^{-ivx'}\mathrm{d}x'\\
&=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\sum_{k=0}^{\infty}\frac{\left(|x+\frac{x'}{2}|+|x-\frac{x'}{2}|\right)^k}{k!}u_0\left(x+\frac{x'}{2},x-\frac{x'}{2}\right)e^{-ivx'}\mathrm{d}x'\\
&=\sum_{k=0}^{\infty}\frac{(2|x|^2-\frac{1}{2}\Delta_v)^k}{k!}\mathcal{W}(u_0)(x,v)=e^{2|x|^2-\frac{1}{2}\Delta_v}f_0(x,v).
\end{aligned}
$$

Using the Winger transformation from L^2 to L^2, we gain that $e^{2|x|^2-\frac{1}{2}\Delta_v}f_0(x,v)\in L^2$.
Proof of adequacy: Using Lemmas 3.1 and 3.2, it holds that both the Winger transformation and the inverse Winger transformation are one-to-one mappings from L^2 to L^2. Since $e^{2|x|^2-\frac{1}{2}\Delta_v}f_0(x,v)\in L^2(\mathbb{R}^{2n}_{xv})$, we get $\mathcal{W}^{-1}(e^{2|x|^2-\frac{1}{2}\Delta_v}f_0)(x,y)\in L^2(R^{2n}_{xy})$ and

$$\mathcal{W}^{-1}\left(e^{2|x|^2-\frac{1}{2}\Delta_v}f_0\right)(x,y)=e^{|x|^2+|y|^2}u_0(x,y),$$

then it follows that $e^{|x|^2+|y|^2}u_0(x,y)\in L^2(\mathbb{R}^{2n}_{xy})$. This completes the proof of Lemma 3.4. □

Proposition 3.1. *For any $k\in\mathbb{N}$,*

$$\mathcal{W}\left((P^{e_1}-P_{e_1})^k u\right)(x,v)=(2x_1-2tv_1)^k\mathcal{W}(u)(x,v), \tag{3.1}$$

$$\mathcal{W}\left((P^{e_1}+P_{e_1})^k u\right)(x,v)=(D_{v_1}+tD_{x_1})^k\mathcal{W}(u)(x,v). \tag{3.2}$$

Proof. Denote, for $\alpha,\beta\in\mathbb{N}^n$,

$$P^\alpha=(x+it\partial_x)^\alpha,\quad P_\beta=(-y+it\partial_y)^\beta.$$

For $\alpha=e_1,D_v=i\partial_v$, we have

$$\begin{aligned}\mathcal{W}(P^{e_1}u)(x,v)=&\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\left(x_1+\frac{x_1'}{2}\right)u\left(x+\frac{x'}{2},x-\frac{x'}{2}\right)e^{-iv\cdot x'}\mathrm{d}x'\\&+\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}itu_{x_1}\left(x+\frac{x'}{2},x-\frac{x'}{2}\right)e^{-iv\cdot x'}\mathrm{d}x'\\=&\left(x_1+\frac{D_{v_1}}{2}\right)\mathcal{W}(u)(x,v)+\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}itu_{x_1}\left(x+\frac{x'}{2},x-\frac{x'}{2}\right)e^{-iv\cdot x'}\mathrm{d}x',\end{aligned}$$

and for $\beta=e_1,D_v=i\partial_v$,

$$\begin{aligned}\mathcal{W}(P_{e_1}u)(x,v)=&-\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\left(x_1-\frac{x_1'}{2}\right)u\left(x+\frac{x'}{2},x-\frac{x'}{2}\right)e^{-iv\cdot x'}\mathrm{d}x'\\&+\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}itu_{y_1}\left(x+\frac{x'}{2},x-\frac{x'}{2}\right)e^{-iv\cdot x'}\mathrm{d}x'\\=&\left(\frac{D_{v_1}}{2}-x_1\right)\mathcal{W}(u)(x,v)+\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}itu_{y_1}\left(x+\frac{x'}{2},x-\frac{x'}{2}\right)e^{-iv\cdot x'}\mathrm{d}x'.\end{aligned}$$

