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THE NONLINEAR BOUNDARY VALUE PROBLEMS OF THREE ELEMENTS FOR THE FIRST ORDER QUASILINEAR ELLIPTIC SYSTEMS

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Abstract In this paper, we discuss the nonlinear boundary value problems of three elements with two shifts for the first order quasilinear elliptic systems and the related solvability by using the continuity method.

Key Words The first order quasilinear elliptic systems; nonlinear boundary value problems; continuity method.
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1. Introduction

A great number of results have been got about the boundary value problems for the first order elliptic systems[1–4].

In this paper we discuss the nonlinear boundary value problems of three elements with two shifts for the first order quasilinear elliptic systems.

Assume that Γ is a simple smooth closed curve in the complex plane E, denote by G^+ the simple connected region surrounded, denote $G^- = F \setminus G^+$, $F(z,w)$ is a complex function of complex variable z and the complex function $w(z)$, the function $g(t, w_1(t), w_2(t))$ is defined on $\Gamma \times E \times E$. And $\alpha(t)$, $\beta(t)$ are positive and opposite shifts respectively, satisfying

(a) $\alpha(\alpha(t)) \equiv \beta(\beta(t)) \equiv t, \quad \alpha(\beta(t)) \equiv \beta(\alpha(t)) \quad$ for $t \in \Gamma$;

(b) $\alpha'(t)$ and $\beta'(t)$ are Hölder continuous on Γ.

$G_1(t)$ and $G_2(t)$ are Hölder continuous functions and different from zero on Γ. Find a piecewise regular solution $w(z)$ of the first order quasilinear elliptic systems

$$w_{\bar{z}} = F(z, w) \qquad \text{for } z \in E \setminus \Gamma$$

such that it satisfies the boundary conditions

$$w^+(t) = G_1(t)w^-(\alpha(t)) + G_2(t)w^+(\beta(t)) + g(t, w^+(t), w^-(t)) \quad \text{for } t \in \Gamma$$

$$|w(z)| = O(|z|^m) \text{ for } z \to +\infty \text{ and the integer } m$$

For the related notations see [1], we assume

(c) For each fixed point z in E, $F(z,w)$ has second order continuous partial derivatives with respect to w and $\overline{w}$, which are uniformly bounded in the norm $L_{p,2}[\cdot, E]$. Denote by C this bound, $F(z,0) \in L_{p,2}(z)$, $p > 2$.

(d) For arbitrary $w_1(t), w_2(t) \in H_\nu(\Gamma)$, $g(t, w_1(t), w_2(t)) \in H_\nu(\Gamma)$, $\nu = 1 - 2/p$; and there exists a nonnegative constant M such that for any $w_1^{(1)}(t)$, $w_1^{(2)}(t)$, $w_2^{(1)}(t)$, $w_2^{(2)}(t) \in H_\nu(\Gamma)$

$$\begin{aligned} &C_\nu[g(t, w_1^{(1)}(t),\ w_2^{(1)}(t)) - g(t, w_1^{(2)}(t),\ w_2^{(2)}(t)),\ \Gamma] \\ &\qquad \le M\{C_\nu[w_1^{(1)}(t) - w_1^{(2)}(t),\ \Gamma] + C_\nu[w_2^{(1)}(t) - w_2^{(2)}(t),\ \Gamma]\} \end{aligned} \tag{1}$$

2. Linear Boundary Value Problems of Three Elements

Theorem 1 *Under the conditions that a more respective inequality is added, the piecewise regular solutions in the complex plane of the boundary value problems of generalized analytic functions*

$$\begin{cases} w_{\bar z} + Aw + B\overline{w} = 0 & \text{for } z \in E \setminus \Gamma \\ w^+(t) = w^-(\alpha(t)) + G(t)w^+(\beta(t)) & \text{for } t \in \Gamma \\ w^-(\infty) = 0 \end{cases} \tag{2}$$

$$\begin{cases} w_{\bar z} + Aw + B\overline{w} = 0 & \text{for } z \in E \setminus \Gamma \\ w^-(t) = w^+(\alpha(t)) + G(t)w^-(\beta(t)) & \text{for } t \in \Gamma \\ w^-(\infty) = 0 \end{cases} \tag{3}$$

are all unique zero solutions, where $A(z)$, $B(z) \in L_{p,2}(E)$, $p > 2$. $G(t)$ is a Hölder continuous function on Γ.

