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REGULARITY OF SOLUTION FOR FULLY NONLINEAR SECOND ORDER ELLIPTIC EQUATION WITH GRADIENT CONSTRAINT

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Abstract The existence and regularity of the solution for the fully nonlinear second order elliptic equation with gradient constraint are discussed in this paper. We prove the existence of the strong solution by choice of special penalty function, and obtain the local $W^{2,\infty}$ estimate of the solution by means of the theory of viscosity solution with the auxiliary function method.
Key Words Regularity and existence; gradient constraint; viscosity solution.
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1. Introduction

Let $K = \{v \in H_0^1(\Omega) \mid |\nabla v| \le 1 \text{ a.e. } \Omega\}$. For $\forall f \in H^{-1}(\Omega)$, there is a function $u \in K$ such that

$$\int_\Omega \nabla u \nabla(v-u)dx \ge \int_\Omega f(v-u)dx, \quad \forall v \in K$$

This is a simple elliptic variational inequality with a gradient constraint. The elastic-plastic torsion is well-known when f =constant (See [1, 2]). A general elliptic or parabolic equation with gradient constraint also appears in the study of the optimal stochastic control and differential games (See [3, 4]) and in the references contained in these papers.

In [3], L.C. Evans considered the problem of solving a second order elliptic equation subject to a pointwise constraint on the size of the gradient, i.e.

$$\begin{cases} \max(Lu - f, (|\nabla u| - g)) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is a bounded domain in $\mathbf{R}^n$, f and g are the functions given, $Lu = -a_{ij}(x)u_{ij} + b_i(x)u_i + c(x)u$, $c(x) \ge 0$ is a linear second order elliptic operator.

In [3], L.C. Evans proved the existence and uniqueness of the solution under some conditions, and obtained the $W^{2,\infty}_{\text{loc}}$ of the solution under assumption $Lu = -\Delta u$ or a_{ij} =constants for $1 \le i, j \le n$.

In [5], M. Wiegner obtained the $W^{2,\infty}_{\text{loc}}$ estimate of the solution when $a_{ij} = a_{ij}(x)$.

In [6], H. Ishii & S.K. Koike discussed the boundary regularity and uniqueness of the solution, and obtained the $W^{2,\infty}$ estimate of the solution. It is that the $W^{2,\infty}$ estimate is the best estimate one can expect (for the regularity of the solutions to the obstacle problems).

In [7], [1] and [5], G. Gerhardt, R. Jensen, G.H. Williams et al., considered the regularity of the solution for quasilinear variational inequality with gradient constraint.

In [2], [8] and [9], T.N. Razhkovskaya considered the existence and regularity of the solutions for the quasilinear variational inequality and unilateral problem of elliptic systems with the gradient constraint.

In this paper, we will study the full nonlinear second order elliptic equation with gradient constraint, i.e.

$$\begin{cases} -F(x,u,\nabla u,\nabla^2 u) \le 0 & \text{in } \Omega \\ |\nabla u| \le g & \text{in } \Omega \\ F(x,u,\nabla u,\nabla^2 u)(|\nabla u| - g) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where Ω is a smooth domain in $\mathbf{R}^n$, F is a smooth function on $\Gamma = \Omega \times \mathbf{R} \times \mathbf{R}^n \times M^n$, g is an obstacle function given, M^n denotes the space of $n \times n$ symmetric matrices, and ∇u and $\nabla^2 u$ denote respectively the gradient and Hessian matrix of u.

We show the existence and regularity of the strong solution of (1.1) by choosing of special penalty functions and obtain the $W^{2,\infty}_{\text{loc}}$ estimate of the solution using the theory of viscosity solution with the auxiliary function method.

First, we will assume that Dirichlet's problem:

$$\begin{cases} F(x,u,\nabla u,\nabla^2 u) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

it has at least one classical solution.

We assume that there are $\lambda = \lambda(x,z,p)$ and $\Lambda = \Lambda(x,z,p)$ such that

1° $\lambda I \le (F_{ij}) \le \Lambda I$ on Γ, $\lambda > 0$, $\lambda^{-1}\Lambda \le \mu_1(|z|)$,

2° $|F(x,z,p,0)| \le \mu_2(|z|)(1+|p|^2)$,

3° $|F_x|(1+|p|)^{-1} + |F_z| + |F_p|(1+|p|) \le \lambda\mu_3(|z|)(1+|p|^2+|r|)$,

4° $F(x,z,p,0)$ sign $z \le \lambda\overline{\mu}(1+|p|)$,