We get then

$$\begin{aligned}&\mathcal{W}\big((P^{e_1}-P_{e_1})u\big)(x,v)\\=&2x_1\ \mathcal{W}(u)(x,v)+\frac{it}{(2\pi)^n}\int_{\mathbb{R}^n}\left(u_{x_1}\left(x+\frac{x'}{2},x-\frac{x'}{2}\right)-u_{y_1}\left(x+\frac{x'}{2},x-\frac{x'}{2}\right)\right)e^{-iv\cdot x'}\mathrm{d}x'\\=&2x_1\ \mathcal{W}(u)(x,v)+\frac{it}{(2\pi)^n}\int_{\mathbb{R}^n}2\partial_{x_1'}\left(u\left(x+\frac{x'}{2},x-\frac{x'}{2}\right)\right)e^{-iv\cdot x'}\mathrm{d}x'\\=&2x_1\ \mathcal{W}(u)(x,v)-\frac{2it}{(2\pi)^n}\int_{\mathbb{R}^n}u\left(x+\frac{x'}{2},x-\frac{x'}{2}\right)\partial_{x_1'}\left(e^{-iv\cdot x'}\right)\mathrm{d}x'\\=&(2x_1-2tv_1)\mathcal{W}(u)(x,v),\end{aligned}\tag{3.3}$$

and

$$\begin{aligned}&\mathcal{W}\big((P^{e_1}+P_{e_1})u\big)(x,v)\\=&D_{v_1}\ \mathcal{W}(u)(x,v)+\frac{it}{(2\pi)^n}\int_{\mathbb{R}^n}\left(u_{x_1}\left(x+\frac{x'}{2},x-\frac{x'}{2}\right)+u_{y_1}\left(x+\frac{x'}{2},x-\frac{x'}{2}\right)\right)e^{-iv\cdot x'}\mathrm{d}x'\\=&D_{v_1}\ \mathcal{W}(u)(x,v)+\frac{it}{(2\pi)^n}\int_{\mathbb{R}^n}\partial_{x_1}\left(u\left(x+\frac{x'}{2},x-\frac{x'}{2}\right)\right)e^{-iv\cdot x'}\mathrm{d}x'\\=&(D_{v_1}+tD_{x_1})\mathcal{W}(u)(x,v).\end{aligned}$$

We only prove $\mathcal{W}\big((P^{e_1}-P_{e_1})^k u\big)(x,v)=(2x_1-2tv_1)^k\mathcal{W}(u)(x,v)$. $\mathcal{W}\big((P^{e_1}+P_{e_1})^k u\big)(x,v)=(D_{v_1}+tD_{x_1})^k\mathcal{W}(u)(x,v)$ is proved similarly, and we omit it here.

We prove (3.1) by induction with respect to k. (1) When $k=1$, By (3.3), the above proposition holds. (2) When $k=n$, we assume that proposition 3.1 holds. (3) When $k=n+1$, using (2) and (3.3) it follows that

$$\begin{aligned}
&\mathcal{W}\big((P^{e_1}-P_{e_1})^{n+1}u\big)(x,v)\\
=&\mathcal{W}\big((P^{e_1}-P_{e_1})(P^{e_1}-P_{e_1})^{n}u\big)(x,v)\\
=&(2x_1-2tv_1)\mathcal{W}((P^{e_1}-P_{e_1})^{n}u)(x,v),\\
=&(2x_1-2tv_1)(2x_1-2tv_1)^n\mathcal{W}(u)(x,v),\\
=&(2x_1-2tv_1)^{n+1}\mathcal{W}(u)(x,v).
\end{aligned}$$

