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Ground State Solutions for a Semilinear Elliptic Equation Involving Concave-Convex Nonlinearities

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Abstract. This work is devoted to the existence and multiplicity properties of the ground state solutions of the semilinear boundary value problem $-\Delta u = \lambda a(x)u|u|^{q-2} + b(x)u|u|^{2^*-2}$ in a bounded domain coupled with Dirichlet boundary condition. Here 2^* is the critical Sobolev exponent, and the term ground state refers to minimizers of the corresponding energy within the set of nontrivial positive solutions. Using the Nehari manifold method we prove that one can find an interval Λ such that there exist at least two positive solutions of the problem for $\lambda \in \Lambda$.

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1 Introduction

We consider the following semilinear elliptic equation:

$$\begin{cases} -\Delta u = \lambda a(x)u|u|^{q-2} + b(x)u|u|^{2^*-2}, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a smooth bounded domain, $\lambda > 0$, $1 \leq q < 2$, and $2^* = 2N/(N-2)$ is the critical Sobolev exponent and the weight functions a,b are satisfying the following conditions:

(A) $a^+ = \max\{a,0\} \not\equiv 0$ and $a \in L^{r_q}(\Omega)$ where $r_q = \frac{r}{r-q}$ for some $r \in (q, 2^*-1)$, with in addition $a(x) \geq 0$ a.e in Ω in case $q = 1$;

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(B) $b^+=\max\{b,0\}\not\equiv 0$ and $b\in C(\overline{\Omega})$.

Tsung-fang Wu [1]has investigated the following equation:

$$\begin{cases} -\Delta u=\lambda a(x)u^q+b(x)u^p, & x\in\Omega,\\ u\geq 0,\ u\not\equiv 0, & x\in\Omega,\\ u=0, & x\in\partial\Omega, \end{cases} \tag{1.2}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $0\leq q<1<p<2^*-1$ ($2^*=\frac{2N}{N-2}$ if $N\geq 3$, $2^*=\infty$ if $N=2$), $\lambda>0$ and the weight functions a,b satisfy the following conditions:

(A′) $a^+=\max\{a,0\}\not\equiv 0$ and $a\in L^{r_q}(\Omega)$ where $r_q=\frac{r}{r-(q+1)}$ for some $r\in(q+1,2^*)$, with in addition $a(x)\geq 0$ a.e in Ω in case $q=0$;

(B′) $b^+=\max\{b,0\}\not\equiv 0$ and $b\in L^{s_p}(\Omega)$ where $s_p=\frac{s}{s-(p+1)}$ for some $s\in(p+1,2^*)$.

If the weight functions $a\equiv b\equiv 1$, Ambrosetti-Brezis-Cerami [2] studied Eq. (1.2). They established that there exists $\lambda_0>0$ such that Eq. (1.2) attains at least two positive solutions for $\lambda\in(0,\lambda_0)$, has a positive solution for $\lambda=\lambda_0$ and no positive solution exists for $\lambda>\lambda_0$. Wu [3] found that if the weight functions a changes sign in $\overline{\Omega}$, $b\equiv 1$ and λ is sufficiently small in Eq. (1.2), then Eq. (1.2) has at least two positive solutions.

Throughout this paper we denote $H_0^1(\Omega)$ the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|=\|u\|_{H_0^1(\Omega)}=\left(\int_\Omega|\nabla u|^2\mathrm{d}x\right)^{1/2}.$$

The function $u\in H_0^1(\Omega)$ is said to be a weak solution of the Eq. (1.1), if u satisfies

$$\int_\Omega\left(\nabla u\nabla v-|u|^{2^*-2}uv-\lambda|u|^{q-2}uv\right)\mathrm{d}x=0,\quad \forall v\in H_0^1(\Omega).$$

The energy functional corresponding to Eq. (1.1) is defined as follows:

$$J_\lambda(u):=\frac{1}{2}\int_\Omega|\nabla u|^2\mathrm{d}x-\frac{1}{2^*}\int_\Omega b(x)|u|^{2^*}\mathrm{d}x-\frac{\lambda}{q}\int_\Omega a(x)|u|^q\mathrm{d}x,$$

and then J_λ is well defined on $H_0^1(\Omega)$. It is well-known that the solutions of Eq. (1.1) are the critical points of the functional J_λ.