Proof Assume that $w(z)$ is a solution of the boundary value problem (2). By using the expression $w(z) = \Phi(z)e^{\omega(z)}$[1] we can obtain that $\Phi(z)$ satisfies

$$\begin{cases} \Phi^+(t) = e^{\omega^-(\alpha(t)) - \omega^+(t)}\Phi^+(\alpha(t)) + G(t)e^{\omega^+(\beta(t)) - \omega^+(t)}\Phi^+(\beta(t)) \text{ for } t \in \Gamma \\ \Phi^-(\infty) = 0 \end{cases} \tag{4}$$

Taking $H_1(t) = e^{\omega^-(\alpha(t)) - \omega^+(t)}$, $H_2(t) = e^{\omega^+(\beta(t)) - \omega^+(t)}G(t)$, we have $\operatorname{ind}_\Gamma H_1(t) = 0$ and $H_2^*(\tau)H_2^*(\beta_1(\tau)) = G^*(\tau)G^*(\beta_1(\tau))$ for $\tau \in L$. For the variable τ, the curve L, the shifts $\beta_1(\tau)$, $H_2^*(\tau)$, $G^*(\tau)$, see [5].

Let $X_1(z) = \exp\Big\{\dfrac{1}{2\pi i}\displaystyle\int_L \ln[H_1^*(\tau)]\dfrac{d\tau}{\tau - z}\Big\}$. From the properties of $\omega(z)$[1], it follows that there exists a positive constant M_ν depending only on the curve Γ, the shifts $\alpha(t)$, $\beta(t)$ and $L_{p,2}(|A| + |B|)$ such that $|X_1^-(\beta_1(\tau))/X_1^-(\tau)| < M_\nu$ for $\tau \in L$. So if we take $X_2(z)=\exp\Big\{\frac{1}{2\pi i}\int_L \ln\Big[\frac{1}{1-G^*(\tau)G^*(\beta_1(\tau))}\Big]\frac{d\tau}{\tau-z}\Big\}$ then when $G(t)$ satisfies the inequality

$$\begin{cases} 1-G^*(\tau)G^*(\beta_1(\tau))\neq 0 \\ \Big|G^*(\tau)\frac{X_2^-(\beta_1(\tau))}{X_2^-(\tau)}\Big|\max_{\tau\in L}\sqrt{|\beta_1'(\tau)|}<\frac{1}{M_\nu e^{2M_pL_{p,2}(|A|+|B|)}}\min\Big\{\frac{2}{1+M_q},1\Big\} \end{cases} \tag{5}$$

we have

$$\begin{cases} 1-H_2^*(\tau)H_2^*(\beta_1(\tau))\neq 0 \\ \Big|H_2^*(\beta_1(\tau))\frac{H_2^*(\tau)}{H_1^*(\tau)}\Big|\Big|\frac{X^-(\beta_1(\tau))}{X^-(\tau)}\Big|\max_{\tau\in L}\sqrt{|\beta_1'(\tau)|}<\min\Big\{\frac{2}{1+M_g},\ 1\Big\} \end{cases}$$

where for the positive constant M_q [1] and for M [5], $X(z)=X_1(z)X_2(z)$. By Theorem 3 in [5], the boundary value problem (4) has a unique zero solution, thereby $w(z)\equiv 0$.

We are concerned with the boundary value problem of analytic function

$$\begin{cases} \Phi^-(t)=\Phi^+(\alpha(t))+G(t)\Phi^-(\beta(t)) \quad \text{for } t\in\Gamma \\ \Phi^-(\infty)=0 \end{cases} \tag{6}$$

Denote $\tilde{G}(t)=G(\alpha(t))$, $\beta_2(\tau)=\omega_1^-(\beta(\alpha(\omega_{1,-1}^+(\tau))))$, and for the notation $w_1^\pm$, $w_{1,-1}^\pm$ see [5]. By using the method and the related conformal mapping established in [5], we can obtain that if $\tilde{G}(t)$ satisfies the inequality $|\tilde{G}^*(\tau)|\max_{\tau\in L}\sqrt{|\beta_2'(\tau)|}<\min\Big\{\frac{2}{1+M_q},1\Big\}$, then (6) has only a unique zero solution $\Phi(z)=0$. Starting with this solvability results, by an analogue argument to the boundary value problem (2), we can prove that there exist a positive constant M_s depending only on the curve Γ, the shifts $\alpha(t)$, $\beta(t)$ and $L_{p,2}(|A|+|B|)$ such that when $G(t)$ satisfies the inequality

$$|G^*(\tau)|\max_{\tau\in L}\sqrt{|\beta_2'(\tau)|}<\frac{1}{M_s e^{2M_pL_{p,2}(|A|+|B|)}}\min\Big\{\frac{2}{1+M_q},\ 1\Big\} \tag{7}$$

the boundary value problem (3) has only a unique zero solution.

