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VERY WEAK SOLUTIONS OF p- LAPLACIAN TYPE EQUATIONS WITH VMO COEFFICIENTS*

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Abstract In this note we obtain a new *a priori* estimate for the very weak solutions of p-Laplacian type equations with VMO coefficients when p is close to 2, and then prove that the very weak solutions of such equations are the usual weak solutions. Our approach is based on the Hodge decomposition and the L^p-estimate for the corresponding linear equations. And this also provides a simpler proof for the results in [1].

Key Words p-Laplacian type; VMO coefficients; very weak solution.
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1. Introduction

Let $1 < p < \infty$. We consider the very weak solutions of the quasilinear equation

$$\operatorname{div}\left((ADu \cdot Du)^{(p-2)/2} ADu\right) = 0 \quad \text{in} \quad \mathbf{R}^n \tag{1.1}$$

where $A = (A_{ij}(x))_{n\times n}$ is a symmetric matrix with measurable coefficients satisfying the uniform ellipticity condition

$$\nu^{-1}|\xi|^2 \le A(x)\xi \cdot \xi \le \nu|\xi|^2 \tag{1.2}$$

for all $\xi \in \mathbf{R}^n$ and almost every $x \in \mathbf{R}^n$. Here $\nu \ge 1$ is a constant and the dot denotes the standard inner product of $\mathbf{R}^n$.

The equation (1.1) arises naturally in many different contexts. In the case $p = n$ the equation (1.1) plays a key role in the theory of quasiconformal mappings. If A is the identity matrix, then it reduces to the known p-harmonic equation. If $p = 2$, it is a linear equation.

We recall the definition of the very weak solution for the equation (1.1).

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Definition 1.1 *A function $u \in W^{1,r}_{\rm loc}(\mathbf{R}^n), \max\{1, p-1\} \le r \le p$, is called a very weak solution of the equation* (1.1) *if*

$$\int_{\mathbf{R}^n} (ADu \cdot Du)^{(p-2)/2} ADu \cdot D\psi dx = 0 \tag{1.3}$$

for every $\psi \in C_0^\infty(\mathbf{R}^n)$.

We are concerned with the question whether the very weak solutions of the equation (1.1) are the usual weak solutions, that is, whether the very weak solutions belong to the Sobolev space $W^{1,p}_{\rm loc}$. Generally speaking, the answer is not true even in the linear case $p = 2$ (see [2]). However, there is still some hope when the coefficients in the equation (1.1) satisfy certain nice condition as well as the uniformly ellipticity condition (1.2). The case with discontinuous coefficients is much interesting. Therefore, we are keen on considering the problem with the discontinuous coefficients. A natural weakness of the case with smooth coefficients is to assume that the coefficients of the matrix A are of vanishing mean oscillation (VMO).

We say that a locally integrable function f is of bounded mean oscillation (BMO) if $f_B|f - f_B|dy$ is uniformly bounded as B ranges over all balls in $\mathbf{R}^n$, here $f_B = f_B f(y)dy = |B|^{-1} \int_B f(y)dy$ denotes the integral mean over the ball B. If, in addition, we require that these averages tend to zero uniformly as the radii tend to zero, we say that f is of vanishing mean oscillation and denotes $f \in$ VMO, see [3]. Uniformly continuous functions and $W^{1,n}$ functions are of vanishing mean oscillation. In general, the functions of vanishing mean oscillation need not be continuous.

We state our main result as below.

Theorem 1.2 *Suppose that $A_{ij} \in VMO$, $i, j = 1, \cdots, n$, satisfy the condition* (1.2). *For every $\eta, 0 < \eta < 1$, there is a positive number $\delta = \delta(\eta, n, \nu, A_\#) \le \eta$ such that every very weak solution $u \in W^{1,r}_{\rm loc}, 1 + \eta \le r \le p$, of the equation* (1.1) *is the usual weak solution provided that $|p - 2| \le \delta$. Namely, $u \in W^{1,p}_{\rm loc}$ and it satisfies the equality* (1.3). *Here $A_\#$ denotes the VMO modulus of the coefficients $A_{ij}, i, j = 1, \cdots, n$.*