We denote by S_l the best Sobolev constant for the embedding of $H_0^1(\Omega)$ in $L^l(\Omega)$, where $1\leq l\leq 2^*$. We define the Palais-Smale (or (PS)-) sequences, (PS)-values, and (PS)-conditions in $H_0^1(\Omega)$ for J_λ as follows:

Definition 1.1. (i) *For $c\in\mathbb{R}$, a sequence u_n is a $(PS)_c$-sequence in $H_0^1(\Omega)$ for J_λ if $J_\lambda(u_n)=c+o_n(1)$ and $J'_\lambda(u_n)=o_n(1)$ strongly in $H^{-1}(\Omega)$ as $n\to\infty$.*

(ii) *$c\in\mathbb{R}$ is a (PS)-value in $H_0^1(\Omega)$ for J_λ if there exists a $(PS)_c$-sequence in $H_0^1(\Omega)$ for J_λ.*

(iii) *J_λ satisfies the $(PS)_c$-condition in $H_0^1(\Omega)$ if any $(PS)_c$-sequence u_n in $H_0^1(\Omega)$ for J_λ contains a convergent subsequence.*

We define the following constants:

$$S := \inf_{u\in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 \mathrm{d}x}{\left(\int_\Omega b(x)\,|\,u\,|^{2^*}\mathrm{d}x\right)^{\frac{2}{2^*}}}, \tag{1.3}$$

$$\lambda_0 := \frac{q}{2}\Big(\frac{2-q}{2^*-q}\Big)^{\frac{2-q}{2^*-2}}\Big(\frac{2^*-2}{2^*-q}\Big) S^{\frac{2^*(2-q)}{2(2^*-2)}} \,\|\, a\,\|_{L^{r_q}}^{-1} S_r^{-q}. \tag{1.4}$$

Our main result is the following.

Theorem 1.1. *Assume that the conditions* (A) *and* (B) *hold; then there exists an interval* Λ *such that for* $\lambda\in\Lambda$, *Eq.* (1.1) *has at least two positive solutions.*

We omit dx in the integration for convenience. This paper is organized as follows. In Section 2, we give some properties of the Nehari manifold. In Sections 3 and 4 we prove Theorem 1.1.

2 The Nehari manifold

As the energy functional J_λ is not bounded below on $H_0^1(\Omega)$, considering the functional on the Nehari manifold

$$\mathcal{M}_{\geq} = \{u \in H_0^1(\Omega)\setminus\{0\} : \langle J'_\lambda(u),u\rangle = 0\}$$

is of interest. So, $u\in\mathcal{M}_{\geq}$ if and only if

$$\langle J'_\lambda(u),u\rangle = \|\,u\,\|^2 - \int_\Omega b(x)\,|\,u\,|^{2^*} - \lambda\int_\Omega a(x)\,|\,u\,|^q = 0. \tag{2.1}$$

It has to be considered that $\mathcal{M}_{\geq}$ contains every nonzero solution of Eq. (1.1). Furthermore, we have the following result.

Lemma 2.1. *The energy functional* J_λ *is coercive and bounded below on* $\mathcal{M}_{\geq}$.

Proof. If $u\in\mathcal{M}_{\geq}$, then by (1.3),(2.1) and the Hölder and Young inequalities, we have

$$\begin{aligned} J_\lambda(u) &= \frac{2^*-2}{22^*}\|\,u\,\|^2 - \lambda\left(\frac{2^*-q}{2^*q}\right)\int_\Omega a(x)\,|\,u\,|^q \\ &\geq \frac{1}{N}\|\,u\,\|^2 - \lambda\left(\frac{2^*-q}{2^*q}\right)\|\,u\,\|^q\|\,a\,\|_{L^{r_q}} S_r^{-q}. \end{aligned} \tag{2.2}$$