Assume that $G(t)$ satisfies (5) and (7) throughout this section.

Theorem 2 *The integral equation*

$$\begin{aligned} K_\mu(t)\equiv \mu(t)&+\frac{1}{2\pi i}\int_\Gamma[\Omega_1(t,\tau)-\Omega_1(\alpha(t),\ \alpha(\tau))\alpha'(\tau)]\mu(\tau)d\tau \\ &-\frac{1}{2\pi i}\int_\Gamma[\Omega_2(t,\tau)-\Omega_2(\alpha(t),\ \alpha(\tau))\overline{\alpha'(\tau)\mu(\tau)}]d\overline{\tau} \\ &+\frac{1}{2\pi i}\int_\Gamma[\Omega_1(\beta(t),\beta(\tau))\beta'(\tau)G(t)-\Omega_1(\alpha(t),\ \alpha(\tau))\alpha'(\tau)G(\tau)]\mu(\beta(\tau))d\tau \\ &-\frac{1}{2\pi i}\int_\Gamma[\Omega_2(\beta(t),\beta(\tau))\overline{\beta'(\tau)}G(t)-\Omega_2(\alpha(t),\ \alpha(\tau))\overline{\alpha'(\tau)G(\tau)}]\overline{\mu(\beta(\tau))}d\overline{\tau} \\ =g_1(t)& \end{aligned} \tag{8}$$

has a unique solution for any $g_1(t) \in H_\nu(\Gamma)$, *where* Ω_1 *and* Ω_2 *are fundamental kernels.*

Proof By using the results in [2] and the estimate for integral kernels $\Omega_1(\beta(t), \beta(\tau))$ $\beta'(\tau)G(t) - \Omega_1(\alpha(t), \alpha(\tau))\alpha'(\tau)G(\tau)$, $\Omega_2(\beta(t), \beta(\tau))\overline{\beta'(\tau)}G(t) - \Omega_2(\alpha(t), \alpha(\tau))\overline{\alpha'(\tau)G(\tau)}$, we obtain that the kernels of integral equation (8) are of weak oddness.

Next we prove that the homogeneous equation $K_\mu = 0$ of (8) has a unique zero solution. Assume that $\mu(t)$ is a solution of $K_\mu = 0$, which is Hölder continuous on Γ obviously, and make the following complex function:

$$w(z) = \begin{cases} \frac{1}{2\pi i}\int_\Gamma \Omega_1(z,t)\mu(t)dt - \Omega_2(z,t)\overline{\mu(t)}d\bar{t} & \text{for } z \in G^+ \\ \frac{1}{2\pi i}\int_\Gamma \Omega_1(z,t)\mu(\alpha(t))dt - \Omega_2(z,t)\overline{\mu(\alpha(t))}d\bar{t} & \\ \quad + \frac{1}{2\pi i}\int_\Gamma \Omega_1(z,t)G(\alpha(t))\mu(\beta(\alpha(t)))dt - \Omega_2(z,t)\overline{G(\alpha(t))\mu(\beta(\alpha(t)))}d\bar{t} & \\ \qquad \text{for } z \in G^- & \end{cases} \tag{9}$$

then $w(z)$ satisfies the boundary value problem (2). By Theorem 1 we get $w(z) \equiv 0$, hence $\mu(t)$ stands for the boundary value of the function $w_1^-(z)$ which is generalized analytic in G^-, vanishes at the infinite point and is continuous on $G^- \cup \Gamma$, while $\mu(\alpha(t)) + G(\alpha(t))\mu(\beta(\alpha(t)))$ stands for that of the function $w_1^+(z)$ which is generalized analytic in G^+ and continuous on $G^+ \cup \Gamma$. Both of them satisfy the relation $w_1^-(t) = w_1^+(\alpha(t)) - G(t)w_1^-(\beta(t))$ for $t \in \Gamma$. By Theorem 1 we obtain $w_1(z) \equiv 0$, thereby $\mu(t) \equiv 0$.

Finally, by introducing the corresponding systems of equation (8) and using the methods of Section 8 in [6], we can prove that the integral equation (8) has a unique solution for any $g_1(t) \in H_\nu(\Gamma)$ if and only if its homogeneous equation has a unique zero solution. Therefore the result of Theorem 2 holds.

