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INITIAL BOUNDARY VALUE PROBLEM FOR GENERALIZED 2D COMPLEX GINZBURG–LANDAU EQUATION*

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Abstract In this paper we study an initial boundary value problem for a generalized complex Ginzburg–Landau equation with two spatial variables (2D). Applying the notion of the ε-regular map we show the unique existence of global solutions for initial data with low regularity and the existence of the global attractor.

Key Words Generalized 2D Ginzburg–Landau equation; initial boundary value problem; ε-regular map; global solution; global attractor.

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1. Introduction

The Ginzburg–Landau equation (GLE) describes various pattern formation and the onset of instabilities in nonequilibrium fluid dynamical systems, as well as in the theory of phase transitions and superconductivity and has drawn great attention to many scientists. The existence of weak and strong solutions, the global attractors and their relative dynamical issues, have been studied by many authors, see, e.g. [1–3] and references therein. A 1D generalized (derivative) GLE has been derived by Doelman [4, 5]) and the global existence of solutions and long time behavior have been studied in [6–8].

In this paper we study the initial boundary value problem for the generalized 2D GLE on a bounded regular domain $\Omega \subset \mathbf{R}^2$

$$u_t = \gamma u + (1 + i\nu)\Delta u - (1 + i\mu)|u|^{2\sigma}u + \lambda_1 \cdot \nabla(|u|^2 u) + (\lambda_2 \cdot \nabla u)|u|^2, \tag{1.1}$$

$$u(t, x) = 0, \qquad t \geq 0, \quad x \in \partial\Omega, \tag{1.2}$$

$$u(0, x) = u_0(x), \qquad x \in \Omega. \tag{1.3}$$

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Here λ_1, λ_2 are constant vectors with complex components. The most interesting case for the derivative GLE is $\sigma = 2$. Guo and Wang [9] proved the existence of a finite dimensional global attractor. One of their assumptions on σ is $\sigma \geq 3$; the initial data is in H^2. The Cauchy problem was studied in [7] and the lower bound on σ becomes $\sigma \geq \frac{1+\sqrt{10}}{2}$. Later the results were improved in [10, 11]. The initial data are required to have one order (weak) derivatives and the conditions on σ, ν and μ are reduced to

(**A1**) either (i) $\sigma > 2$ or (ii) $\sigma = 2$, $|\lambda_1|$ and $|\lambda_2|$ are suitably small;

(**A2**) $-1 - \nu\mu < \frac{\sqrt{2\sigma+1}}{\sigma}|\nu - \mu|$.

The main purpose of this paper is to study the existence and uniqueness of the global solution with initial data belonging to some fractional power Sobolev space $H^s(\Omega)$, $s < 1$, the existence of the global attractor, and the existence of a time-periodic solution as well. We shall prove

Main Theorem *Let σ, ν, μ satisfy* (**A1**) *and* (**A2**), $s \in (1 - \frac{1}{2\sigma}, 1)$. *Then for any* $u_0 \in X^1 = D((-\Delta)^{s/2}) \subset H^s(\Omega)$, (1.1)–(1.3) *possess a unique solution u satisfying* $u \in C([0, \infty); X^1) \bigcap C((0, \infty); H^2 \cap H_0^1(\Omega))$. *When σ is an integer,* $u \in C^\infty((0, \infty) \times \overline{\Omega})$. *Moreover,* (1.1)(1.2) *possesses a global attractor $\mathcal{A}$ which is compact in H_0^1 and attracts bounded subsets of H_0^1 and points of X^1.*

This paper is arranged as follows. First we prove in Section 2 the local existence of solutions. The idea comes from the so-called ε-regular map and ε-regular solution developed in [12]. Then in Section 3 we refine the estimates in [10] to show the uniform boundedness of solutions for large time and the existence of the global attractor.

2. Local Existence

We put (1.1)–(1.3) into a functional setting

$$u_t + Au = F(u), \qquad u(0) = u_0,$$

$A = -(1 + i\nu)\Delta : D(A) = H^2 \cap H_0^1(\Omega) \subset L^2(\Omega) \to L^2(\Omega)$, $F(u) = \gamma u + (1 + i\mu)|u|^{2\sigma}u + (\lambda_1 \cdot \nabla)(|u|^2 u) + (\lambda_2 \cdot \nabla u)|u|^2$. It is known that A is a sectorial operator and generates an analytic semigroup on L^2. The fractional power of A, A^β, with the domain of definition $E^\beta = D(A^\beta)$, for any $\beta \in \mathbf{R}$, has the following properties [13].

$$\begin{aligned} &E^\beta \hookrightarrow H^{2\beta}(\Omega), \quad \beta \geq 0; E^\beta = H^{2\beta} \cap H_0^1(\Omega), \quad \frac{1}{2} \leq \beta \leq 1, \\ &E^\beta = H_0^1(\Omega), \quad 0 \leq \beta \leq 1/4, E^\beta \hookleftarrow L^s(\Omega), \quad -\frac{1}{2} < \beta \leq 0, \quad s \geq \frac{4}{2 - 4\beta}. \end{aligned}$$

The realization of A in E^β (still denoted by A) is an isometry from $E^{1+\beta}$ to E^β and is also sectorial on E^β.

Let $s \in (1 - \frac{1}{2\sigma}, 1)$, $X^\alpha = E^{\alpha + \frac{1}{2}s - 1}$, $\varepsilon = \frac{1}{2}(1 - s)$, $\gamma = (2\sigma + 1)\varepsilon$. Then A is a sectorial operator on X^0 with the domain of definition X^1, $0 < 1 - \frac{1}{2}s - \gamma < \frac{1}{2}$, and