This completes the proof of Proposition 3.1. □

Similar to the calculation method of (3.1) and (3.2), we have

$$\mathcal{W}\big((P^{e_k}-P_{e_k})u\big)(x,v)=(2x_k-2tv_k)\mathcal{W}(u)(x,v),$$

and

$$\mathcal{W}\big((P^{e_k}+P_{e_k})u\big)(x,v)=(D_{v_k}+tD_{x_k})\mathcal{W}(u)(x,v).$$

Owing to proposition 3.1, by calculation we gain

$$\mathcal{W}\big((P^{e_1}-P_{e_1})^{\alpha_1}\cdots(P^{e_n}-P_{e_n})^{\alpha_n}u\big)(x,v)=(2x-2tv)^{\alpha}\mathcal{W}(u)(x,v). \tag{3.4}$$

and

$$\mathcal{W}\big((P^{e_1}+P_{e_1})^{\alpha_1}\cdots(P^{e_n}+P_{e_n})^{\alpha_n}u\big)(x,v)=(D_v+tD_x)^{\alpha}\mathcal{W}(u)(x,v). \tag{3.5}$$

We study now the free transport equation.

Proposition 3.2. *Assume that* $e^{2|x|^2-\frac{1}{2}\Delta_v}f_0\in L^2(\mathbb{R}^{2n}_{x,v})$*, then the Cauchy problem* (1.1) *admits a unique solution* $f(t,x,v)$*, which satisfies:*

$$\begin{aligned}
&\Big\|(2x_1-2tv_1)^k f(t)\Big\|^2_{L^2(\mathbb{R}^{2n}_{x,v})}\le A^{k+1}k!,\\
&\Big\|(\partial_{v_1}+t\partial_{x_1})^k f(t)\Big\|^2_{L^2(\mathbb{R}^{2n}_{x,v})}\le A^{k+1}k!.
\end{aligned}$$

Proof. Using Lemma 3.4, we have that,

$$e^{|x|^2+|y|^2}u_0\in L^2\big(\mathbb{R}^{2n}_{x,y}\big),$$

where $u_0=\mathcal{W}^{-1}(f_0)$. By Proposition 2.1 and Lemma 3.3, it follows that the Cauchy problem (1.1) admits a unique solution $f(t,x,v)$. Using now (2.9), for any $\alpha,\beta\in\mathbb{N}^n,t\neq 0$,

$$\|P^{\alpha}P_{\beta}u\|^2_{L^2(\mathbb{R}^{2n}_{x,y})}\le\frac{c_{\alpha,\beta}\alpha!\beta!}{2^{(|\alpha|+|\beta|)}}. \tag{3.6}$$

For simplicity of writing later, let's calculate the following result.

Replacing α with $\alpha-j$ and β with j respectively in (3.6), by Schwartz inequality we have

$$\sum_{|j|=0}^{|\alpha|} C_\alpha^j \|P^{\alpha-j}P_j u\|_{L^2(\mathbb{R}^{2n}_{xy})} \le \sum_{|j|=0}^{|\alpha|} C_\alpha^j \Big(\frac{c_{\alpha-j,j}(\alpha-j)!j!}{2^{|\alpha|}}\Big)^{\frac{1}{2}} \qquad \text{(by (3.6))}$$

$$= \frac{(\alpha!)^{\frac{1}{2}}}{2^{\frac{|\alpha|}{2}}} \sum_{|j|=0}^{|\alpha|} (C_\alpha^j)^{\frac{1}{2}} (c_{\alpha-j,j})^{\frac{1}{2}} \le \frac{(\alpha!)^{\frac{1}{2}}}{2^{\frac{|\alpha|}{2}}} \left(\sum_{|j|=0}^{|\alpha|} C_\alpha^j\right)^{\frac{1}{2}} \left(\sum_{|j|=0}^{|\alpha|} c_{\alpha-j,j}\right)^{\frac{1}{2}} \le (\alpha!)^{\frac{1}{2}} \big(A^{|\alpha|+1}\big)^{\frac{1}{2}}, \qquad (3.7)$$