From Theorem 2, we can obtain

Corollary 1 *The boundary value problem*

$$\begin{cases} w_{\bar{z}} + Aw + B\overline{w} = F_1 & \text{for } z \in E \setminus \Gamma \\ w^+(t) = w^-(\alpha(t)) + G(t)w^+(\beta(t)) + g_1(t) & \text{for } t \in \Gamma \\ w^-(\infty) = 0 & \end{cases} \tag{10}$$

has a unique piecewise regular solution which can be expressed in the form

$$w(z) = w_1(z) + \mathcal{F}(z)$$

where $F_1(z) \in L_{\hat{p},2}(E)$, $\mathcal{F}(z) = -\frac{1}{\pi}\int\int_E [\Omega_1(z,\varsigma)F_1(\varsigma) + \Omega_2(z,\varsigma)\overline{F_1(\varsigma)}]d\xi d\eta$; $w_1(z)$ *can be expressed in the form* (9) *and its integral density* $\mu(t)$ *is a solution of the integral equation*

$$K_\mu(t) = \tilde{g}_1(t)$$

where $\tilde{g}_1(t) = g_1(t) + \tilde{\mathcal{F}}(t)$, $\tilde{\mathcal{F}}(t) = \mathcal{F}(\alpha(t)) + G(t)\mathcal{F}(\beta(t)) - \mathcal{F}(t)$.

Corollary 2 *The piecewise regular solution $w(z)$ of the boundary value problem* (10) *satisfies the following estimate*

$$\begin{aligned} C_\nu[w^+,\ G^+ \cup \Gamma] &\le N\{C_\nu[g, \Gamma] + L_{p,2}[F, E]\} \\ C_\nu[w^-,\ G^- \cup \Gamma] &\le N\{C_\nu[g, \Gamma] + L_{p,2}[F, E]\} \end{aligned} \tag{11}$$

where N is constant depending only on the curve Γ, the function $G(t)$ the shifts $\alpha(t)$, $\beta(t)$ and the coefficients $A(z)$, $B(z)$.

3. Solving the Nonlinear Boundary Value Problem of Three Elements

We are concerned with the boundary value problem

$$\begin{cases} w_{\bar z} = F(z, w) & \text{for } z \in E \setminus \Gamma \\ w^+(t) = G_1(t)w^-(\alpha(t)) + G_2(t)w^+(\beta(t)) + g(t, w^+(t),\ w^-(t)) & \text{for } t \in \Gamma \\ |w(z)| = O(|z|^m) & \text{for } |z| \to +\infty \text{ and the integer } m \end{cases} \tag{12}$$

By using the standard function of the Haseman boundary value problem for analytic function corresponding to $G_1(\alpha(t))$[2,6] and the method in [4], we can prove that solving the boundary value problem (12) is equivalent to finding the piecewise regular solution in E of the boundary value problem

$$\begin{cases} w_{\bar z} = F(z, w) & \text{for } z \in E \setminus \Gamma \\ w^+(t) = w^-(\alpha(t)) + G(t)w^+(\beta(t)) + g(\,t, w^+(t), w^-(t)) & \text{for } t \in \Gamma \\ w^-(\infty) = 0 \end{cases} \tag{13}$$

In what follows we assume that $G(t)$ satisfies the inequalities (5) and (7) with $2C$ replacing $L_{p,2}(\,|A| + |B|\,)$, where the constant C has been given in Section 1.

Theorem 3 *When $F(z, w)$ and $g(t,\ w_1(t),\ w_2(t))$ satisfy conditions* (c) *and* (d) *in Section* 1 *and the constant M in* (1) *is sufficiently small, the boundary value problem* (13) *has a unique solution which can be constructed by using the continuity method.*

Proof By using the estimate (11) and properties of $F(z, w)$, $g(t,\ w_1(t),\ w_2(t))$, we can prove that the solution of the boundary value problem (13) is unique when M satisfies $2MN < 1$.

Next we prove the existence of the solution, we are concerned with the boundary value problem with a parameter μ, $0 \le \mu \le 1$,

$$\begin{cases} w_{\bar{z}} = F(z, w) & \text{for } z \in E \setminus \Gamma \\ w^+(t) = w^-(\alpha(t)) + G(t)w^+(\beta(t)) + \mu g(t, w^+(t), w^-(t)) & \text{for } t \in \Gamma \\ w^-(\infty) = 0 \end{cases} \tag{14}$$

Obviously, the case of $\mu = 1$ is the boundary value problem (13) and the case of $\mu = 0$ is the boundary value problem of the analytic function, which has a unique zero solution[5].