When $p = 2$, we have

Corollary 1.3 *Suppose that $A_{ij} \in VMO$, $i, j = 1, \cdots, n$, satisfy the condition* (1.2). *Then every very weak solution $u \in W^{1,r}_{\rm loc}$ with $r > 1$ of equation*

$$\operatorname{div}(ADu) = 0$$

is the usual weak solution. Namely, $u \in W^{1,2}_{loc}$ and it satisfies

$$\int_{\mathbf{R}^n} ADu \cdot D\psi dx = 0$$

for every $\psi \in C_0^\infty(\mathbf{R}^n)$.

However, the above result is not true even in the linear case if the coefficents $A_{ij}(x), i, j = 1, \cdots, n$ are only bounded and measurable since Serrin provided a well-known counter example more than thirty years ago ([2]). In the beginning of nineties, Iwaniec, Sbordone and Lewis were engaged in studying the very weak solutions of p-Laplacian type equations and made great contributions. Motivated by their pioneering work [4] and [5], many mathematicians such as Giachetti, Leonetti, Schianchi and Fiorenza, etc have generalized their results to much more general cases (see [6], [7] and [8]). However, all their results depend on the hypothesis that the Sobolev exponent r of the very weak solution is close to the natural Sobolev exponent p. The novelty of our result is that r might be far away from p. But now we require the assumption that p is close to 2. Even so, our problem is highly nonlinear when $p \neq 2$. It is still a challenging problem for the arbitrary $p > 1$.

Our approach is based on the wellknown Hodge decomposition (See [4]) and the powerful L^p-estimate in the linear case (See [9]). Nevertheless, we need modify the Hodge decomposition to fit our problem. The new feature in our proof is that we only use the Hodge decomposition once and employ a result of the linear problem. In some sense, we give a linearization proof for the nonlinear problem (1.1). Compared with the tricky proof in [1], our new proof is much shorter and more natural since we require that the exponent p is quite close to 2. This assumption somehow suggests that the results in the linear case may be utilized. On the other hand, we would like to point out that the trick in [1] works only for p-harmonic operator and does not work for the equation (1.1) with variable coefficients. Another different case $r > p$ has been discussed in [10].

In the following we use C_1, C_2, C_3 to denote the constants depending only on the data η, n, ν and $A_{\#}, C$ to denote the generic constants, which may change even in the same line.

2. *A Priori* Estimates

In this section we will prove an important *a priori* estimate and then explain the ideas to prove Theorem 1.2, which has been clearly presented in [4].

Since the matrix $A(x)$ for almost every $x \in \mathbf{R}^n$ is symmetric and positive definite, there exists a symmetric, positive definite matrix $G(x)$ such that $A(x) = G(x)^2$. Recalling the condition (1.2), we have, for every $\xi \in \mathbf{R}^n$ and almost every $x \in \mathbf{R}^n$,

$$\nu^{-1/2}|\xi| \leq |G(x)\xi| \leq \nu^{1/2}|\xi| \tag{2.1}$$

And now, the equation (1.1) reads

$$\operatorname{div}(|GDu|^{p-2}ADu) = 0 \quad \text{in } \mathbf{R}^n \tag{2.2}$$

First we slightly modify the Hodge decomposition (Theorem 3 in [4]) to suit our case.

Theorem 2.1 *Let the matrix $G(x)$ for almost every $x \in \mathbf{R}^n$ be defined as above and Ω be a regular domain in $\mathbf{R}^n$ (See [4]). Let $r > 1$ and $-1 < \varepsilon < r-1$. Then for every $w \in W_0^{1,r}(\Omega)$, there exist $\varphi \in W_0^{1,r/(1+\varepsilon)}(\Omega)$ and a divergence free $H \in L^{r/(1+\varepsilon)}(\Omega, \mathbf{R}^n)$ such that*

$$|G(x)\nabla w(x)|^{\varepsilon}\nabla w(x) = \nabla\varphi(x) + H(x) \tag{2.3}$$

and

$$\|H\|_{r/(1+\varepsilon),\Omega} \le C_r(\Omega, n, \nu)|\varepsilon|\|\nabla w\|_{r,\Omega}^{1+\varepsilon} \tag{2.4}$$

We are going to prove this theorem in next section.