since $\sum_{|j|=0}^{|\alpha|} C_\alpha^j = 2^{|\alpha|}$, where $\alpha=(\alpha_1,\cdots,\alpha_n), j=(j_1,\cdots,j_n)$ and $(c_{\alpha-j,j}) \in \ell^1$. Using now Proposition 3.1 and Lemma 3.2, We have

$$\Big\|(2x_1-2tv_1)^k f(t)\Big\|_{L^2(R^{2n}_{xv})} = \Big\|\mathcal{W}\big((P^{e_1}-P_{e_1})^k u\big)\Big\|_{L^2(\mathbb{R}^{2n}_{xv})}$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}} \Big\|(P^{e_1}-P_{e_1})^k u\Big\|_{L^2(\mathbb{R}^{2n}_{xy})} = \frac{1}{(2\pi)^{\frac{n}{2}}} \left\|\sum_{j=0}^{k} C_k^j (P^{e_1})^{k-j}(-1)^j (P_{e_1})^j u\right\|_{L^2(\mathbb{R}^{2n}_{xy})}$$

$$\le \sum_{j=0}^{k} C_k^j \|(P^{e_1})^{k-j}(P_{e_1})^j u\|_{L^2(\mathbb{R}^{2n}_{xy})} \le \big(A^{k+1}\big)^{\frac{1}{2}} (k!)^{\frac{1}{2}}$$

and

$$\Big\|(\partial_{v_1}+t\partial_{x_1})^k f(t)\Big\|^2_{L^2(R^{2n}_{xv})} = \Big\|\mathcal{W}\big((P^{e_1}+P_{e_1})^k u\big)\Big\|_{L^2(\mathbb{R}^{2n}_{xv})}$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}} \Big\|(P^{e_1}+P_{e_1})^k u\Big\|_{L^2(\mathbb{R}^{2n}_{xy})} = \frac{1}{(2\pi)^{\frac{n}{2}}} \left\|\sum_{j=0}^{k} C_k^j (P^{e_1})^{k-j} (P_{e_1})^j u\right\|_{L^2(\mathbb{R}^{2n}_{xy})}$$

$$\le \sum_{j=0}^{k} C_k^j \Big\|(P^{e_1})^{k-j}(P_{e_1})^j u\Big\|_{L^2(\mathbb{R}^{2n}_{xy})} \le \big(A^{k+1}\big)^{\frac{1}{2}} (k!)^{\frac{1}{2}}.$$

We get then

$$\Big\|(2x_1-2tv_1)^k f(t)\Big\|^2_{L^2(\mathbb{R}^{2n}_{xv})} \le A^{k+1} k!,$$

and

$$\Big\|(\partial_{v_1}+t\partial_{x_1})^k f(t)\Big\|^2_{L^2(\mathbb{R}^{2n}_{xv})} \le A^{k+1} k!.$$

This completes the proof of the Proposition 3.2. $\square$

Proof of Theorem 1.1: By Proposition 3.2, we gain a unique solution $f(t,x,v)$ of the Cauchy problem (1.1). By (3.4), Lemma 3.2, $[P^\alpha,P_\beta]=0$ and (3.7), we have

$$\begin{aligned}
&\|(2x-2tv)^\alpha f(t)\|_{L^2(R^{2n}_{xv})}=\big\|\mathcal{W}\big((P^{e_1}-P_{e_1})^{\alpha_1}\cdots(P^{e_n}-P_{e_n})^{\alpha_n}u\big)\big\|_{L^2(\mathbb{R}^{2n}_{xv})}\\
=&\frac{1}{(2\pi)^{\frac{n}{2}}}\big\|(P^{e_1}-P_{e_1})^{\alpha_1}\cdots(P^{e_n}-P_{e_n})^{\alpha_n}u\big\|_{L^2(\mathbb{R}^{2n}_{xy})}\\
\le&\left\|\left(\sum_{j_1=0}^{\alpha_1}C^{j_1}_{\alpha_1}(P^{e_1})^{\alpha_1-j_1}(-P_{e_1})^{j_1}\right)\cdots\left(\sum_{j_n=0}^{\alpha_n}C^{j_1}_{\alpha_n}(P^{e_n})^{\alpha_n-j_n}(-P_{e_n})^{j_n}\right)u\right\|_{L^2(\mathbb{R}^{2n}_{xy})}\\
\le&\sum_{|j|=0}^{|\alpha|}C^j_\alpha\|P^{\alpha-j}P_ju\|_{L^2(\mathbb{R}^{2n}_{xy})}\le(\alpha!)^{\frac12}\big(A^{|\alpha|+1}\big)^{\frac12}.
\end{aligned}$$

