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APPROXIMATE INERTIAL MANIFOLDS TO THE NEWTON-BOUSSINESQ EQUATIONS

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Abstract Approximate inertial manifolds are related to the study of long time behaviour of dissipative partial differential equations. In this paper, we construct two approximate inertial manifolds for the two dimensional Newton-Boussinesq Equations. The orders of approximations of these manifolds to the global attractor are derived.

Key Words Nonlinear Galerkin methods; long time integration; approximate inertial manifolds; Newton-Boussinesq equations.

Classification 35B40; 35P10.

1. Introduction

The study of the long time behaviour of the solutions of dissipative partial differential equations is a major problem of mathematical physics, directly related to the understanding of turbulence. In general, the long time behaviour of systems can be characterized by the existence of global attractors. Associated to the global attractor, approximate inertial manifolds have been introduced recently by Foias Manley and Temam [1]. We recall that an approximate inertial manifold is a finite dimensional smooth manifold such that every solution enters its thin neighborhood in a finite time. In particular, the global attractor is contained in this neighborhood. The existence of approximate inertial manifolds has been obtained for many dissipative partial differential equations. In this respect, we refer readers to Temam [2]; Marion [3]; Jolly, Kevrekidis and Titi [4], and references therein. In the present paper, we construct two approximate inertial manifolds for the two dimensional Newton-Boussinesq equations.

The paper is divided into three parts. In Section 2, we recall a few facts about the equations considered. Also we write the flow function ψ (resp. temperature θ) as $\psi_1+\psi_2$ (resp. $\theta_1+\theta_2$) corresponding to the first m modes and the other modes; uniform priori estimates of ψ_2 and θ_2 we deduce are also contained in this section. The first author of this paper introduces nonlinear Galerkin methods for the two dimensional Newton-Boussinesq equations in [5], which are concerned with a nonlinear mapping F. Our aim in Section 3 is to show $\Sigma_1 = \text{graph}\,(F)$ is an approximate inertial manifold. Finally, in Section 4, by contraction principle, we give another implicit approximate inertial manifold Σ_2, and we prove that Σ_2 can provide higher order approximation to the global attractor than Σ_1. By the way, we introduce the nonlinear Galerkin methods based on Σ_2 and obtain its convergence results.

2. The Equations and Properties of Solutions

2.1 The equations

We consider the following two dimensional Newton-Boussinesq equations:

$$\partial_t\xi + u\partial_x\xi + v\partial_y\xi = \Delta\xi - \frac{R_a}{P_r}\partial_x\theta$$

$$\Delta\psi = \xi, \quad u = \psi_y, v = -\psi_x$$

$$\partial_t\theta + u\partial_x\theta + v\partial_y\theta = \frac{1}{P_r}\Delta\theta$$

where $\vec{u} = (u, v)$ is velocity vector, θ is temperature, ψ is flow function, ξ is vortex, $P_r > 0$ is Prandtl number, and $R_a > 0$ is Rayleigh number. The above equations can be rewritten as

$$\frac{\partial}{\partial t}\Delta\psi + J(\psi, \Delta\psi) = \Delta^2\psi - \frac{R_a}{P_r}\frac{\partial\theta}{\partial x} \tag{2.1}$$

$$\frac{\partial\theta}{\partial t} + J(\psi, \theta) = \frac{1}{P_r}\Delta\theta \tag{2.2}$$

where

$$J(u, v) = u_y v_x - u_x v_y$$

The equations are supplemented with the initial condition

$$\psi\,|_{t=0} = \psi_0(x, y), \quad \theta\,|_{t=0} = \theta_0(x, y) \tag{2.3}$$

and the periodic boundary condition

$$\begin{cases} \psi(x + 2D, y, t) = \psi(x, y, t), & \psi(x, y + 2D, t) = \psi(x, y, t) \\ \theta(x + 2D, y, t) = \theta(x, y, t), & \theta(x, y + 2D, t) = \theta(x, y, t) \end{cases} \tag{2.4}$$

For the functional setting of the problem, let us introduce the operator $Au = -\Delta u$ on $H = L^2(\Omega)$ equipped with its usual scalar product $(\cdot, \cdot)$ and norm $|\cdot|$, where $D(A) =$