Next we recall a result of the linear problem (See [9]). Let B_R denote a ball with radius $R > 0$ in R^n and $f \in L^q(B_R, \mathrm{R}^n), q > 1$. Consider the Dirichlet problem

$$\begin{cases} \operatorname{div}(AD\varphi) = \operatorname{div} f \\ \varphi \in W_0^{1,q}(B_R) \end{cases} \tag{2.5}$$

where $A = A(x)$ is the matrix defined as above.

Theorem 2.2 *The Dirichlet problem* (2.5) *has a unique solution and moreover, there exists a constant* $C = C(n, q, \nu, R, A_\#)$ *such that*

$$\|\nabla\varphi\|_{q,B_R} \le C\|f\|_{q,B_R} \tag{2.6}$$

Here $A_\#$ *denotes the VMO modulus of the matrix* A.

From the proof of Theorem 2.2 in [9], we find that the constant C may be independent of the radius R if $0 < R \le 1$.

An examination of [4] reveals that all results in [4] are based on its Theorem 5.1. Instead of rewriting all the results in [4], we prove Theorem 2.3 for the equation (2.2) in details when p is close to 2. This theorem is a counterpart of Theorem 5.1 in [4] in our case.

With the help of the above theorems, we are ready to prove Theorem 2.3 where a new *a priori* estimate is provided.

Theorem 2.3 *Let* $0 < \eta < 1$ *and* $0 < R \le 1$. *Suppose that* $w \in W_0^{1,r}(B_R), 1+\eta \le r \le p$, *is a very weak solution of the nonhomogeneous equation*

$$\operatorname{div}(|G(g + \nabla w)|^{p-2}A(g + \nabla w)) = \operatorname{div} h \tag{2.7}$$

where $g \in L^r(B_R, R^n)$ *and* $h \in L^{r/(p-1)}(B_R, R^n)$ *are the vector functions and the matrices* $A = A(x)$ *and* $G = G(x)$ *are defined as above. Then there exist positive numbers* $\delta = \delta(\eta, n, \nu, \|A\|_\#) \le \eta$ *and* $C = C(\eta, n, \nu, \|A\|_\#)$ *such that*

$$\|\nabla w\|_{r,B_R} \le C(\|g\|_{r,B_R} + \|h\|_{r/(p-1),B_R}^{1/(p-1)}) \tag{2.8}$$

provided that

$$|p - 2| \le \delta \tag{2.9}$$

Before proving Theorem 2.3, we point out that all the constants involved in Theorem 2.1 and Theorem 2.2 are independent of the radius R. This fact is clear by rescaling.

Proof of Theorem 2.3 First we restrict p such that

$$1 + \frac{\eta}{2} < 1 + \eta \le r \le p \le 2 + \frac{\eta}{2} \tag{2.10}$$

Hence

$$1+\frac{\eta}{4}\le\frac{r}{p-1}\le\frac{2+\eta}{\eta} \tag{2.11}$$

By using Theorem 2.1 with $\varepsilon = p-2$, we obtain $\varphi \in W_0^{1,r/(p-1)}(B_R)$ and $H \in L^{r/(p-1)}(B_R, \mathbf{R}^n)$ such that

$$|G\nabla w|^{p-2}\nabla w = \nabla\varphi + H \tag{2.12}$$

and

$$\|H\|_{r/(p-1)} \le C_1|p-2|\|\nabla w\|_r^{p-1} \tag{2.13}$$

Here $C_1 = C(\eta, n, \nu)$ is determined by the formula in [4] and the inequality (2.11).