By (3.5), Lemma 3.2, $[P^\alpha,P_\beta]=0$ and (3.7), we have

$$\begin{aligned}
&\|(\partial_v+t\partial_x)^\alpha f(t)\|_{L^2(R^{2n}_{xv})}=\big\|\mathcal{W}\big((P^{e_1}+P_{e_1})^{\alpha_1}\cdots(P^{e_n}+P_{e_n})^{\alpha_n}u\big)\big\|_{L^2(\mathbb{R}^{2n}_{xv})}\\
=&\frac{1}{(2\pi)^{\frac{n}{2}}}\big\|(P^{e_1}+P_{e_1})^{\alpha_1}\cdots(P^{e_n}+P_{e_n})^{\alpha_n}u\big\|_{L^2(\mathbb{R}^{2n}_{xy})}\\
\le&\left\|\left(\sum_{j_1=0}^{\alpha_1}C^{j_1}_{\alpha_1}(P^{e_1})^{\alpha_1-j_1}(P_{e_1})^{j_1}\right)\cdots\left(\sum_{j_n=0}^{\alpha_n}C^{j_1}_{\alpha_n}(P^{e_n})^{\alpha_n-j_n}(P_{e_n})^{j_n}\right)u\right\|_{L^2(\mathbb{R}^{2n}_{xy})}\\
\le&\sum_{|j|=0}^{|\alpha|}C^j_\alpha\|P^{\alpha-j}P_ju\|_{L^2(\mathbb{R}^{2n}_{xy})}\le(\alpha!)^{\frac12}\big(A^{|\alpha|+1}\big)^{\frac12}.
\end{aligned}$$

Thus we have

$$\big\|(x-tv)^\alpha f(t)\big\|^2_{L^2(\mathbb{R}^{2n}_{x,v})}\le A^{|\alpha|+1}\alpha!$$

and

$$\big\|(\partial_v+t\partial_x)^\alpha f(t)\big\|^2_{L^2(\mathbb{R}^{2n}_{x,v})}\le A^{|\alpha|+1}\alpha!\ .$$

This completes the proof of Theorem 1.1.

4 Holomorphic extension

By Proposition 2.1, we have $g(t,x,y)=e^{-i\frac{|x|^2-|y|^2}{2t}}u(t,x,y)$ is analytic for $(x,y)\in\mathbb{R}^4$. According to (2.10), the analytical radius of $g(t,x,y)$ at each points $(x,y)\in\mathbb{R}^4$ is $\sqrt{2}t,\forall t\neq0$. Let $z=(x,y)+i(w,\theta)\in\mathbb{C}^4$, we extend $g(t,x,y)$ to the complex plane.

$$g(t,z)=\sum_{|\alpha|+|\beta|=0}^{\infty}\frac{\partial^{(\alpha,\beta)}_{xy}g(t,x,y)}{\alpha!\beta!}(i(w,\theta))^{(\alpha,\beta)}.\tag{4.1}$$

Next, we show that the function $g(t,z)$ is holomorphic on $z\in\mathbb{C}^4$.