If the boundary value problem (14) has a solution for certain $\mu = \mu_0$, $0 \le \mu_0 \le 1$, then it can be proved that there exists a constant δ independent of μ_0 such that for any μ, $\mu_0 \le \mu \le \mu_0 + \delta$, the boundary value problem (14) has a unique solution.

Denote by $w_0(z)$ the solution to (14) for $\mu = \mu_0$, then we have

$$C_\nu[w_0, E] \le \frac{2N}{1 - 2MN}\{C_\nu[g(t, 0, 0), \Gamma] + L_{p,2}[F(z, 0), E]\}$$

Now we shall construct an approximation sequence $\{w_m(z) : m \ge 1\}$ as follows:

$$\begin{cases} w_{m,\bar{z}} = \mu[F_w(z, w_{m-1})(w_m - w_{m-1}) + F_{\overline{w}}(z, w_{m-1})(\overline{w}_m - \overline{w}_{m-1}) + F(z, w_{m-1})] \\ \qquad\qquad \text{for } z \in E \setminus \Gamma \\ w_m^+(t) = w_m^-(\alpha(t)) + G(t)w_m^+(\beta(t)) + \mu g(t w_{m-1}^+(t), w_{m-1}^-(t)) \quad \text{for } t \in \Gamma \\ w_m^-(\infty) = 0 \end{cases}$$

This boundary value problem has a unique solution. By the estimate (11), the uniform boundedness of $C_\nu[w_m, E]$ can be derived from the boundedness of $C_\nu[w_0, E]$.

In order to examine the convergence of the approximation sequence, we consider the difference:

$$\eta_{m-1}(z) = w_m(z) - w_{m-1}(z) \qquad \text{for } m \ge 1$$

Then we have

$$\begin{cases} \eta_{0,\bar{z}} = \mu F_w(z, w)\eta_0 + \mu F_{\overline{w}}(z, w_0)\overline{\eta}_0 + (\mu - \mu_0)F(z, w_0) & \text{for } z \in E \setminus \Gamma \\ \eta_0^+(t) = \eta_0^-(\alpha(t)) + G(t)\eta_0^+(\beta(t)) + (\mu - \mu_0)g(t, w_0^+(t), w_0^-(t)) & \text{for } t \in \Gamma \\ \eta_0^-(\infty) = 0 \end{cases}$$

$$\begin{cases} \eta_{m-1,\bar{z}} = A_{m-1}(z)\eta_{m-1} + B_{m-1}(z)\overline{\eta}_{m-1} + C_{m-1}(z) & \text{for } z \in E \setminus \Gamma \\ \eta_{m-1}^+(t) = \eta_{m-1}^-(\alpha(t)) + G(t)\eta_{m-1}^+(\beta(t)) + g_{m-1}(t) & \text{for } t \in \Gamma \\ \eta_{m-1}(\infty) = 0 & \text{for } m > 1 \end{cases}$$

Here

$$A_{m-1}(z) = \mu F_w(z, w_{m-1})$$

$$B_{m-1}(z) = \mu F_w(z, w_{m-1})$$

$$C_{m-1}(z) = \mu[F(z, w_{m-1}) - F(z, w_{m-2}) - F_w(z, w_{m-2})\eta_{m-2} - F_{\overline{w}}(z, w_{m-2})\overline{\eta}_{m-2}]$$

$$g_{m-1}(t) = \mu[\, g(t, w^+_{m-1}(t), w^-_{m-1}(t)) - g(\, t, w^+_{m-2}(t), w^-_{m-2}(t))]$$

Noting the uniform boundedness of $\{w_m(z) : \ m \geq 1\}$ and using (11) we have

$$C_\nu[\eta_0, E] \leq (\mu - \mu_0)N_1,$$

$$C_\nu[\eta_{m-1}, E] \leq \mu\{N_2 + N_3 C_\nu[\eta_{m-2}, E]\} C_\nu[\eta_{m-2}, E] \quad \text{for } m > 1$$

where the constants $N_2 = 2MN$, N_1, N_3 are independent of μ and μ_0. It is only required that $\mu - \mu_0 < (1 - 2MN)/(N_1 N_3)$ so as to guarantee that the sequence $\{w_m : m \geq 1\}$ will be convergent. Using the method in [4], we can prove that the limit function of $\{w_m(z) : \ m \geq 1\}$ is exactly the solution of the boundary value problem (14).

In this way we obtain the step length $\delta = \mu - \mu_0 < (1 - 2MN)/N_1 N_3$ which is independent of μ and μ_0. Therefore, starting with $\mu = 0$, repeating the finite process we obtain the solution of the boundary value problem (13).

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