It follows from (2.7) that

$$\operatorname{div}(|G(\nabla w)|^{p-2}A(\nabla w)) = \operatorname{div}(|G(\nabla w)|^{p-2}A(\nabla w) - |G(g+\nabla w)|^{p-2}A(g+\nabla w) + h).$$

In view of (2.12), it follows that

$$\operatorname{div}(A\nabla\varphi) = \operatorname{div}(GF - AH + h)$$

where

$$F = |G(\nabla w)|^{p-2}G(\nabla w) - |G(g+\nabla w)|^{p-2}G(g+\nabla w)$$

Invoking Theroem 2.2 with $q = r/(p-1)$ we have

$$\|\nabla\varphi\|_{r/(p-1)} \le C_2(\|F\|_{r/(p-1)} + \|H\|_{r/(p-1)} + \|h\|_{r/(p-1)}) \tag{2.14}$$

Here $C_2 = C(\eta, n, \nu, \|A\|_{\#})$ is determined by the estimate in [9] and the inequality (2.11). First we estimate $\|F\|_{r/(p-1)}$.

When $p \ge 2$, we observe

$$||\xi|^{p-2}\xi - |\eta|^{p-2}\eta| \le C(|\xi|^{p-2} + |\eta|^{p-2})|\xi - \eta|$$

and obtain

$$|F| \le C(|\nabla w|^{p-2} + |\nabla w + g|^{p-2})|g| \le C(|\nabla w|^{p-2} + |g|^{p-2})|g|$$

By Hölder inequality and Young's inequality we estimate

$$\begin{aligned}\|F\|_{r/(p-1)} &\le C_3(\||\nabla w|^{p-2}|g|\|_{r/(p-1)} + \||g|^{p-1}\|_{r/(p-1)})\\ &\le C_3(\|\nabla w\|_r^{p-2}\|g\|_r + \|g\|_r^{p-1})\\ &\le C_3\frac{p-2}{p-1}\|\nabla w\|_r^{p-1} + C_3\frac{p}{p-1}\|g\|_r^{p-1}\\ &\le C_3|p-2|\|\nabla w\|_r^{p-1} + pC_3\|g\|_r^{p-1}\end{aligned} \tag{2.15}$$

When $1 < p < 2$, we note

$$||\xi|^{p-2}\xi - |\eta|^{p-2}\eta| \le C|\xi - \eta|^{p-1}$$

and obtain

$$|F| \le C|g|^{p-1}$$

Therefore,

$$\|F\|_{r/(p-1)} \le C_3\|g\|_r^{p-1} \tag{2.16}$$

Using (2.12), (2.13), (2.14) together with (2.15) or (2.16), we have

$$\begin{aligned}\||G\nabla w|^{p-2}\nabla w\|_{r/(p-1)} \le& \|\nabla\varphi\|_{r/(p-1)} + \|H\|_{r/(p-1)}\\ \le& C_2\|F\|_{r/(p-1)} + (C_2+1)\|H\|_{r/(p-1)} + C_2\|h\|_{r/(p-1)}\\ \le& (C_2+1)(C_1+C_3)|p-2|\|\nabla w\|_r^{p-1} + pC_2C_3\|g\|_r^{p-1}\\ &+ C_2\|h\|_{r/(p-1)}\end{aligned} \tag{2.17}$$

Recalling (2.1) and (2.10) we have

$$\||G\nabla w|^{p-2}\nabla w\|_{r/(p-1)} \ge C_0\|\nabla w\|_r^{p-1} \tag{2.18}$$

where $C_0 = C(\eta, \nu)$.

Combining (2.17) with (2.18), we obtain (2.8) provided that

$$|p-2| < \frac{C_0}{(C_2+1)(C_1+C_3)}$$

This completes the proof of Theorem 2.3.

Based on this crucial theorem, Theorem 1.2 will follows. Roughly speaking, we first let $r = r_1 = 1+\eta > 1$. Next we localize the equation (1.1) to obtain another equation for the localized solution $w = \mu^q u \in W_0^{1,r_1}(B_R)$ with $q = p/(p-1)$, where $\mu = \mu(x)$ is a cutoff function in any pair $(B_{R/2}, B_R)$ (See [4,10]). Then we apply the estimate (2.8) to deduce a weak Reverse Hölder inequality

$$\left(\fint_{B_{R/2}} |\nabla u|^r\right)^{1/r} \le \frac{C}{R}\left(\fint_{B_R} |u|^r\right)^{1/r} + C\left(\fint_{B_R} |\nabla u|^s\right)^{1/s}$$