Proposition 4.1. *Let u be a solution of Cauchy problem* (1.2)*, under the assumption of Proposition* 2.1*, then the function $g(t,z)$ defined by* (4.1) *is holomorphic for $z\in\mathbb{C}^4$ and $t\neq 0$.*

Proof. Let $z=(z_1,z_2,z_3,z_4)=(x_1+iw_1,x_2+iw_2,y_1+i\theta_1,y_2+i\theta_2)\in\mathbb{C}^4$. We just need to cheek Cauchy-Riemann equation:

$$\frac{\partial g(t,z)}{\partial \bar{z}_j}=0, \quad \forall z\in\mathbb{C}^4,\ t\neq 0,\ j=1,2,3,4.$$

We compute only the case of $j=1$. Using the definition of $g(t,z)$ in (4.1) and simple calculation we have that

$$\frac{\partial g(t,z)}{\partial x_1}=\sum_{|\alpha|+|\beta|=0}^{\infty}\frac{\partial_{x_1}\partial_{xy}^{(\alpha,\beta)}g(t,x,y)}{\alpha!\beta!}(i(w,\theta))^{(\alpha,\beta)}$$

and

$$\begin{aligned}\frac{\partial g(t,z)}{\partial w_1}&=i\sum_{\substack{|\alpha|+|\beta|=1\\ \alpha_1\geq 1}}^{\infty}\frac{\partial_{xy}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)}g(x,y)}{(\alpha_1-1)!\alpha_2!\beta!}(iw_1)^{\alpha_1-1}(iw_2)^{\alpha_2}(i\theta_1)^{\beta_1}(i\theta_2)^{\beta_2}\\ &=i\sum_{|\alpha|+|\beta|=0}^{\infty}\frac{\partial_{xy}^{(\alpha_1+1,\alpha_2,\beta_1,\beta_2)}g(x,y)}{\alpha_1!\alpha_2!\beta!}(iw_1)^{\alpha_1}(iw_2)^{\alpha_2}(i\theta_1)^{\beta_1}(i\theta_2)^{\beta_2}\\ &=i\sum_{|\alpha|+|\beta|=0}^{\infty}\frac{\partial_{x_1}\partial_{xy}^{(\alpha,\beta)}g(t,x,y)}{\alpha!\beta!}(i(w,\theta))^{(\alpha,\beta)}.\end{aligned}$$

We gain that

$$\begin{aligned}&\frac{\partial g(t,x_1+iw_1,x_2+iw_2,y_1+i\theta_1,y_2+i\theta_2)}{\partial(x_1-iw_1)}\\ =&\frac{\partial g}{\partial x_1}(t,x_1+iw_1,x_2+iw_2,y_1+i\theta_1,y_2+i\theta_2)\cdot\frac{\partial x_1}{\partial(x_1-iw_1)}\\ &\quad+\frac{\partial g}{\partial w_1}(t,x_1+iw_1,x_2+iw_2,y_1+i\theta_1,y_2+i\theta_2)\cdot\frac{\partial w_1}{\partial(x_1-iw_1)}\\ =&\frac{\partial g}{\partial x_1}(t,x_1+iw_1,x_2+iw_2,y_1+i\theta_1,y_2+i\theta_2)\\ &\quad+i\frac{\partial g}{\partial w_1}(t,x_1+iw_1,x_2+iw_2,y_1+i\theta_1,y_2+i\theta_2)=0.\end{aligned}$$

So we gain

$$\frac{\partial g(t,z)}{\partial \bar{z_1}}=0.$$

This completes the proof of Proposition 4.1. $\square$

We prove now (1.5) in Theorem 1.2. Let

$$u(t,x+iw,y+i\theta)=g(t,x+iw,y+i\theta)e^{i\frac{(x+iw)^2-(y+i\theta)^2}{2t}}, \tag{4.2}$$

where the function $g(t,x+iw,y+i\theta)$ is defined by (4.1).