Thus, J_λ is coercive and bounded below on M_λ. □

The Nehari manifold is closely associated with the behavior of the function of the form $\varphi_u: t \to J_\lambda(tu)$ for $t>0$. Such maps are known as fibering maps and were suggested by Brown and Zhang [4]. For $u \in H_0^1(\Omega)$, we have

$$\begin{aligned}
&\varphi_u(t) = \frac{t^2}{2}\|u\|^2 - \frac{t^{2^*}}{2^*}\int_\Omega b(x)|u|^{2^*} - \lambda\frac{t^q}{q}\int_\Omega a(x)|u|^q;\\
&\varphi_u'(t) = t\|u\|^2 - t^{2^*-1}\int_\Omega b(x)|u|^{2^*} - \lambda t^{q-1}\int_\Omega a(x)|u|^q;\\
&\varphi_u''(t) = \|u\|^2 - (2^*-1)t^{2^*-2}\int_\Omega b(x)|u|^{2^*} - \lambda(q-1)t^{q-2}\int_\Omega a(x)|u|^q.
\end{aligned}$$

It is easy to see that for $u \in H_0^1(\Omega)\backslash\{0\}$ and $t>0$, $\varphi_u'(t)=0$ if and only if $tu \in \mathcal{M}_\lambda$, in other words, the critical points of φ_u correspond to the points on the Nehari manifold. Particularly, $\varphi_u'(1)=0$ if and only if $u \in \mathcal{M}_\lambda$. Therefore, we are allow to divide $\mathcal{M}_\lambda$ into three parts corresponding to local minima, local maxima and points of inflection. Therefore, we define

$$\begin{aligned}
&\mathcal{M}_\geq^+ = \{u \in \mathcal{M}_\geq : \varphi_u''(1)>0\}; \qquad \mathcal{M}_\lambda^0 = \{u \in \mathcal{M}_\geq : \varphi_u''(1)=0\};\\
&\mathcal{M}_\geq^- = \{u \in \mathcal{M}_\geq : \varphi_u''(1)<0\},
\end{aligned}$$

and note that if $u \in \mathcal{M}_\geq$, that is $\varphi_u'(1)=0$, then

$$\begin{aligned}
\varphi_u''(1) &= (2-q)\|u\|^2 - (2^*-q)\int_\Omega b(x)|u|^{2^*}\\
&= (2-2^*)\|u\|^2 - \lambda(q-2^*)\int_\Omega a(x)|u|^q. \qquad (2.3)
\end{aligned}$$

Now we conclude some basic properties of $\mathcal{M}_\geq^+$, $\mathcal{M}_\lambda^0$ and $\mathcal{M}_\geq^-$.

Lemma 2.2. *Assume that u_0 is a local minimizer for J_λ on $\mathcal{M}_\geq$ and $u_0 \notin \mathcal{M}_\lambda^0$. Then $J_\lambda'(u_0)=0$ in $H^{-1}(\Omega)$ (the dual space of $H_0^1(\Omega)$).*

Proof. See [2, Theorem 2.3]. □

Let $\Lambda = (0,\lambda_0)$ where λ_0 is the same as in (1.4), then we have the following result.

Lemma 2.3. *If $\lambda \in \Lambda$, then $\mathcal{M}_\lambda^0 = \emptyset$.*

Proof. Suppose the contrary. Then there exists $\lambda \in \Lambda$ such that $\mathcal{M}_\lambda^0 \neq \emptyset$. Then for $u \in \mathcal{M}_\lambda^0$ by (1.3) and (2.3), we have

$$\frac{2-q}{2^*-q}\|u\|^2 = \int_\Omega b(x)|u|^{2^*} \leq S^{-\frac{2^*}{2}}\|u\|^{2^*},$$

and so

$$\|u\| \geq \left(\frac{2-q}{2^*-q}\right)^{\frac{1}{2^*-2}} S^{\frac{2^*}{2(2^*-2)}}.$$

Similarly, using (1.3), (2.3), and the Hölder and Young inequalities, we have

$$\| u \|^2 = \lambda \frac{2^*-q}{2^*-2} \int_\Omega a(x) |u|^q \le \lambda \frac{2^*-q}{2^*-2} \| u \|^q \| a \|_{L^{r_q}} S_r^q.$$

Hence

$$\lambda \ge \left(\frac{2-q}{2^*-q} \right)^{\frac{2-q}{2^*-2}} \left(\frac{2^*-2}{2^*-q} \right) S^{\frac{2^*(2-q)}{2(2^*-2)}} \| a \|_{L^{r_q}}^{-1} S_r^{-q} > \lambda_0,$$

which is a contradiction. This completes the proof. □

We consider the function $\psi_u : \mathbb{R}^+ \to \mathbb{R}$ defined by

$$\psi_u(t) = t^{1-q} \varphi_u'(t) + \lambda \int_\Omega a(x) |u|^q, \quad \text{for } t > 0.$$

The following result explains the behavior of the graph of ψ_u.

