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AN IRROTATIONAL AND INCOMPRESSIBLE FLOW AROUND A BODY IN SPACE

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Abstract We consider the problem of an irrotational and incompressible flow around a body in space. The basic existence is proved by formulating the problem into a variational problem. We also show that the solution is unique, and the maximum speed is attained on the body's boundary.
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1. The objective of this note is to solve the problem of an irrotational and incompressible flow around a body in space. We prove the basic existence by formulating the problem into a variational problem. We also prove the general uniqueness, and that the maximum speed is attained on the body's boundary.

One point we would like to stress is that a two-dimensional flow around a profile is substantially different from a flow around a body in space of dimensions $n \geq 3$. This substantial difference is reflected in the substantially new method we developed. In fact, a large part of this paper also serves to highlight the technique developed by Professor Dong G.C. and the author in our work on the subsonic flows around a body in space.

2. Let Γ be a bounded smooth surface, describing the boundary of a uniformly moving body in n-dimensional space $R^n (n \geq 3)$. Suppose the exterior region Ω is filled with an incompressible, invisible fluid, water for example. Moving with the body, we observe the fluid flow around the body with uniform speed at infinity while the body itself is still. Denote $\vec{v} = (v_1, \cdots, v_n)$ as the velocity vector filed of the fluid in Ω. We are interested in $\vec{v}$ when the flow is in a steady state.

For further simplicity, we assume Ω is a simply connected region. By our assumptions, there is a velocity potential u, unique up to an added constant, such that

$$\vec{v} = Du = (D_1 u, \cdots, D_n u)$$

From the conservation of mass, the velocity potential satisfies $\Delta u = 0$ in Ω. On the boundary Γ, there is no normal speed. Thus $\partial u / \partial \nu = 0$ where $\nu = (\nu_1, \cdots, \nu_2)$ is the outer normal of Γ. At infinity, Du is asymptotically a constant vector. Without loss of generality we let this constant vector be $(u^\infty, 0, \cdots, 0)$ with u^∞ being the speed of the flow at infinity.

In summary, the velocity potential $u(x)$ satisfies

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma \\ Du|_\infty = (u^\infty, 0, \cdots, 0) \end{cases} \tag{1}$$

The objective of this paper is to prove that (1) is a well posed problem. That is, with a given u^∞, there is a solution u, unique up to an added constant, satisfying (1).

3. We mention an interesting work in [1] at first. The purpose is to illustrate, by considering a simpler problem, that how to well pose a problem like (1) can be quite intricate. Besides many other results, those authors showed that if the space dimension $n > 3$, then

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = \phi & \text{on } \Gamma \\ u|_\infty = l \end{cases}$$

is a well posed problem for any given function ϕ and constant l; if $n = 2$, then

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = \phi & \text{on } \Gamma \\ u & \text{is bounded in } \Omega \end{cases}$$

is a well posed problem for any given function ϕ. As it turns out, there is a rather interesting analogous phenomenon for (1) with respect to the dimensions.

4. Before we start to work on (1), we prove a few inequalities and define a functional space to do some neccessary preparation. This overlaps with a section in [2], but we include it here for completeness.

We prove a tricky lemma at first.

Lemma 1 *Let B_R be a ball in $\boldsymbol{R}^n (n \geq 3)$ centered at the origin with radius R, let $B_R^c = \boldsymbol{R}^n \setminus B_R$. Then for any $R \geq 0$, any function $\phi \in C_0^\infty(\boldsymbol{R}^n)$ which is infinitely smooth and compactly supported,*

$$\int_{B_R^c} \frac{v^2}{|x|^2} dx \leq \left(\frac{2}{n-2}\right)^2 \int_{B_R^c} |Dv|^2 dx, \quad for\ n \geq 3 \tag{2}$$

Proof Along any ray originated from the origin, let $D_r v$ be the radial derivative