For the rest of Theorem 1.2, we need to prove the following identity, for $t\neq 0$,

$$\begin{aligned}&\frac{1}{(2t^2\pi)^2}\int_{\mathbb{R}^8} e^{-\frac{w^2+\theta^2}{2t^2}}e^{\frac{2(xw-y\theta)}{t}}|u(t,(x,y)+i(w,\theta))|^2\mathrm{d}x\mathrm{d}y\mathrm{d}w\mathrm{d}\theta\\ =&\int_{\mathbb{R}^4} e^{2(|x|^2+|y|^2)}|\varphi(x,y)|^2\mathrm{d}x\mathrm{d}y.\end{aligned}$$

We prove now a lemma.

Lemma 4.1. *Let u be a solution of Cauchy problem* (1.2)*, under the assumption of proposition* 2.1*, then*

$$\begin{aligned}&\int_{\mathbb{R}^4}\frac{1}{(2t^2\pi)^2}e^{-\frac{|w|^2+|\theta|^2}{2t^2}}\int_{\mathbb{R}^4} e^{\frac{2(x\cdot w-y\cdot\theta)}{t}}|u(t,(x,y)+i(w,\theta))|^2\mathrm{d}x\mathrm{d}y\mathrm{d}w\mathrm{d}\theta\\ =&\sum_{|\alpha|+|\beta|=0}^{\infty}\frac{(2t^2)^{(|\alpha|+|\beta|)}}{\alpha!\beta!}\int_{\mathbb{R}^4}\left|\partial_{xy}^{(\alpha,\beta)}\left(e^{-i\frac{|x|^2-|y|^2}{2t}}u(t,x,y)\right)\right|^2\mathrm{d}x\mathrm{d}y.\end{aligned} \tag{4.3}$$

Proof. By Proposition 4.1 and (4.2), one gains that

$$g(t,x+iw,y+i\theta)=u(t,x+iw,y+i\theta)e^{-i\frac{(x+iw)^2-(y+i\theta)^2}{2t}}$$

is holomorphic function on $\mathbb{C}^4$. By using (4.1) and expanding $\alpha_1,\alpha_2,\beta_1,\beta_2$ into odd and even numbers in turn, and using the perfect square formula to expand, the following series of squares also using the perfect square formula to expand, one has

$$\begin{aligned}&\int_{\mathbb{R}^8}\frac{1}{(2t^2\pi)^2}e^{-\frac{|w|^2+|\theta|^2}{2t^2}}|g(t,(x,y)+i(w,\theta))|^2\mathrm{d}x\mathrm{d}y\mathrm{d}w\mathrm{d}\theta\\ =&\sum_{|\alpha|+|\beta|=0}^{\infty}\left(\frac{2^{2|\alpha|}2^{2|\beta|}}{(2\alpha)!(2\beta)!}\int_{\mathbb{R}^4}|\partial_{xy}^{(\alpha,\beta)}g(t,x,y)|^2\mathrm{d}x\mathrm{d}y\right.\\ &\left.\times\int_{\mathbb{R}^4}\frac{1}{(2t^2\pi)^2}e^{-\frac{w_1^2+w_2^2+\theta_1^2+\theta_2^2}{2t^2}}w_1^{2\alpha_1}w_2^{2\alpha_2}\theta_1^{2\beta_1}\theta_2^{2\beta_2}\mathrm{d}w\mathrm{d}\theta\right).\end{aligned}$$

Since

$$\frac{1}{\sqrt{2t^2\pi}}\int_{\mathbb{R}}e^{-\frac{r^2}{2t^2}}r^{2k}\mathrm{d}r=\frac{(2k)!}{2^{2k}k!}\left(2t^2\right)^k,$$

we have

$$\int_{\mathbb{R}^4}\frac{1}{(2t^2\pi)^2}e^{-\frac{w_1^2+w_2^2+\theta_1^2+\theta_2^2}{2t^2}}w_1^{2\alpha_1}w_2^{2\alpha_2}\theta_1^{2\beta_1}\theta_2^{2\beta_2}\mathrm{d}w\mathrm{d}\theta$$