Lemma 2.4. *For sufficiently small* λ, ψ_u *is strictly increasing on* $(0, t_{\max}(u))$ *and strictly decreasing on* $(t_{\max}(u), \infty)$ *with* $\lim_{t\to\infty} \psi_u(t) = -\infty$, *where*

$$t_{\max}(u) = \left(\frac{(2-q) \| u \|^2}{(2^*-q) \int_\Omega b(x) |u|^{2^*}} \right)^{\frac{1}{2^*-2}} > 0.$$

Proof. Clearly $tu \in \mathcal{M}_\lambda$ if and only if

$$\psi_u(t) = \lambda \int_\Omega a(x) |u|^q.$$

Moreover,

$$\psi_u'(t) = (2-q) t^{1-q} \| u \|^2 - (2^*-q) t^{2^*-q-1} \int_\Omega b(x) |u|^{2^*}, \quad \text{for } t > 0, \tag{2.4}$$

and so it is easy to see that, if $tu \in \mathcal{M}_\lambda$, then

$$t^{q-1} \psi_u'(t) = \varphi_u''(t).$$

Hence, $tu \in \mathcal{M}_\lambda^+$ (or $tu \in \mathcal{M}_\lambda^-$) if and only if $\psi_u'(t) > 0$ ($\psi_u'(t) < 0$).

For $u \in H_0^1(\Omega) \backslash \{0\}$, by (2.4), ψ_u has a unique critical point at $t = t_{\max}(u)$; which is mentioned above. Clearly ψ_u is strictly increasing on $(0, t_{\max}(u))$ and strictly decreasing on $(t_{\max}(u), \infty)$ with $\lim_{t\to\infty} \psi_u(t) = -\infty$. □

Remark 2.1. Note that if $\lambda \in \Lambda$, then

$$\begin{aligned}\psi_u(t_{\max}(u)) &= \left[\left(\frac{2-q}{2^*-q}\right)^{\frac{2-q}{2^*-2}} - \left(\frac{2-q}{2^*-q}\right)^{\frac{2^*-q}{2^*-2}}\right] \frac{\|u\|^{\frac{2(2^*-q)}{2^*-2}}}{\left(\int_\Omega b(x)|u|^{2^*}\right)^{\frac{2-q}{2^*-2}}} \\ &= \|u\|^q \left(\frac{2^*-2}{2^*-q}\right)\left(\frac{2-q}{2^*-q}\right)^{\frac{2-q}{2^*-2}} \left(\frac{\|u\|^{2^*}}{\int_\Omega b(x)|u|^{2^*}}\right)^{\frac{2-q}{2^*-2}} \\ &\geq \|u\|^q \left(\frac{2^*-2}{2^*-q}\right)\left(\frac{2-q}{2^*-q}\right)^{\frac{2-q}{2^*-2}} S^{\frac{2^*(2-q)}{2(2^*-2)}} \\ &> \frac{2}{q}\lambda \|u\|^q \|a\|_{L^{r_q}} S_r^{+q} \geq \frac{2}{q}\lambda \int_\Omega a(x)|u|^q.\end{aligned}$$

Moreover, we have the following lemma.

Lemma 2.5. *Let $\lambda \in \Lambda$. For each $u \in H_0^1(\Omega)\backslash\{0\}$, we have the following.*
(i) *There exist unique $0 < t^+ = t^+(u) < t_{\max}(u) < t^- = t^-(u)$ such that $t^+u \in \mathcal{M}_\lambda^+$, $t^-u \in \mathcal{M}_\lambda^-$, φ_u is decreasing on $(0,t^+)$, increasing on (t^+,t^-) and decreasing on (t^-,∞), and*

$$J_\lambda(t^+u) = \inf_{0 \leq t \leq t_{\max}(u)} J_\lambda(tu); \quad J_\lambda(t^-u) = \sup_{t \geq t^+} J_\lambda(tu).$$

(ii) $\mathcal{M}_\lambda^- = \left\{u \in H_0^1(\Omega)\backslash\{0\} : \frac{1}{\|u\|} t^-\left(\frac{u}{\|u\|}\right) = 1\right\}$.