where $s = \max\{1, p-1, nr/(n+r)\} < r$. Now Using the Gehring's Lemma (See [11]) we know that there exists a constant $\varepsilon > 0$ such that $u \in W_{\rm loc}^{1,r_2}$, where $r_2 = r_1 + \varepsilon$. And we note that ε depends only on the data $\eta, n, \nu, A_\#$, not on r. If $r_2 \ge p$, we have done. If $r_2 < p$, we set $r = r_2$ and do the same procedure again to get another $r_3 = r_2 + \varepsilon = r_1 + 2\varepsilon$. So we improve the Sobolev exponent of the solution of u by 2ε. After finitely many steps, we eventually improve its Sobolov exponent to p. Therefore, we finish the proof of Theorem 1.2. We refer readers to [4] for details.

3. The Proof of Theorem 2.1

Theorem 2.1 is the direct conclusion of Theorem 3.1 and Corollary 3.2. We just need to choose the following operator T to be the operator $\mathcal{H}_\Omega$ defined in Section 2 of [4].

Let $1 \le r_1 < r_2$. If $f(x) \in L^r(\mathbf{R}^n)$ for some $r \in (r_1, r_2)$, then we define

$$(S^\varepsilon f)(x) = \left(\frac{|G(x)f(x)|}{\|f\|_r}\right)^\varepsilon f(x)$$

In view of (2.1), we have

$$\begin{aligned}|(S^\varepsilon f)(x)| &\le \max\{\nu^{-\varepsilon/2}, \nu^{\varepsilon/2}\}\frac{|f(x)|^{1+\varepsilon}}{\|f\|_r^\varepsilon}\\ &\le \max\{\nu^{(r/r_1-1)/2}, \nu^{(1-r/r_2)/2}\}\frac{|f(x)|^{1+\varepsilon}}{\|f\|_r^\varepsilon} \qquad (3.1)\end{aligned}$$

Theorem 3.1 *Let $T : L^s \to L^s$ be a bounded linear operator for all $s \in [r_1, r_2]$. For every $r \in (r_1, r_2)$ and $r/r_2 - 1 \le \varepsilon \le r/r_1 - 1$, the estimate*

$$\|TS^\varepsilon(f) - S^\varepsilon(Tf)\|_{r/(1+\varepsilon)} \le C_r|\varepsilon|\|f\|_r$$

holds for each $f \in L^r$, where

$$C_r = \frac{2r(r_2 - r_1)\max\{\nu^{(r/r_1-1)/2}, \nu^{(1-r/r_2)/2}\}}{(r - r_1)(r_2 - r)} \sup_{r_1 \le s \le r_2} \|T\|_s$$

Corollary 3.2 *Under the hypotheses stated above, if moreover $Tf = 0$, then*

$$\|T(|Gf|^\varepsilon f)\|_{r/(1+\varepsilon)} \le C_r|\varepsilon|\|f\|_r^{1+\varepsilon}$$

For the convenience of the readers and the completeness of this paper, we give a complete proof of Theorem 3.1, which is basically a copy of the proof of Theorem 4 in [4].

Proof of Theorem 3.1

Denote

$$\rho = \frac{(r - r_1)(r_2 - r)}{r^2(r_2 - r_1)} \qquad (3.2)$$

By using (3.1) we easily prove that

$$\begin{aligned}\|TS^\varepsilon(f) - S^\varepsilon(Tf)\|_{r/(1+\varepsilon)} &\le \|TS^\varepsilon(f)\|_{r/(1+\varepsilon)} + \|S^\varepsilon(Tf)\|_{r/(1+\varepsilon)}\\ &\le \max\{\nu^{(r/r_1-1)/2}, \nu^{(1-r/r_2)/2}\}(\|T\|_{r/(1+\varepsilon)}\|f\|_r + \|T\|_r\|f\|_r)\\ &\le 2\max\{\nu^{(r/r_1-1)/2}, \nu^{(1-r/r_2)/2}\} \sup_{r_1 \le s \le r_2} \|T\|_s\|f\|_r \le C_r\varepsilon\|f\|\end{aligned}$$

if $|\varepsilon| \ge r\rho$. So we only need to consider the case $|\varepsilon| \le r\rho$.