$$
\begin{aligned}
&=(2t^2)^{|\alpha|+|\beta|}\frac{(2\alpha_1)!}{2^{2\alpha_1}\alpha_1!}\frac{(2\alpha_2)!}{2^{2\alpha_2}\alpha_2!}\frac{(2\beta_1)!}{2^{2\beta_1}\beta_1!}\frac{(2\beta_2)!}{2^{2\beta_2}\beta_2!}\\
&=(2t^2)^{|\alpha|+|\beta|}\frac{(2\alpha)!(2\beta)!}{2^{2|\alpha|}2^{2|\beta|}\alpha!\beta!},
\end{aligned}
$$

we get then

$$
\begin{aligned}
&\int_{\mathbb{R}^8}\frac{1}{(2t^2\pi)^2}e^{-\frac{|w|^2+|\theta|^2}{2t^2}}|g(t,(x,y)+i(w,\theta))|^2\mathrm{d}x\mathrm{d}y\mathrm{d}w\mathrm{d}\theta\\
=&\sum_{|\alpha|+|\beta|=0}^{\infty}\frac{(2t^2)^{|\alpha|+|\beta|}}{\alpha!\beta!}\int_{\mathbb{R}^4}\left|\partial_{xy}^{(\alpha,\beta)}g(t,x,y)\right|^2\mathrm{d}x\mathrm{d}y.
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
&\int_{\mathbb{R}^8}\frac{1}{(2t^2\pi)^2}e^{-\frac{|w|^2+|\theta|^2}{2t^2}}|g(t,(x,y)+i(w,\theta))|^2\mathrm{d}x\mathrm{d}y\mathrm{d}w\mathrm{d}\theta\\
=&\int_{\mathbb{R}^8}\frac{1}{(2t^2\pi)^2}\left(e^{-\frac{|w|^2+|\theta|^2}{2t^2}}\times\left|u(t,x+iw,y+i\theta)e^{-i\frac{(x+iw)^2-(y+i\theta)^2}{2t}}\right|^2\right)\mathrm{d}x\mathrm{d}y\mathrm{d}w\mathrm{d}\theta\\
=&\int_{\mathbb{R}^8}\frac{1}{(2t^2\pi)^2}e^{-\frac{|w|^2+|\theta|^2}{2t^2}}e^{\frac{2(x\cdot w-y\cdot\theta)}{t}}|u(t,(x,y)+i(w,\theta))|^2\mathrm{d}x\mathrm{d}y\mathrm{d}w\mathrm{d}\theta,
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{|\alpha|+|\beta|=0}^{\infty}\frac{(2t^2)^{|\alpha|+|\beta|}}{\alpha!\beta!}\int_{\mathbb{R}^4}\left|\partial_{xy}^{(\alpha,\beta)}g(t,x,y)\right|^2\mathrm{d}x\mathrm{d}y\\
=&\sum_{|\alpha|+|\beta|=0}^{\infty}\frac{(2t^2)^{(|\alpha|+|\beta|)}}{\alpha!\beta!}\int_{\mathbb{R}^4}\left|\partial_{xy}^{(\alpha,\beta)}\left(e^{-i\frac{|x|^2-|y|^2}{2t}}u(t,x,y)\right)\right|^2\mathrm{d}x\mathrm{d}y.
\end{aligned}
$$

This completes the proof of Lemma 4.1. □

Finally, using Proposition 2.2 and (4.3), we gain

$$
\begin{aligned}
&\int_{\mathbb{R}^4}\frac{1}{(2t^2\pi)^2}e^{-\frac{w^2+\theta^2}{2t^2}}\int_{\mathbb{R}^4}e^{\frac{2(x\cdot w-y\cdot\theta)}{t}}|u(t,(x,y)+i(w,\theta))|^2\mathrm{d}x\mathrm{d}y\mathrm{d}w\mathrm{d}\theta\\
=&\int_{\mathbb{R}^4}e^{2(|x|^2+|y|^2)}|\varphi(x,y)|^2\mathrm{d}x\mathrm{d}y.
\end{aligned}
$$

This completes the proof of Theorem 1.2.

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