(iii) *There exist a continuous bijection between $U = \{u \in H_0^1(\Omega)\backslash\{0\} : \|u\| = 1\}$ and $\mathcal{M}_\lambda^-$. In particular, t^- is a continuous function for $u \in H_0^1(\Omega)\backslash\{0\}$.*

Proof. See [5, Lemma 2.6]. □

3 The existence of a ground state

By Lemma 2.3, we can write

$$\mathcal{M}_{\geq} = \mathcal{M}_{\geq}^+ \cup \mathcal{M}_{\geq}^-,$$

for all $\lambda \in \Lambda$. Furthermore, by Lemma 2.5 it follows that $\mathcal{M}_\lambda^+$ and $\mathcal{M}_\lambda^-$ are non-empty and by Lemma 2.1 we may define

$$\alpha_\lambda = \inf_{u \in \mathcal{M}_{\geq}} J_\lambda(u); \quad \alpha_\lambda^+ = \inf_{u \in \mathcal{M}_{\geq}^+} J_\lambda(u); \quad \alpha_\lambda^- = \inf_{u \in \mathcal{M}_{\geq}^-} J_\lambda(u).$$

Then we have the following result.

Theorem 3.1. *If $\lambda \in \Lambda$ then*
(i) $\alpha_\lambda^+ < 0$;
(ii) $\alpha_\lambda^- > d_0$, for some $d_0 > 0$.
In particular, we have $\alpha_\lambda = \alpha_\lambda^+$.

Proof. (i) Let $u \in \mathcal{M}_\lambda^+$. By (2.4),

$$\frac{2-q}{2^*-q}\|u\|^2 > \int_\Omega b(x)|u|^{2^*},$$

and so

$$\begin{aligned} J_\lambda(u) &= \left(\frac{1}{2}-\frac{1}{q}\right)\|u\|^2 + \left(\frac{1}{q}-\frac{1}{2^*}\right)\int_\Omega b(x)|u|^{2^*} \\ &< \left[\left(\frac{1}{2}-\frac{1}{q}\right) + \left(\frac{1}{q}-\frac{1}{2^*}\right)\left(\frac{2-q}{2^*-q}\right)\right]\|u\|^2 \\ &= -\frac{(2^*-2)(2-q)}{22^*q}\|u\|^2 < 0. \end{aligned}$$

Therefore, $\alpha_\lambda^+ < 0$.
(ii) Let $u \in \mathcal{M}_\lambda^-$. By (2.3),

$$\frac{2-q}{2^*-q}\|u\|^2 < \int_\Omega b(x)|u|^{2^*}.$$

Moreover, by (1.3) we have

$$\int_\Omega b(x)|u|^{2^*} \le S^{-\frac{2^*}{2}}\|u\|^{2^*}.$$

This implies

$$\|u\| > \left(\frac{2-q}{2^*-q}\right)^{\frac{1}{2^*-2}} S^{\frac{N}{4}}, \quad \text{for } u \in \mathcal{M}_\ge^-. \tag{3.1}$$

By (2.3) and (3.1), we have

$$\begin{aligned} J_\lambda(u) &\ge \|u\|^q\left[\frac{1}{N}\|u\|^{2-q} - \lambda\left(\frac{2^*-q}{2^*q}\right)\|a\|_{L^{r_q}} S_r^q\right] \\ &> \left(\frac{2-q}{2^*-q}\right)^{\frac{q}{2^*-2}} S^{\frac{qN}{4}}\left[\frac{1}{N}\left(\frac{2-q}{2^*-q}\right)^{\frac{2-q}{2^*-2}} S^{\frac{(2-q)N}{4}} - \lambda\left(\frac{2^*-q}{2^*q}\right)\|a\|_{L^{r_q}} S_r^q\right]. \end{aligned}$$

