JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **35**, No. 4, pp. 307-319

doi: 10.4208/jpde.v35.n4.1
December 2022

Doubly Nonlinear Degenerate Parabolic Equations with a Singular Potential for Greiner Vector Fields

HAN Junqiang*

School of Mathematics and Statistics, Northwestern Polytechnical University, Xi'an 710129, China.

Received 11 June 2021; Accepted 24 March 2022

Abstract. The purpose of this paper is to investigate the nonexistence of positive solutions of the following doubly nonlinear degenerate parabolic equations:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \nabla_k \cdot \left(u^{m-1} |\nabla_k u|^{p-2} \nabla_k u \right) + V(w) u^{m+p-2}, & \text{in } \Omega \times (0,T), \\ u(w,0) = u_0(w) \geqslant 0, & \text{in } \Omega, \\ u(w,t) = 0, & \text{on } \partial\Omega \times (0,T), \end{cases}$$

where Ω is a Carnot-Carathéodory metric ball in $\mathbb{R}^{2n+1}$ generated by Greiner vector fields, $V \in L^1_{loc}(\Omega)$, $k \in \mathbb{N}$, $m \in \mathbb{R}$, $1 < p < 2n+2k$ and $m+p-2 > 0$. The better lower bound p^* for $m+p$ is found and the nonexistence results are proved for $p^* \leqslant m+p < 3$.

AMS Subject Classifications: 35K55, 35K65

Chinese Library Classifications: O175.29

Key Words: Doubly nonlinear degenerate parabolic equations; Greiner vector fields; positive solutions; nonexistence; Hardy inequality.

1 Introduction

In this paper we are concerned with the nonexistence of positive solutions of the following doubly nonlinear degenerate parabolic equations:

$$\begin{cases} \dfrac{\partial u}{\partial t} = Pu + V(w) u^{m+p-2}, & \text{in } \Omega \times (0,T), \\ u(w,0) = u_0(w) \geq 0, & \text{in } \Omega, \\ u(w,t) = 0, & \text{in } \partial\Omega \times (0,T), \end{cases} \tag{1.1}$$

*Corresponding author. *Email address:* southhan@163.com (J. Q. Han)

where Ω is a Carnot-Carathéodory metric ball in $\mathbb{R}^{2n+1}$ generated by Greiner vector fields, $T>0$, $V\in L^1_{loc}(\Omega)$ is a singular potential function and $u_0(w)$is not identically zero. The doubly nonlinear operator P is

$$Pu=\nabla_k\cdot\left(u^{m-1}|\nabla_k u|^{p-2}\nabla_k u\right). \tag{1.2}$$

We assume that $k\in\mathbb{N}$, $m\in\mathbb{R}$, $1<p<Q=2n+2k$ (the homogeneous dimension) and $m+p-2>0$. $\nabla_k:=(X_1,\dots,X_n,Y_1,\dots,Y_n)$ is the subelliptic gradient, where X_j and Y_j are known as Greiner vector fields (see [1,2]):

$$X_j=\frac{\partial}{\partial x_j}+2ky_j|z|^{2k-2}\frac{\partial}{\partial l},\quad Y_j=\frac{\partial}{\partial y_j}-2kx_j|z|^{2k-2}\frac{\partial}{\partial l},\qquad j=1,\dots,n, \tag{1.3}$$

$x,y\in\mathbb{R}^n$, $z=(x,y)\in\mathbb{R}^{2n}$, $l\in\mathbb{R}$, $w=(z,l)\in\mathbb{R}^{2n+1}$. When $m=1$, $p=2$, $k=1$, P is reduced to the sub-Laplacian operator $\triangle_{H^n}$ defined on the Heisenberg group H^n, which is the simplest nontrival example of Carnot groups. When m=1, p=2, k=2,3,..., P is the Greiner operator $\triangle_k$ (see [3]), which arises in a diverse area of mathematics including boundary value problems in several complex variables, harmonic analysis, quantum mechanics of anharmonic oscillators and electromagnetic fields.

The equation of type (1.1) arises in many applications such as fields of mechanics, physics, biology, non-Newtonian fluids and gas flow in porous media (see [4–8] and references therein) . Therefore, much effort has been put on seeking solutions of it. Kombe studied this problem on Euclidean space and the Heisenberg group in [9]. The results are extended to Carnot groups in [10] and Baouendi-Grushin operators in [11]. In this paper, we deal with the same problem for Greiner vector fields.

By taking $m=1$ and $p=2$ in (1.1), the heat equation with a singular potential on Euclidean space is considered in the pioneering article of Baras and Goldstein [12]. More precisely, they proved that the following linear heat equation

$$\begin{cases}\dfrac{\partial u}{\partial t}=\Delta u+\dfrac{c}{|x|^2}u, & \text{in } \Omega\times(0,T),\\ u(x,0)=u_0(x)\geq 0, & \text{in } \Omega,\\ u(x,t)=0, & \text{on } \partial\Omega\times(0,T),\end{cases} \tag{1.4}$$

has no nonnegative solutions except $u\equiv 0$ if $c>C^*(n)=((n-2)/2)^2$, whereas positive weak solutions do exist if $c\leq C^*(n)$. Thus, $C^*(n)=((n-2)/2)^2$ is the critical value for the existence of positive solutions to (1.4). $C^*(n)$ is also sharp constant in Hardy inequality. It is well understood that the existence and nonexistence of positive solutions for (1.4) has strong connections with Hardy inequality.

If $m=1$ and $V=c/|x|^p$ in (1.1), the p-Laplace heat equation

$$\begin{cases}\dfrac{\partial u}{\partial t}=\Delta_p u+\dfrac{c}{|x|^p}u^{p-1}, & \text{in } \Omega\times(0,T),\\ u(x,0)=u_0(x)\geq 0, & \text{in } \Omega,\\ u(x,t)=0, & \text{on } \partial\Omega\times(0,T),\end{cases} \tag{1.5}$$