Let $q = r/(r-1)$ be the conjugate number of r. For each complex number $w = u+iv$ in the strip

$$\frac{1}{r_2} - \frac{1}{r} \le u \le \frac{1}{r_1} - \frac{1}{r} \tag{3.3}$$

define

$$(R_w f)(x) = \left(\frac{|G(x)f(x)|}{\|f\|_r}\right)^{rw} f(x), \quad (Q_w g)(x) = \left(\frac{|g(x)|}{\|g\|_q}\right)^{-q\bar{w}} g(x) \tag{3.4}$$

Denote

$$r_w = \frac{r}{1+ru}, \text{ and } q_w = \frac{q}{1-qu}$$

Recalling (2.1) and (3.3), we have

$$|(R_w f)| \le \|f\|_r^{-ru}|Gf|^{ru}|f| \le \max\{\nu^{(r/r_1-1)/2}, \nu^{(1-r/r_2)/2}\}\|f\|_r^{-ru}|f|^{1+ru}$$

and

$$|Q_w g| = \|g\|_q^{qu}|g|^{1-qu}$$

So $R_w : L^r(\mathbf{R}^N) \to L^{r_w}(\mathbf{R}^N)$ and $Q_w : L^q(\mathbf{R}^N) \to L^{q_w}(\mathbf{R}^N)$ satisfy

$$\|R_w f\|_{r_w} \le \max\{\nu^{(r/r_1-1)/2}, \nu^{(1-r/r_2)/2}\}\|f\|_r, \quad \|Q_w g\|_{q_w} = \|g\|_q \tag{3.5}$$

Then we observe that

$$\phi(w) = \int_{\mathbf{R}^n} ((TR_w f)(x) - (R_w Tf)(x)) \cdot \overline{Q_w g}(x) dx \tag{3.6}$$

is a holomorphic on the strip defiend in (3.3). Using Hölder inequality and (3.4) we have

$$\begin{aligned}
|\phi(w)| &\le \|TR_w f - R_w Tf\|_{r_w}\|Q_w g\|_{q_w} \\
&\le \max\{\nu^{(r/r_1-1)/2}, \nu^{(1-r/r_2)/2}\}(\|T\|_{r_u} + \|T\|_r)\|f\|_r\|g\|_q \\
&\le 2\max\{\nu^{(r/r_1-1)/2}, \nu^{(1-r/r_2)/2}\} \sup_{r_1\le s\le r_2} \|T\|_s\|f\|_r\|g\|_q
\end{aligned}$$

Since $\phi(0) = 0$, we use Schwarz's lemma in the disk $\{w; |w| \le \rho\}$, where ρ is defined in (3.2) to obtain

$$|\phi(w)| \le 2\frac{|w|}{\rho}\max\{\nu^{(r/r_1-1)/2}, \nu^{(1-r/r_2)/2}\} \sup_{r_1\le s\le r_2} \|T\|_s\|f\|_r\|g\|_q$$

In particular, since $|\varepsilon| \le r\rho$, we have

$$\left|\phi\left(\frac{\varepsilon}{r}\right)\right| \le C_r|\varepsilon|\|f\|_r\|g\|_q$$

Noting that $S^\varepsilon(f) = R^\varepsilon_r(f)$ and $\|Q_{\frac{\varepsilon}{r}} g\|_{r/(r-1-\varepsilon)} = \|g\|_q$, we find

$$\|TS^\varepsilon(f) - S^\varepsilon(Tf)\|_{r/(1+\varepsilon)} = \sup_{\|g\|_q=1} \int_{\mathbf{R}^n} ((TR_w f)(x) - (R_w Tf)(x)) \cdot \overline{Q_w g}(x)dx$$

$$= \sup_{\|g\|_q=1} \left|\phi\left(\frac{\varepsilon}{r}\right)\right| \le C_r|\varepsilon|\|f\|_r$$

Therefore, we complete the proof of Theorem 3.1.

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