Thus, if $\lambda \in \Lambda$, then $J_\lambda(u) > d_0$ for all $u \in \mathcal{M}_\ge^-$, for some positive constant d_0. □

Remark 3.1. (i) If $\lambda \in \Lambda$, then by (1.3), (2.3), and the Hölder and Young inequalities, for each $u \in \mathcal{M}_\ge^+$ we have

$$\begin{aligned} \|u\|^2 &< \lambda\frac{2^*-q}{2^*-2}\int_\Omega a(x)|u|^q \le \lambda\frac{2^*-q}{2^*-2}\|u\|^q\|a\|_{L^{r_q}} S_r^q \\ &\le \lambda_0\frac{2^*-q}{2^*-2}\|u\|^q\|a\|_{L^{r_q}} S_r^q, \end{aligned} \tag{3.2}$$

and so

$$\|u\| < \left(\lambda_0 \frac{2^*-q}{2^*-2}\|a\|_{L^{r_q}} S_r^q\right)^{\frac{1}{2-q}}, \quad \text{for all } u \in \mathcal{M}_{\geq}^{+}.$$

(ii) If $\lambda \in \Lambda$, then by Lemma 2.5(i) and Theorem 3.1(ii), for each $u \in \mathcal{M}_{\geq}^{-}$ we have

$$J_\lambda(u) = \sup_{t \geq 0} J_\lambda(tu).$$

Then we have the following results.

Proposition 3.1. *If $\lambda \in \Lambda$, then*
(i) *there exists a $(PS)_{\alpha_\lambda}$-sequence $u_n \subset \mathcal{M}_\lambda$ in $H_0^1(\Omega)$ for J_λ;*
(ii) *there exists a $(PS)_{\alpha_\lambda^-}$-sequence $u_n \subset \mathcal{M}_\lambda^-$ in $H_0^1(\Omega)$ for J_λ.*

Proof. See [6, Proposition 9]. □

Now, we establish the existence of local minimum for J_λ on $\mathcal{M}_\lambda^+$.

Theorem 3.2. *If $\lambda \in \Lambda$, then J_λ has a minimizer u_λ in $\mathcal{M}_\lambda^+$ and it satisfies the following.*
(i) $J_\lambda(u_\lambda) = \alpha_\lambda = \alpha_\lambda^+$.
(ii) *u_λ is a positive solution of Eq.* (1.1).
(iii) $\|u_\lambda\| \to 0$ *as* $\lambda \to 0^+$.

Proof. By Proposition 3.1(i), there is a minimizing sequence u_n for J_λ on $\mathcal{M}_\lambda$ such that

$$J_\lambda(u_n) = \alpha_\lambda + o_n(1) \quad \text{and} \quad J_\lambda'(u_n) = o_n(1) \quad \text{in } H^{-1}(\Omega). \tag{3.3}$$

Since J_λ is coercive on $\mathcal{M}_\lambda$ (see Lemma 2.1), we get that u_n is bounded in $H_0^1(\Omega)$. Going if necessary to a subsequence, we can assume that there exists $u_\lambda \in H_0^1(\Omega)$ such that

$$\begin{cases} u_n \rightharpoonup u_\lambda & \text{weakly in } H_0^1(\Omega), \\ u_n \to u_\lambda & \text{almost everywhere in } \Omega, \\ u_n \to u_\lambda & \text{strongly in } L^s(\Omega) \text{ for all } 1 \leq s < 2^*. \end{cases} \tag{3.4}$$

Thus, we have

$$\lambda \int_\Omega a(x)|u_n|^q = \lambda \int_\Omega a(x)|u_\lambda|^q + o_n(1) \quad \text{as } n \to \infty. \tag{3.5}$$

First, we claim that u_λ is a nonzero solution of Eq. (1.1). By (3.3) and (3.4), it is easy to see that u_λ is a solution of Eq. (1.1). From $u_\lambda \in \mathcal{M}_\lambda$ and (2.2), we deduce that

$$\lambda \int_\Omega a(x)|u_n|^q = \frac{q(2^*-2)}{2(2^*-q)}\|u_n\|^2 - \frac{2^* q}{2^*-q} J_\lambda(u_n). \tag{3.6}$$

Let $n \to \infty$ in (3.6), by (3.3), (3.5) and $\alpha_\lambda < 0$, we get

$$\lambda \int_\Omega a(x)|u_\lambda|^q \geq -\frac{2^* q}{2^*-q}\alpha_\lambda > 0.$$

Thus, $u_\lambda \in \mathcal{M}_\lambda$ is a nonzero solution of Eq. (1.1). Now we prove that $u_n \to u_\lambda$ strongly in $H_0^1(\Omega)$ and $J_\lambda(u_\lambda)=\alpha_\lambda$. By (3.6), if $u\in\mathcal{M}_\lambda$, then

$$J_\lambda(u)=\frac{1}{N}\|u\|^2-\lambda\frac{2^*-q}{2^*q}\int_\Omega a(x)|u|^q. \tag{3.7}$$

First we show that $J_\lambda(u_\lambda)=\alpha_\lambda$. It suffices to recall that $u_n, u_\lambda \in \mathcal{M}_\lambda$; by (3.7) and using weakly lower semi continuity of J_λ we get

$$\begin{aligned}\alpha_\lambda \le J_\lambda(u_\lambda) &= \frac{1}{N}\|u_\lambda\|^2-\lambda\frac{2^*-q}{2^*q}\int_\Omega a(x)|u_\lambda|^q\\ &\le \liminf_{n\to\infty}\left(\frac{1}{N}\|u_n\|^2-\lambda\frac{2^*-q}{2^*q}\int_\Omega a(x)|u_n|^q\right)\\ &\le \liminf_{n\to\infty} J_\lambda(u_n)=\alpha_\lambda.\end{aligned}$$

This implies that $J_\lambda(u_\lambda)=\alpha_\lambda$ and $\lim\limits_{n\to\infty}\|u_n\|^2=\|u_\lambda\|^2$. Let $v_n=u_n-u_\lambda$; then by Brézis-Lieb lemma [7] we have

$$\|v_n\|^2=\|u_n\|^2-\|u_\lambda\|^2+o_n(1).$$

Thus $u_n\to u_\lambda$ strongly in $H_0^1(\Omega)$. Moreover, we have $u_\lambda\in\mathcal{M}_\lambda^+$. If, on the contrary, $u_\lambda\in\mathcal{M}_\lambda^-$, then by Lemma 2.5, there are unique t_0^+ and t_0^- such that $t_0^+u_\lambda\in\mathcal{M}_\lambda^+$ and $t_0^-u_\lambda\in\mathcal{M}_\lambda^-$. In particular, we have $t_0^+<t_0^-=1$. Since

$$\frac{\mathrm{d}}{\mathrm{d}t}J_\lambda(t_0^+u_\lambda)=0 \quad \text{and} \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2}J_\lambda(t_0^+u_\lambda)>0,$$

there exists $t_0^+<t^-\le t_0^-$ such that $J_\lambda(t_0^+u_\lambda)<J_\lambda(t^-u_\lambda)$. By Lemma 2.5(i),

$$J_\lambda(t_0^+u_\lambda)<J_\lambda(t^-u_\lambda)\le J_\lambda(t_0^-u_\lambda)=J_\lambda(u_\lambda),$$

which is a contradiction. Since $J_\lambda(u_\lambda)=J_\lambda(|u_\lambda|)$ and $|u_\lambda|\in\mathcal{M}_\lambda^+$, by Lemma 2.2, we may assume that u_λ is a nonzero nonnegative solution of Eq. (1.1). By the Harnack inequality [8] we deduce that $u_\lambda>0$ in Ω. Finally, by (3.2), we have

$$\|u_\lambda\|^{2-q}<\lambda\frac{2^*-q}{2^*-2}\|a\|_{L^{r_q}}S_r^q,$$

and so $\|u_\lambda\|\to 0$ as $\lambda\to 0^+$. □

4 Proof of Theorem 1.1

In this section, we establish the existence of a local minimum for J_λ on $\mathcal{M}_\lambda^-(\Omega)$.

Theorem 4.1. *Let $\lambda_0>0$ as in* (1.4), *then for $\lambda\in(0,\lambda_0)$, J_λ has a minimizer U_λ in $\mathcal{M}_\lambda^-(\Omega)$ and it satisfies*
(i) $J_\lambda(U_\lambda)=\alpha_\lambda^-(\Omega)$;
(ii) U_λ is a solution of Eq. (1.1).

Proof. By proposition 3.1(ii), there exists a minimizing sequence u_n for J_λ on $\mathcal{M}_\lambda^-(\Omega)$ such that

$$J_\lambda(u_n)=\alpha_\lambda^-(\Omega)+o_n(1), \qquad J'_\lambda(u_n)=o_n(1), \text{ in } H^{-1}(\Omega).$$

Since J_λ is coercive on $\mathcal{M}_\lambda$ (see Lemma 2.1), we get that u_n is bounded in $\mathcal{M}_\lambda^-(\Omega)$. From this and by compact embedding Theorem, there exists a subsequence of u_n and $U_\lambda\in\mathcal{M}_\lambda^-$ such that

$$\begin{cases} u_n\rightharpoonup U_\lambda & \text{weakly in } H_0^1(\Omega),\\ u_n\to U_\lambda & \text{strongly in } L^r(\Omega) \text{ for all } 1\le r<2^*,\\ u_n\rightharpoonup U_\lambda & \text{weakly in } L^{2^*}(\Omega).\end{cases}$$

Since

$$o_n(1)=\langle J'_\lambda(u_n),\eta\rangle=\langle J'_\lambda(U_\lambda),\eta\rangle+o_n(1), \quad \text{for all } \eta\in H_0^1(\Omega),$$

and

$$\begin{aligned} 0>\varphi''_{u_n}(1)&=(2-q)\,\|u_n\|^2-(2^*-q)\int_\Omega b(x)\,|u_n|^{2^*}\\ &\ge(2-q)\,\|U_\lambda\|^2-(2^*-q)\int_\Omega b(x)\,|U_\lambda|^{2^*},\end{aligned}$$

thus we get $U_\lambda\in\mathcal{M}_\lambda^-(\Omega)$ is a nonzero solution of Eq. (1.1). We now prove that $u_n\to U_\lambda$ strongly in $H_0^1(\Omega)$. Suppose otherwise; then $\|U_\lambda\|<\liminf_{n\to\infty}\|U_n\|$ and so

$$\begin{aligned}\langle J'_\lambda(U_\lambda),U_\lambda\rangle&=\|U_\lambda\|^2-\lambda\int_\Omega a(x)\,|U_\lambda|^q-\int_\Omega b(x)\,|U_\lambda|^{2^*}\\ &<\liminf_{n\to\infty}\left(\|U_n\|^2-\lambda\int_\Omega a(x)\,|u_n|^q-\int_\Omega b(x)\,|u_n|^{2^*}\right)\\ &=\liminf_{n\to\infty}\langle J'_\lambda(u_n),u_n\rangle=0.\end{aligned}$$

This contradicts $U_\lambda\in\mathcal{M}_\lambda^-(\Omega)$. Hence $u_n\to U_\lambda$ strongly in $H_0^1(\Omega)$. This implies

$$J_\lambda(u_n)\to J_\lambda(U_\lambda)=\alpha_\lambda^-(\Omega), \qquad \text{as } n\to\infty.$$

Since $J_\lambda(U_\lambda)=J_\lambda(|U_\lambda|)$ and $|U_\lambda|\in\mathcal{M}_\lambda^-(\Omega)$ by Lemma 2.2 we may assume that U_λ is a nonzero nonnegative solution of Eq. (1.1). Finally, by the Harnack inequality [8], we deduce that $U_\lambda>0$ in Ω. □

Now, we complete the proof of Theorem 1.1: By Theorems 3.2, 4.1, Eq. (1.1) has two solutions u_λ,U_λ such that $u_\lambda\in\mathcal{M}_\lambda^+(\Omega)$, $U_\lambda\in\mathcal{M}_\lambda^-(\Omega)$. Since $\mathcal{M}_\lambda^+(\Omega)\cap\mathcal{M}_\lambda^-(\Omega)=\emptyset$, this implies that u_λ and U_λ are different.

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